Journal of Mathematical Sciences: Advances and Applications
Volume 14, Number 1, 2012, Pages 27-34

A NEW METHOD FOR FINDING THE SOLUTION OF THE SECOND ORDER LINEAR DIFFERENTIAL EQUATION

G. N. MILIARAS

American University of Athens
Andrianiou 9, P. Psychico, 11525
Athens
Greece
e-mail: gmiliara@yahoo.gr

Abstract

In this paper, we give a general way to obtain a solution of the second order linear differential equation of the form

$$\frac{d^2y(x)}{dx^2} + f(x)\frac{dy(x)}{dx} + g(x)y(x) = h(x),$$

where y is an unknown function of the variable x, and f, g, h are any real continuous functions on a closed interval $[a, b]$.

1. Introduction

Consider the second order linear differential equation of the form

$$\frac{d^2y(x)}{dx^2} + f(x)\frac{dy(x)}{dx} + g(x)y(x) = h(x), \qquad \text{(E)}$$

2010 Mathematics Subject Classification: 34A05, 34A35.

Keywords and phrases: second order linear differential equation, power series.

Received February 4, 2012

where y is an unknown function of the variable x, and f, g, h are any real continuous functions on a closed interval $[a, b]$.

Equation (E) is a very old and well known problem [3-6, 9, 11, 13, 14], and has a variety of applications in science and engineering (for example, vibration of springs, electric circuits) [1, 2, 7-10, 12]. We can always obtain analytical solutions if we already know a solution of the homogeneous equation, or if the coefficient functions are developable in power series, as Cauchy proved, or if the coefficient functions are constants.

It has been proven that, the problem in its general form, i.e., when the coefficient functions are any continuous functions cannot be solved with finite integrations.

In the present paper, we give a solution, in a series form, of the homogeneous equation

$$\frac{d^2y_0(x)}{dx^2} + f(x)\frac{dy_0(x)}{dx} + g(x)y_0(x) = 0, \qquad (\mathrm{E}_0)$$

because, it is well known that in order to solve Equation (E), it is enough to find a solution of Equation (E_0). Hence, if y_0 is a solution of (E_0), then a solution of (E) is

$$y(x) = y_0(x)\left[\int y_0^{-2} e^{-\int f(x)dx}\left(\int h(x)\, e^{\int f(x)dx} + c_1\right)dx + c_2\right]. \qquad (1.1)$$

2. The Proposed Method

Setting

$$z(x) = \frac{y_0'(x)}{y_0(x)}, \qquad (2.1)$$

we easily obtain that Equation (E_0) becomes the following Riccati equation:

$$z'(x) + z^2(x) + f(x)z(x) + g(x) = 0. \qquad (\mathrm{E}_0')$$

Clearly,

$$y_0(x) = e^{\int z(x)\,dx}. \tag{2.2}$$

Consider now, the partial differential equation (PDE)

$$u_x(x,\,t) + u^2(x,\,t) + tf(x)u(x,\,t) + tg(x) = 0, \qquad (\mathrm{E}_1)$$

where $0 \le t \le 1$. Consequently, it is clear that $u(x, 1)$ is a solution of (E_0').

By Equation (E_1), for $t = 0$, we have

$$u_x(x,\,0) + u^2(x,\,0) = 0.$$

This implies that either $u(x, 0) = 0$ or $u(x, 0) = \dfrac{1}{x + c}$, where c is a real constant. Without loss of generality, we may assume that $u(x, 0) = 0$ as an initial condition.

Note: At this point, we have to mention that with the symbol "$\int$ „ of the integration we also include the constant of the integration without writing it. Also, the symbol "$u^{(n)}$„ denotes the partial derivative with respect to t of order n.

It is clear that since the functions tf and tg have partial derivatives with respect to t of any order so does u.

Based on this fact and taking into account that, if F and G are functions that have derivative of any order, we have

$$(F \cdot G)^{(n)} = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\, F^{(k)} G^{(n-k)},$$

differentiating Equation (E_1) n-times, we take

$$u_x^{(1)} + 2uu^{(1)} + tfu^{(1)} + fu + g = 0,$$

$$\vdots$$

$$\vdots$$

and

$$u_x^{(n)} + \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\, u^{(k)} u^{(n-k)} + tfu^{(n)} + nfu^{(n-1)} = 0,$$

or

$$u_x^{(n)} + 2uu^{(n)} + \sum_{k=1}^{n-1} \frac{n!}{k!\,(n-k)!}\, u^{(k)} u^{(n-k)} + tfu^{(n)} + nfu^{(n-1)} = 0.$$

These equations are linear of first order with respect to $u^{(1)}$ and $u^{(n)}$. Furthermore, we know that if

$$w'(x) + F(x)w(x) = G(x)$$

is a linear ordinary differential equation of first order, then

$$w(x) = e^{-\int F(x)\,dx}\left[\int e^{\int F(x)\,dx} G(x)\,dx + C\right].$$

So solving the linear equations with respect to $u^{(1)}$ and $u^{(n)}$, we obtain

$$u^{(1)} = -\frac{\int (g + fu) e^{\int (tf+2u)\,dx}\,dx}{e^{\int (tf+2u)\,dx}}, \tag{2.3}$$

and

$$u^{(n)} = -\frac{\int \left(nfu^{(n-1)} + \sum_{k=1}^{n-1} \frac{n!}{k!\,(n-k)!}\, u^{(k)} u^{(n-k)} \right) e^{\int (tf-2u)\,dx}\,dx}{e^{\int (tf-2u)\,dx}}. \tag{2.4}$$

Clearly, for $t = 0$, we have $u(x, 0) = 0$, $u^{(1)}(x, 0) = -\int g(x)\,dx$ and for $n > 1$,

$$u^{(n)}(x, 0) = -n\int f(x) u^{(n-1)}(x, 0) - \sum_{k=1}^{n-1} \frac{n!}{k!\,(n-k)!} \int u^{(k)}(x, 0) u^{(n-k)}(x, 0). \tag{2.5}$$

Now since tf and tg have convergent Taylor series expansions for $t = 0$ (obviously they coincide with tf and tg, respectively), so does $u(x, t)$ for $0 \le t \le 1$. Hence,

$$u(x, t) = \sum_{n=1}^{\infty} \frac{u^{(n)}(x, 0)}{n!} t^n.$$

Put

$$z(x) = u(x, 1) = \sum_{n=1}^{\infty} \frac{u^{(n)}(x, 0)}{n!},$$

this is the solution of (E_0'), and in view of (2.2), the solution of (E_0) is the following function:

$$y_0(x) = e^{\int \sum_{n=1}^{\infty} \frac{u^{(n)}(x,0)}{n!} dx}. \tag{2.6}$$

This final formula is giving a form of the solution of (E_0), in order to find the exact solution, we need to plug what we have found in the equation. We need to do this because we have infinitely many integrations, and therefore, we have an infinite number of constants, all but two of those have to be computed, and, perhaps to be eliminated.

The value of this formula is only theoretical since it does not seem to be of any practical value at all.

For example, if we consider the equation

$$y''(x) + Ay'(x) + By(x) = 0, \tag{2.7}$$

where A and B are real numbers, this is giving us the Riccati equation

$$z'(x) + z^2(x) + Az(x) + B = 0. \tag{2.8}$$

We can easily get

$$u^{(1)}(x, 0) = -Bx + c_1,$$

$$u^{(2)}(x, 0) = -B^2 \frac{x^3}{3} + (AB - Bc_1^2)x + c_2,$$

and continuing this process, we always end up to a constant arbitrary term c_n for every $u^{(n)}(x, 0)$ and each $u^{(n)}(x, 0)$ also depends on the previous constants $c_1, c_2, \cdots, c_{n-1}$.

So each coefficient of x^n is actually an arbitrary constant, notice that $u^{(n)}(x, 0)$ is a polynomial of x. Let

$$z(x) = \sum_{n=0}^{\infty} a_n x^n,$$

z satisfies the Equation (2.8) and $a_0 = c_1 + c_2 + \cdots$.

The only restriction for $c_1, c_2, \cdots, c_n, \cdots$ is that the sum $\sum_{n=1}^{\infty} c_n$ has to converge in $\mathbb{R}$ and therefore $c_n \to 0$. Now, if we plug the series form of z in the Equation (2.8), we will have

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' + \left(\sum_{n=0}^{\infty} a_n x^n\right)^2 + A\sum_{n=0}^{\infty} a_n x^n + B = 0,$$

so we should have the coefficients of x^n equal to zero for every n, so

$$a_1 + a_0^2 + Aa_0 + B = 0,$$

$$2a_2 + 2a_0a_1 + Aa_1 = 0,$$

$$3a_3 + a_1^2 + Aa_2 = 0,$$

$$\vdots$$

and so on. Giving arbitrary values to a_0, we can compute a_1 and then a_2, and we can continue forever. We notice that every a_n can be computed a_n is actualy a function $a_0, a_1, \cdots, a_{n-1}$, so finally all coefficients are functions of a_0, this means that since each a_n can be computed, we can find a solution after doing infinitely many computations.

Now, if the trinomial $w^2 + Aw + B = 0$ has real roots, then set $a_0 = \rho$ a root of the trinomial and then we clearly obtain $a_1 = a_2 = \cdots = a_{n-1} = a_n = \cdots = 0,$ and we get the known solutions. Since $z = \rho$ is an obvious solution of (2.8), and therefore $y_0(x) = e^{\rho x}$ is an obvious solution of (2.7), respectively. But what if the trinomial does not have real roots? Since a_0 is an arbitrary constant, if we put $a_0 = 0,$ we obtain

$$a_1 = -B,$$

$$a_2 = \frac{1}{2} AB,$$

$$a_3 = -\frac{1}{3} B^2 - \frac{1}{6} A^2 B,$$

and so on, but still we cannot get a final formula.

If f and g are not constants, then the solution is much more complicated.

That is why this formula has only theoretical value, but we hope that some analysts will find it interesting.

References

[1] W. F. Ames, Nonlinear Partial Differential Equations in Engineering, Vol. 1, Academic Press, New York, 1967.

[2] W. F. Ames, Nonlinear Partial Differential Equations in Engineering, Vol. 2, Academic Press, New York, 1972.

[3] G. Baumann, Symmetry Analysis of Differential Equations with Mathematica, Springer-Verlag, New York, 2000.

[4] M. Braun, Differential Equations and their Applications, HSpringer-Verlag, New York, 1975.

[5] E. Coddington, An Introduction to Ordinary Differential Equations, GPrentice-Hall, New York, 1961.

[6] L. Dresner, Similarity Solutions of Nonlinear Partial Differential Equations, Pitman, Boston, 1983.

[7] J. M. Hill, Differential Equations and Groups Methods for Scientists and Engineers, CRC Press, Boca Raton, 1992.

[8] N. H. Ibragimov (Editor), CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 2, Applications in Engineering and Physical Sciences, CRC Press, Boca Raton, 1995.

[9] G. Kristensson, Second Order Differential Equations, 1st Edition, Springer, 2010.

[10] A. D. Polyanin and V. F. Zaitsev, Handbook of Nonlinear Partial Differential Equations, Chapman & Hall/CRC Press, Boca Raton, 2004.

[11] M. M. Smirnov, Second Order Partial Differential Equations, Groningen, Noordhoff, 1966.

[12] G. Stephenson, Partial Differential Equations for Scientists and Engineers, Longman Group Ltd., London, 1985.

[13] C. Wylie, Differential Equations, LMcGraw-Hill, Tokyo, 1979.

[14] D. Zwillinger, Handbook of Differential Equations, 3rd Edition, Academic Press, Boston, 1997.

■