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PRODUCTS OF RADIAL TOEPLITZ OPERATORS ON WEIGHTED BERGMAN SPACE

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Abstract

In this paper, we find necessary and sufficient conditions for the product of two Toeplitz operators of certain symbols to be a Toeplitz operator and give a formula of the product on weighted Bergman space.

1. Introduction

Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equal to 1. For $\alpha > -1$, we denote by dA_α the measure $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$. For $1 \le p < +\infty$, the space $L^p(\mathbb{D}, dA_\alpha)$ is a Banach space. The weighted Bergman space A^2_α is the closed subspace of analytic functions in the Hilbert space $L^2(\mathbb{D}, dA_\alpha)$. For each $z \in \mathbb{D}$, the application

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$$L_z : A^2_\alpha \longrightarrow \mathbb{C},$$
$$f \longmapsto f(z)$$

is continuous and can be represented as $L_z(f) = \langle f, K^{(\alpha)}_z \rangle_\alpha$, where

$$K^{(\alpha)}_z(w) = \frac{1}{(1 - w\bar{z})^{2+\alpha}} = \sum_{n=0}^{\infty} \frac{\Gamma(n + 2 + \alpha)}{n!\,\Gamma(2 + \alpha)} (w\bar{z})^n, \quad z, w \in \mathbb{D}.$$

This means that, if P_α is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto A^2_α, then P_α can be defined by

$$(P_\alpha f)(w) = \langle f, K^{(\alpha)}_w \rangle_\alpha = \int_{\mathbb{D}} f(z) \frac{1}{(1 - \overline{w}z)^{2+\alpha}} dA_\alpha(z).$$

For $u \in L^\infty(\mathbb{D}, dA_\alpha)$, we define the Toeplitz operator with symbol u, $T_u : A^2_\alpha \to A^2_\alpha$ by the equation

$$T_u(f)(z) = P_\alpha(uf)(z) = \int_{\mathbb{D}} u(w) f(w) \overline{K^{(\alpha)}_z(w)} dA_\alpha(w). \tag{1.1}$$

The operators defined in this way are the simplest and most natural Toeplitz operators (since the product of an L^∞ and an L^2 function is always a well defined element of L^2). But for reasons which will become evident, we prefer to consider a more general class of Toeplitz operators.

Let u be any finite complex measure on $\mathbb{D}$. In analogy with Equation (1.1), we can define an operator T_u on A^2_α by

$$T_u f(z) = \int_{\mathbb{D}} f(w) \overline{K^{(\alpha)}_z(w)} d_\alpha u(w). \tag{1.2}$$

If $d_\alpha u(z) = F(z) dA_\alpha(z)$ for some $F \in L^1(\mathbb{D}, dA_\alpha)$, then, we simply write $T_u = T_F$. This operator is always defined on the polynomials and the image of any polynomial is always an analytic function on the disc. We are interested in the case, where densely defined operator is bounded in the A^2_α norm. This happens often. For example, if u has compact support, then T_u is not only bounded, but also compact.

Thus, if $F \in L^1(\mathbb{D}, dA_\alpha)$ and there is an $r \in (0, 1)$ such that F is (essentially) bounded on the annulus $\{z : r < |z| < 1\}$, then F is equal to the sum of an L^1 function with compact support and an L^∞ function and so T_F is a bounded operator. There is, unfortunately, no characterization of the functions in $L^1(\mathbb{D}, dA_\alpha)$, which correspond to bounded operators. This motivates the following definition:

Definition 1.1. Let $F \in L^1(\mathbb{D}, dA_\alpha)$.

(a) We say that F is a T-function if the Equation (1.1), with $u = F$, defines a bounded operator on A^2_α.

(b) If F is a T-function, we write T_F for the continuous extension of the operator defined by the Equation (1.1). We say that T_F is a Toeplitz operator, if and only if T_F is defined in this way.

(c) If there is an $r \in (0, 1)$ such that F is (essentially) bounded on the annulus $\{z : r < |z| < 1\}$, then we say that F is "nearly bounded".

In [2], the authors get a Brown-Halmos type result. More generally, in [1], Ahern and Cucković consider the product $T_f T_g$ for f and g bounded harmonic functions on the disc such that $f = f_1 + \overline{f_2}$ and $g = g_1 + \overline{g_2}$ with f_1, f_2, g_1, and g_2 are bounded analytic functions. In this paper, we find necessary and sufficient conditions for the product of two Toeplitz operators of certain symbols to be a Toeplitz operator, and give a formula of the product on weighted Bergman space base on [5].

2. Product of Toeplitz Operators with Radial Symbols

An operator that will arise in our study of Toeplitz operators is the Mellin transform, defined for any function $\varphi \in L^1([0, 1], rdr)$, by the formula $\widehat{\varphi_\alpha}(z)$ is the Mellin transform

$$\widehat{\varphi_\alpha}(z) = \int_0^1 \varphi(r)(1-r^2)^\alpha r^{z-1} dr,$$

which is a bounded holomorphic function in the half plane $\{z : \mathrm{Re} z \geq 2\}$.

When considering the product of two Toeplitz operators, we shall often be confronted with the "Mellin" or "multiplicative" convolution of their symbols. We denote the Mellin convolution of two functions f and g by $f *_M g$ and we define it to be

$$(f *_M g)(r) = \int_r^1 f(\frac{r}{t}) g(t) \frac{dt}{t}.$$

The multiplication $*_M$ is related to the normal convolution by the change of variable discussed above.

It is easy to see that the Mellin transform converts the convolution product into a pointwise product, i.e., that

$$(\widehat{f *_M g})(r) = \widehat{f}(r)\widehat{g}(r),$$

and that, if f and g are in $L^1([0, 1], rdr)$, then so is $f *_M g$.

Let $\varphi \in L^1(\mathbb{D}, dA_\alpha)$ be a radial function, i.e., suppose that

$$\varphi(z) = \varphi(|z|), \quad z \in \mathbb{D}.$$

Then, if φ is a T-function, the Toeplitz operator with symbol φ acts in a very simple way. In fact, if we define the function φ_r on [0, 1] by

$$\varphi_r(s) = \varphi(s),$$

then a direct calculation shows that

$$\langle T_\varphi(z^k), z^l \rangle_\alpha = \begin{cases} 0, & \text{for} \quad k \neq l, \\ 2(\alpha+1)\widehat{\varphi_\alpha}(2k+2), & \text{for} \quad k = l. \end{cases} \tag{2.1}$$

So that if $k \in \mathbb{N}$,

$$T_\varphi(z^k) = 2(1+\alpha)\frac{\Gamma(k+2+\alpha)}{k!\,\Gamma(2+\alpha)}\int_0^1 \varphi r^{2k+1}(1-|r|^2)^\alpha\, dr z^k$$

$$= 2(1+\alpha)\frac{\Gamma(k+2+\alpha)}{k!\,\Gamma(2+\alpha)}\widehat{\varphi_{\alpha,r}}(2k+2)z^k. \tag{2.2}$$

Thus T_φ is a diagonal operator on A^2_α with coefficient sequence

$$\left(2(1+\alpha)\frac{\Gamma(k+2+\alpha)}{k!\,\Gamma(2+\alpha)}\widehat{\varphi_{\alpha,r}}(2k+2)\right)_{k=0}^{\infty}.$$

This makes it relatively simple to work with the product of two operators with such radial symbols.

Remark. In the following, we shall often identify an integrable radial function φ on the unit disc with the corresponding function φ_r defined on the interval $[0, 1]$. For example, if we speak of the multiplicative convolution $*_M$ of two radial functions φ_1 and φ_2, we mean the radial function φ_3 such that $\varphi_{3,r} = \varphi_{1,r} *_M \varphi_{2,r}$. Similarly, the Mellin coefficients of an integrable radial function φ are defined to be those of the function φ_r.

Now, we define the "radialization" of a function $f \in L^1(\mathbb{D}, dA_\alpha)$ by

$$rad(f)(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}z)dt.$$

It is clear that a function f is a radial, if and only if $rad(f) = f$.

Proposition 2.1. *Let* $\varphi \in L^1(\mathbb{D}, dA_\alpha)$. *Then the following assertions are equivalent*:

(a) *For all* $k \geq 0$, *there exist* $\lambda_k \in \mathbb{C}$ *such that* $T_\varphi(z^k) = \lambda_k z^k$.

(b) φ *is a radial function.*

Proof. Writing out the integrals and changing the order of integration, we see that, for each $n,\ m \in \mathbb{Z}_+$

$$\begin{aligned}\langle T_{rad(\varphi)} z^n,\ z^m \rangle_\alpha &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{D}} \varphi(e^{it}z) z^n \overline{z}^m dA_\alpha(z)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{D}} \varphi(w) e^{-int} w^n e^{imt}\overline{w}^m dA_\alpha(w)dt \\ &= \frac{1}{2\pi}\left(\int_0^{2\pi} e^{i(m-n)t}dt\right)\langle T_\varphi z^n,\ z^m\rangle_\alpha.\end{aligned}$$

Thus,

$$\langle T_{rad(\varphi)} z^n,\ z^m \rangle_\alpha = \begin{cases} 0, & \text{for} \quad n \neq m, \\ \langle T_\varphi z^n,\ z^m\rangle_\alpha, & \text{for} \quad n = m. \end{cases} \tag{2.3}$$

Thus $T_{rad(\varphi)} = T_\varphi$, if and only if (a) is true. And $T_{rad(\varphi)} = T_\varphi$, if and only if $rad(\varphi) = \varphi$.

Corollary 2.2. *Let φ_1 and φ_2 be radial T-functions. If $T_{\varphi_1} T_{\varphi_2} = T_\psi$, then ψ is a radial T-function.*

Proof. By Using Equation (2.2) to calculate $T_{\varphi_1} T_{\varphi_2}(z^k)$, we see that Proposition 2.1 implies that ψ is a radial function. Moreover, T_ψ is clearly a bounded operator.

Theorem 2.3. *Let φ_1 and φ_2 be radial T-functions. Then $T_{\varphi_1} T_{\varphi_2}$ is equal to the Toeplitz operator T_ψ, if and only if ψ is a solution of the equation*

$$\frac{k!\,\Gamma(2+\alpha)}{2(\alpha+1)\,\Gamma(k+2+\alpha)}\widehat{\psi_\alpha}(2k+2) = (\varphi_{\alpha,\widehat{1 *_M \varphi_{\alpha,2}}})(2k+2).$$

Proof.

$$T_\varphi(z^k) = 2(1+\alpha)\frac{\Gamma(k+2+\alpha)}{k!\,\Gamma(2+\alpha)}\int_0^1 \varphi r^{2k+1}(1-|r|^2)^\alpha\, dr z^k$$

$$= 2(1+\alpha)\frac{\Gamma(k+2+\alpha)}{k!\,\Gamma(2+\alpha)}\,\widehat{\varphi_\alpha}(2k+2)z^k,$$

$$T_\psi(z^k) = 2(1+\alpha)\frac{\Gamma(k+2+\alpha)}{k!\,\Gamma(2+\alpha)}\,\widehat{\psi_\alpha}(2k+2)z^k,$$

thus, if

$$T_{\varphi_1}T_{\varphi_2}(z^k) = T_\psi(z^k), \quad k \geq 0,$$

then

$$\widehat{\psi_\alpha}(2k+2) = 2(1+\alpha)\frac{\Gamma(k+2+\alpha)}{k!\,\Gamma(2+\alpha)}\,\widehat{\varphi}_{\alpha,1}(2k+2)\,\widehat{\varphi}_{\alpha,2}(2k+2).$$

Thus, we can have the conclusion that

$$\frac{k!\,\Gamma(2+\alpha)}{2(\alpha+1)\,\Gamma(k+2+\alpha)}\,\widehat{\psi_\alpha}(2k+2) = (\varphi_{\alpha,1}\widehat{*_M}\varphi_{\alpha,2})(2k+2).$$

Here, we discuss the order of the products of Toeplitz operators with quasihomogeneous symbols. Let $\mathfrak{R}$ be the space of square integrable radial functions on $\mathbb{D}$. As before, we identify these functions with the associated functions on $[0, 1]$ that are square integrable with respect to rdr measure. By using that trigonometric polynomials are dense in $L^2(\mathbb{D}, dA_\alpha)$ and that, for $k_1 \neq k_2$, $e^{ik_1\theta}R$ is orthogonal to $e^{ik_2\theta}R$, we see that

$$L^2(\mathbb{D}, dA_\alpha) = \oplus_{k\in\mathbb{Z}} e^{ik\theta}R^\alpha.$$

Definition 2.4. Let φ be a function in $L^1(\mathbb{D}, dA_\alpha)$, which is of the form $e^{ik\theta}f$, where f is a radial function. Then, we say that φ is a quasihomogeneous function of quasihomogeneous degree k.

A direct calculation gives the following lemma, which we shall use often.

Lemma 2.5. *Let* $k, p \in \mathbb{Z}_+$ *and let* φ *be an integrable radial function. Then, if* $e^{ip\theta}\varphi$ *is a T-function, we have*

$$T_{e^{ip\theta}\varphi}(z^k) = 2(\alpha+1)\frac{\Gamma(k+p+2+\alpha)}{(k+p)!\,\Gamma(2+\alpha)}\widehat{\varphi_\alpha}(2k+p+2)\,z^{k+p},$$

and

$$T_{e^{-ip\theta}\varphi}(z^k) = \begin{cases} 0, & \text{if} \quad k < p, \\ 2(\alpha+1)\frac{\Gamma(k-p+2+\alpha)}{(k-p)!\,\Gamma(2+\alpha)}\widehat{\varphi_\alpha}(2k-p+2)z^{k-p}, & \text{if} \quad k \geq p. \end{cases}$$

Proposition 2.6. *A bounded function* φ *is quasihomogeneous of degree* $p \in \mathbb{Z}$, *if and only if for all integers* $n \geq 0$, *there exists* $\lambda_n \in \mathbb{C}$ *such that*

$$T_\varphi(z^n) = \begin{cases} 0, & \text{if} \quad n < \max(-p, 0), \\ \lambda_n z^{n+p}, & \text{if} \quad n \geq \max(-p, 0). \end{cases} \tag{2.4}$$

Proof. Let φ be a quasihomogeneous function of degree p, i.e., $\varphi = e^{ip\theta}f$, where φ is a radial function. The necessity is a direct consequence of Lemma 2.5.

Conversely, suppose that Equation (2.4) is true for any $p \in \mathbb{Z}$. Then for any positive integers n and m;

if $p \geq 0$,

$$\langle T_{\bar{z}^p\varphi}z^n, z^m\rangle_\alpha = \langle T_\varphi z^n, z^{m+p}\rangle_\alpha = \begin{cases} 0, & \text{if} \quad n \neq m, \\ (1+\alpha)\lambda_m B(m+p+\frac{3}{2}, \alpha+1), & \text{if} \quad n = m; \end{cases}$$

and if $p < 0$,

$$\langle T_{z^{-p}\varphi}z^n, z^m\rangle_\alpha = \langle T_\varphi z^{n-p}, z^m\rangle_\alpha = \begin{cases} 0, & \text{if} \quad n \neq m, \\ (1+\alpha)\lambda_{m-p} B(m+\frac{3}{2}, \alpha+1), & \text{if} \quad n = m. \end{cases}$$

Thus, if we note

$$f = \begin{cases} \bar{z}^p \varphi, & \text{if} \quad p \geq 0, \\ z^{-p} \varphi, & \text{if} \quad p < 0, \end{cases}$$

then by Proposition 2.1, f is a radial function. But this easily implies that φ is a quasihomogeneous function of quasihomogeneous degree p.

Proposition 2.7. *Let k_1 and k_2 be greater than, or equal to zero and let φ_1 and φ_2 be quasihomogeneous T-functions in $L^1(\mathbb{D}, dA_\alpha)$ of quasihomogeneous degrees k_1 and $-k_2$, respectively. If there exist a T-function ψ is of quasihomogeneous degree $k_1 - k_2$.*

Proof. Let φ_1, φ_2, and ψ be as above and f_1 and f_2 be radial functions such that $\varphi_1 = e^{ik_1\theta} f_1$ and $\varphi_2 = e^{-ik_2\theta} f_2$. As discussed above, if f is antianalytic or if g is analytic (bounded), then $T_f T_g = T_{fg}$. Thus, since $T_{\varphi_1} T_{\varphi_2} = T_\psi$, we see that

$$T_{\bar{z}^{k_1}\varphi_1} T_{z^{k_2}\varphi_2} = T_{r^{k_1} f_1} T_{r^{k_2} f_2} = T_{\bar{z}^{k_1}} T_{z^{k_2}\psi}.$$

Now, by Corollary 2.2, $\bar{z}^{k_1} z^{k_2} \psi$ is a radial function. This shows that ψ is a quasihomogeneous function of quasihomogeneous degree $k_1 - k_2$.

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