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ON THE NILCLOSURE IN A CATEGORY OF UNSTABLE MODULES

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Abstract

In this paper, we investigate nilclosure of $F'(2)^{\otimes n}/\Sigma_n$ in the quotient category $\mathcal{U}'/Nil'$, where $\mathcal{U}'$ is the category of evenly graded unstable modules, Nil' is the category of evenly graded nilpotent unstable modules, and Σ_n is a symmetric group.

1. Introduction and Main Results

Let p be an odd prime number and $\mathcal{A}$ be the mod-p Steenrod algebra. Let $\mathcal{U}$ be the category of unstable modules over mod-p Steenrod algebra $\mathcal{A}$. In [3], Gabriel constructed the quotient category. Later, Franjou and Schwartz [2] have introduced $\mathcal{U}/Nil$ the quotient category of unstable modules by nilpotent modules. Henn et al. [4] have proved that $\mathcal{U}/Nil$ is equivalent to representation theoretic category $\mathcal{F}_w$, where category $\mathcal{F}_w$

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is the category of analytic functors. Kuhn [7] has given more general description of category $\mathcal{F}_w$ called there 'category of locally finite functors'.

In this paper, we are restricting our attention to full subcategory $\mathcal{U}'$ of $\mathcal{U}$ consisting of unstable modules, which are nontrivial only in even degree. So $F'(2n)$ is free unstable modules over the subalgebra $\mathcal{A}_p$ of $\mathcal{A}$ generated by reduced power P^i, $i > 0$. $F'(2)^{\otimes n}$ is the unstable modules because tensor product of unstable modules with diagonal action is unstable. The symmetric group Σ_n acts on $F'(2)^{\otimes n}$ by permutation of coordinates. Hence, we have

$$F'(2n) \simeq (F'(2)^{\otimes n})^{\Sigma_n},$$

(see [8]).

In the category $\mathcal{F}_w$, there exists naturally arising algebra

$$M_{*,*} = Hom_{\mathcal{F}_w}(S_*, S^*),$$

where S_* and S^* denote the graded symmetric invariants and coinvariants, respectively.

In [7], Kuhn has shown that $M_{*,*}$ is isomorphic to a polynomial algebra generated by the elements $u_{(1,p^k)}$ and $u_{(p^k,1)}$, where $u_{(1,p^k)}$ has bidegree $(1, p^k)$ and $u_{(p^k,1)}$ has bidegree $(p^k, 1)$. Hence, this allows us to compute nilclosure of $(F'(2)^{\otimes n})^{\Sigma_n}$. Using the equivalence,

$$\mathcal{F}_w \simeq \mathcal{U}'/Nil',$$

we have the isomorphism

$$M_{*,j} = \mathcal{N}(F'(2)^{\otimes j})/\Sigma_j .$$

2. Preliminaries

Definition 2.1. An $\mathcal{A}$-module M in the category $\mathcal{U}'$ is called *unstable*, if $P^i(x) = 0$ for $2i > |x|$ and $x \in H^*(X)$, where $|x|$ denotes the degree of x.

Let Φ' denote a functor from $\mathcal{U}'$ to itself. Given $M \in \mathcal{U}'$ define $\Phi' M$ by

$$(\Phi' M)^{2m} = \begin{cases} M^{\frac{2m}{p}}, & \text{if } 2m \equiv 0 \bmod 2p, \\ M^{\frac{2m-2}{p}+1}, & \text{if } 2m \equiv 2 \bmod 2p, \\ 0, & \text{otherwise.} \end{cases}$$

The action of the Steenrod algebra is given by

$$P^i \Phi'(x) = \begin{cases} \Phi' P^{i/p}(x), & \text{if } i \equiv 0 \bmod p, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\beta \Phi'(x) = 0,$$

where $\Phi'(x)$ denotes the element of $\Phi' M$ corresponding to x in M. The map

$$\lambda'_M : \Phi' M \longrightarrow M$$

is defined by $\lambda'_M(\Phi' x) = P^m(x) = P_0(x)$, where the degree of x is $2m$. Define an $\mathcal{A}$-linear map

$$P_{2i} : M^{2m} \longrightarrow M^{2mp-2i(p-1)},$$

by $P_{2i}(x) = P^{m-i}(x)$.

We denote by $\Sigma' : \mathcal{U}' \longrightarrow \mathcal{U}'$ the suspension functor, by $\widetilde{\Phi}'$ (resp., $\widetilde{\Sigma}'$) the right adjoint of Φ' (resp., Σ') and $R^i\widetilde{\Phi}'$ (resp., $R^i\,\widetilde{\Sigma}'$) the derived functor of $\widetilde{\Phi}'$ (resp., $\widetilde{\Sigma}'$). Let

$$\Sigma' i_{2m} : F'(2(m+1)) \to \Sigma' F'(2m)$$

be the map, which sends $i_{2(m+1)}$ to $\Sigma' i_{2m}$.

It is easy to check that tensor product of two unstable modules equipped with the diagonal action is unstable. So consider the unstable module $F'(2)^{\otimes n}$; it is of dimension 2 in degree n. Therefore, up to a scaler multiple, there is one nontrivial map

$$h : F'(2n) \longrightarrow F'(2)^{\otimes n},$$

defined by $h(i_{2n}) = y \otimes y \otimes \cdots \otimes y$. The symmetric group Σ_n acts on $F'(2)^{\otimes n}$ by permuting factor, i.e.,

$$\sigma(y^{p^{\alpha_1}} \otimes y^{p^{\alpha_2}} \otimes \cdots \otimes y^{p^{\alpha_n}}) = y^{p^{\alpha_{\sigma^{-1}(1)}}} \otimes \cdots \otimes y^{p^{\alpha_{\sigma^{-1}(n)}}}.$$

The action Σ_n of $F'(2)^{\otimes n}$ commutes with Steenrod algebra action and therefore the range of h is contained in the invariants $(F'(2)^{\otimes n})^{\Sigma_n}$. So, we have the following lemma:

Lemma 2.2. *The map* $h : F'(2n) \longrightarrow (F'(2)^{\otimes n})^{\Sigma_n}$ *is an isomorphism.*

Definition 2.3. Let M be an unstable module in $\mathcal{U}'$. An element x of an unstable module M is called *nilpotent*, if

$$P^{p^{k\frac{|x|}{2}}} \cdot P^{p^{k-1}\frac{|x|}{2}} \cdots P^{\frac{|x|}{2}}(x) = 0,$$

for some positive integer k. An unstable module M is said to be *nilpotent*, if every element of M is nilpotent.

Now, we can consider the quotient category $\mathcal{U}'/Nil'$. If $r' : \mathcal{U}' \longrightarrow \mathcal{U}'/Nil'$ is the projection functor, then there exists a right adjoint s' to r', i.e., a functor $s' : \mathcal{U}'/Nil' \longrightarrow \mathcal{U}'$ such that

$$Hom_{\mathcal{U}'/Nil'}(r'M, N) \simeq Hom_{\mathcal{U}'}(M, s'N).$$

Definition 2.4. Let $Nil'(M)$ denote the submodule of nilpotent elements of M.

(1) An unstable module M is said to be *reduced*, if $Nil'(M) = 0$.

(2) An unstable module M is called *Nil'-closed*, if the unit of adjoint $\varepsilon : M \longrightarrow s'r'M$ is an isomorphism.

From [1, Proposition 2.1], we have the following lemma:

Lemma 2.5. *An unstable module M is Nil'-closed, if and only if $Ext^i_{\mathcal{U}'}(N, M) = 0$ for any nilpotent module N and $i = 0, 1$.*

As a direct consequence of Lemma 2.5, we note that a product of Nil'-closed unstable $\mathcal{A}_p$-modules is Nil'-closed and that the kernel of a map between Nil'-closed unstable $\mathcal{A}_p$-modules is Nil'-closed. Therefore, any inverse limit of Nil'-closed unstable $\mathcal{A}_p$-modules is still Nil'-closed.

Let $\mathcal{N}(M)$ denote $s'r'M$. So, it has the following universal property: Given an $\mathcal{A}_p$-linear map $g' : M \longrightarrow N$, where N is Nil'-closed, there exists a unique $\mathcal{A}_p$-linear map $\mathcal{N}(g')$ making the following diagram commutative:

$$\begin{array}{ccc} M & \longrightarrow & \mathcal{N}(M) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{N}(g')} \\ N & \xrightarrow{=} & N. \end{array}$$

The following result immediately follows from [9, Theorem 6.3.3] and [9, Theorem 6.3.4].

Proposition 2.6. (1) *An $\mathcal{A}_p$-module $M \in U'$ is Nil'-closed, if and only if the adjoint to λ'*

$$\widetilde{\lambda}' : M \to \widetilde{\Phi}'M$$

is an isomorphism.

(2) *The functor $M \longmapsto colim_k \widetilde{\Phi}'^k M$ and $\mathcal{N}(M)$ are naturally equivalent.*

Let $\mathcal{E}_\infty$ and $\mathcal{E}$ be the category of $\mathbb{F}_p$-vector space and the category of finite dimensional vector space, respectively. Let $\mathcal{F}$ denote the category of covariant functors and natural transformation from $\mathcal{E}$ to $\mathcal{E}_\infty$.

Definition 2.7. (1) A sequence $F \longrightarrow G \longrightarrow H$ is exact if for any elementary abelian p-group V, the sequence $F(V) \longrightarrow G(V) \longrightarrow H(V)$ is exact.

(2) A functor $F \in \mathcal{F}$ is simple, if the only subobjects of F are the 0 and F itself.

(3) A functor $F \in \mathcal{F}$ is finite, if it has a finite composition series with simple subquotients.

(4) A functor $F \in \mathcal{F}$ is locally finite, if it is the union of its finite subfunctors.

Let $\mathcal{F}_w$ denote the full subcategory of locally finite functor. The definition of $\mathcal{F}_w$ as in (4) is due to Kuhn and it is different from the original definition given in [4]. However, it is also shown in [6] that the two definitions are equivalent.

We now define a functor $f' : \mathcal{U}' \longrightarrow \mathcal{F}$ as follows: The functor $f'(M) : \mathcal{E} \longrightarrow \mathcal{E}_\infty$ associated to an unstable $\mathcal{A}_p$-module N is given by the formula

$$f'(M)(V) = \left(Hom_{\mathcal{U}'}(M, H^*V)\right)',$$

where $(Hom_{\mathcal{U}'}(M, H^*V))'$ denotes the continuous dual of profinite $\mathbb{F}_p$-vector space

$$Hom_{\mathcal{U}'}(M, H^*V).$$

The profinite structure of $Hom_{\mathcal{U}'}(M, H^*V)$ comes from the fact that this $\mathbb{F}_p$-vector space is the filtered inverse limit of the finite dimensional $\mathbb{F}_p$-vector space $Hom_{\mathcal{U}'}(M_\alpha, H^*V)$, M_α running through the set of sub-$\mathcal{A}_p$-modules of M finitely generated over $\mathcal{A}$; an element of $(Hom_{\mathcal{U}'}(M, H^*V))'$ is a continuous linear map from $Hom_{\mathcal{U}'}(M, H^*V)$ equipped with this profinite topology to $\mathbb{F}_p$ equipped with its discrete topology.

Let M be an unstable $\mathcal{A}_p$-module and E be $\mathbb{F}_p$-vector space. Let us assume first that M is finitely generated over $\mathcal{A}_p$ and E finite dimensional. It is clear that in this case that one has a natural isomorphism

$$Hom_{\mathcal{U}'}(M, E \otimes H^*(V)) \simeq Hom_{\mathcal{E}_\infty}(f'(M)(V), E)$$

(in this formula E is viewed as an unstable $\mathcal{A}$-module concentrated in degree zero). Taking filtered direct limits, first over E and then over M, shows that this isomorphism holds without assumptions on M and E. Thus, the functor $\mathcal{U}' \longrightarrow \mathcal{E}_\infty$, $M \longmapsto f'(M)(V)$ is left adjoint to functor $\mathcal{E}_\infty \longrightarrow \mathcal{U}'$, $E \longmapsto E \otimes H^*(V)$. This shows that $f'(M)(V)$ identifies with the vector space $(T'_V(M))^0$ of degree zero elements in the unstable $\mathcal{A}$-module $T'_V(M)$, $T'_V : \mathcal{U}' \longrightarrow \mathcal{U}'$ denoting the functor left adjoint to the functor $\mathcal{U}' \longrightarrow \mathcal{U}'$, $N \longmapsto H^*V \otimes N$. So, we can state the properties of the functor f', which is analogue of the result in [8].

Proposition 2.8. (1) *The functor* f' *is exact.*

(2) *The functor* f' *commutes with the tensor products.*

So f' is zero on Nil' and hence f' admits a factorization;

$$\begin{array}{ccc} \mathcal{U}' & \xrightarrow{f'} & \mathcal{F} \\ {\scriptstyle r'}\downarrow & & \uparrow{\scriptstyle \bar{f}'} \\ \mathcal{U}'/Nil' & \xrightarrow{=} & \mathcal{U}'/Nil' \end{array}.$$

By [4, Theorem 1.6.2], we immediately have the following result:

Proposition 2.9. *The functor* $\bar{f}'$ *induces an equivalence of abelian category* $\mathcal{U}'/Nil' \longrightarrow \mathcal{F}_w$.

The functor $m' : \mathcal{F}_w \longrightarrow \mathcal{U}'$, which is corresponding to $s' : \mathcal{U}'/Nil' \longrightarrow \mathcal{U}'$ is a right adjoint of f'. So m' is given as follows:

$$m'(G)^n = Hom_{\mathcal{F}}(H_n, G),$$

and therefore, we have

$$m'f'(M) = \mathcal{N}(M).$$

3. Main Results

Let $(k, \alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$, be a sequence of nonnegative integers. Define an equivalence relation on the set of such sequence by

$$(k, \alpha_1, \alpha_2, \ldots, \alpha_n) \sim (k-1, \alpha_1 - 1, \ldots, \alpha_n - 1),$$

if $\alpha_1 > 0$ and $k > 0$. Then we can call $(k, \alpha_1, \ldots, \alpha_n)$ irreducible, if $k = 0$ or $\alpha_1 = 0$.

Theorem 3.1. *Let S be the set of irreducible sequence. Let B be subset of S with the following property*: *An irreducible sequence* $(k, \alpha_1, \ldots, \alpha_n)$ *is in B, if*

(1) $k = 0$, *or*

(2) $k > 0$ *and the number of* α_i *equal to* j *is divisible by* p^{k-j}, *where* $0 \le j \le k-1$.

Then Nilclosure of $F'(2)^{\otimes n} / \Sigma_n$ *has a* $\mathbb{F}_p$*-basis corresponding to subset* B *of* S.

Proof. Note that a sequence $(k, \alpha_1, \ldots, \alpha_n)$ will correspond to the element $x \in F'(2)^{\otimes n} / \Sigma_n$ such that

$$P^{p^k|x|}(x)P^{p^{k-1}|x|}(x)\cdots P^{|x|}(x) = [y^{p^{\alpha_1}} \otimes y^{p^{\alpha_2}} \otimes \cdots \otimes y^{p^{\alpha_n}}],$$

where x has even degree and $[\cdots]$ denotes the class of the monomial $y^{p^{\alpha_1}} \otimes y^{p^{\alpha_2}} \otimes \cdots \otimes y^{p^{\alpha_n}}$ in $F'(2)^{\otimes n} / \Sigma_n$. Then, the module $F'(2)^{\otimes n} / \Sigma_n$ has a basis corresponding to the orbits of monomials

$$y^{p^{\alpha_1}} \otimes y^{p^{\alpha_2}} \otimes \cdots \otimes y^{p^{\alpha_n}},$$

where $\alpha_1 \ge \alpha_2 \ge \alpha_3 \ge \cdots \ge \alpha_n$. It is clear that these monomials correspond to the irreducible sequences with $k = 0$. First, let us attach Steenrod operations to $F'(2)^{\otimes n} / \Sigma_n$, i.e., the element $x \in F'(2)^{\otimes n} / \Sigma_n$ such that

$$P^{|x|}(x) = [y^{p^{\alpha_1}} \otimes y^{p^{\alpha_2}} \otimes \cdots \otimes y^{p^{\alpha_n}}].$$

Then we get the module M_1 in this way. Using the Proposition 2.6, we have the following:

$$F'(2)^{\otimes n} / \Sigma_n \xrightarrow{\mathcal{N}(f)} \mathcal{N}(F'(2)^{\otimes n} / \Sigma_n).$$

Let $[y^{p^{\alpha_1}} \otimes y^{p^{\alpha_2}} \otimes \cdots \otimes y^{p^{\alpha_n}}]$ be a class in $F'(2)^{\otimes n} / \Sigma_n$ and $\alpha_1 > 0$. Then, we will have

$$P^{|x|}(x) = [y^{p^{\alpha_1}} \otimes y^{p^{\alpha_2}} \otimes \cdots \otimes y^{p^{\alpha_n}}],$$

where $x = y^{p^{\alpha_1 - 1}} \otimes y^{p^{\alpha_2 - 1}} \otimes \cdots \otimes y^{p^{\alpha_n - 1}}$.

If $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_k = 0$, $\alpha_{k+1} > 0$, and k is nonzero in mod-p, then the monomial class

$$[y \otimes y \otimes \cdots \otimes y \otimes y^{p^{\alpha_{k+1}}} \otimes \cdots \otimes y^{p^{\alpha_n}}],$$

does not admit a Steenrod operation because of degree.

Let $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_k = 0$ and k is zero in mod-p. Then consider the coaction map

$$\lambda : M \longrightarrow M \otimes \mathcal{A}^*,$$

where $(M \otimes \mathcal{A}^*)^n = \prod_{k-i=n} M_k \otimes \mathcal{A}_i^*$, is then given by

$$\lambda(m) = \sum m_I \otimes \xi^I,$$

if and only if for any $\theta \in \mathcal{A}$, we have

$$\theta m = \sum < \theta, \xi^I > m_I,$$

where $< .,. >$ denotes the standard Kronecker pairing between $\mathcal{A}$ and $\mathcal{A}^*$.

So using the property of λ, we have

$$\lambda(y_1 \otimes \cdots \otimes y_k \otimes y_{k+1}^{p^{\alpha_{k+1}}} \otimes \cdots \otimes y_n^{p^{\alpha_n}}) = (\sum_{i_1 \geq 0} y_1^{p^{i_1}} \otimes \xi_{i_1}) \cdots$$

$$(\sum_{i_k \geq 0} y_k^{p^{i_k}} \otimes \xi_{i_k}) \cdots (\sum_{i_{k+1} \geq 0} y_{k+1}^{p^{\alpha_{k+1} + i_{k+1}}} \otimes \xi_{i_{k+1}}^{p^{\alpha_{k+1}}}) \cdots$$

$$(\sum_{i_n \geq 0} y_n^{p^{\alpha_n + i_n}} \otimes \xi_{i_n}^{p^{\alpha_n}}).$$

Let $k = \sum_{j=1}^{r} l_j$ and $\rho_i = \sum_{m=1}^{i} l_m$. Then we get

$$\sum_{\substack{i_1,\ldots,i_r\geq 0\\ i_{k+1},\ldots,i_n\geq 0}} (y_1 \otimes \cdots \otimes y_{\rho_1})^{p^{i_1}} \otimes \cdots \otimes (y_{\rho_1+1} \otimes \cdots \otimes y_{\rho_2})^{p^{i_2}}$$

$$\otimes \cdots \otimes (y_{\rho_{r-1}+1} \otimes \cdots \otimes y_k)^{p^{i_r}} \otimes y_{k+1}^{p^{\alpha_{k+1}+i_{k+1}}}$$

$$\otimes \cdots \otimes y_n^{p^{\alpha_n+i_n}} \otimes \xi_{i_1}^{p^{l_1}} \cdots \xi_{i_r}^{p^{l_r}} \xi_{i_{k+1}}^{p^{\alpha_{k+1}}} \cdots \xi_{i_n}^{p^{\alpha_n}}. \qquad (^*)$$

It is clear that terms of the form

$$y_t^{p^r} \otimes y_u^{p^s} \otimes \xi_r\xi_s + y_t^{p^r} \otimes y_u^{p^s} \otimes \xi_r\xi_s,$$

for $r \neq s$ are zero because of the action of $\sum_n$. So terms of the form

$$(y_t^{p^r} \otimes y_u^{p^r}) \otimes (y_v^{p^s} \otimes y_w^{p^s}) \otimes \xi_r^p\xi_s^p + (y_t^{p^s} \otimes y_u^{p^s}) \otimes (y_v^{p^r} \otimes y_w^{p^r}) \otimes \xi_r^p\xi_s^p,$$

for $r \neq s$ are zero. So $(^*)$ holds by induction. We attach to $F'(2)^{\otimes n}/\sum_n$ Steenrod operation of elements of the form

$$y_1 \otimes y_2 \otimes \cdots \otimes y_k \otimes y_{k+1}^{p^{\alpha_{k+1}}} \otimes \cdots \otimes y_n^{p^{\alpha_n}}.$$

The property of λ helps us to define the coaction map

$$\lambda((y_1 \otimes \cdots \otimes y_k \otimes y_{k+1}^{p^{\alpha_{k+1}}} \otimes \cdots \otimes y_n^{p^{\alpha_n}})^{p^{-1}})$$

$$= \sum_{\substack{i_i,\ldots,i_r\geq 0\\ i_{k+1},\ldots,i_n\geq 0}} (y_1 \otimes \cdots \otimes y_{\rho_1})^{p^{i_1-1}} \otimes (y_{\rho_1+1} \otimes \cdots \otimes y_{\rho_2})^{p^{i_2-1}}$$

$$\otimes \cdots \otimes (y_{\rho_{r-1}+1} \otimes \cdots \otimes y_{\rho_k})^{p^{i_r-1}} \otimes \cdots \otimes y_{k+1}^{p^{\alpha_{k+1}+i_{k+1}-1}}$$

$$\otimes \ldots \otimes \xi_{i_1}^{p^{l_1-1}} \cdots \xi_{i_r}^{p^{l_r-1}} \xi_{i_{k+1}}^{p^{\alpha_{k+1}-1}} \cdots \xi_{i_n}^{p^{\alpha_n-1}}. \qquad (^{**})$$

If any of the $i_1, \ldots, i_r$ in $(^{**})$ is zero, then

$$(y_1 \otimes \cdots \otimes y_{\rho_1})^{p^{i_1}-1} \otimes \cdots \otimes (y_{\rho_{r-1}+1} \otimes \cdots \otimes y_{\rho_k})^{p^{i_r}-1} \otimes y_{k+1}^{p^{\beta_{k+1}}} \otimes \cdots \otimes y_n^{p^{\beta_n}}$$

denotes

$$((y_1 \otimes \cdots \otimes y_{\rho_1})^{p^{i_1}} \otimes \cdots \otimes (y_{\rho_{r-1}+1} \otimes \cdots \otimes y_{\rho_k})^{p^{i_r}} \otimes y_{k+1}^{p^{\beta_{k+1}+1}} \otimes \cdots \otimes y_n^{p^{\beta_n+1}})^{p^{-1}}.$$

Next, we shall show that M_1 is an $\mathcal{A}_p$-module, i.e., we need to verify that

$$\theta_1(\theta_2 z) = (\theta_1\theta_2)z,$$

for any $\theta_1, \theta_2 \in \mathcal{A}_p$ and z any of attached elements. In terms of the coaction map λ, condition above can be translated into commutativity of the diagram

$$\begin{array}{ccc} M_1 \otimes \mathcal{A}_p^* \otimes \mathcal{A}_p^* & \xleftarrow{1\otimes\Delta} & M_1 \otimes \mathcal{A}^* \\ {\scriptstyle \lambda\otimes 1}\uparrow & & \uparrow{\scriptstyle \lambda} \\ M_1 \otimes \mathcal{A}^* & \xleftarrow{\lambda} & M_1 \end{array}.$$

So, we show that the following equality:

$$(\lambda \otimes 1)\lambda(z) = (1 \otimes \Delta)\lambda(z)$$

holds, where $z = (y_1 \otimes y_2 \otimes \cdots \otimes y_k)^{p^{-1}} \otimes y_{k+1}^{p^{\alpha_{k+1}}} \otimes \cdots \otimes y_n^{p^{\alpha_n}}$.

First, let us find what the left side of equality is

$$(\lambda \otimes 1)\lambda(z) = (\lambda \otimes 1) \sum_{\substack{i_1,\ldots,i_r \geq 0 \\ i_{k+1},\ldots,i_n \geq 0}} (y_1 \otimes \cdots \otimes y_{\rho_1})^{p^{i_1}-1}$$

$$\otimes \cdots \otimes (y_{\rho_{r-1}+1} \otimes y_2 \otimes \cdots \otimes y_k)^{p^{i_r}-1} \otimes y_{k+1}^{p^{\alpha_{k+1}+i_{k+1}}}$$

$$\otimes \cdots \otimes y_n^{p^{\alpha_n+i_n}} \otimes \xi_{i_1}^{p^{l_1}-1} \cdots \xi_{i_r}^{p^{l_r}-1} \xi_{i_{k+1}}^{p^{\alpha_{k+1}}} \cdots \xi_{i_n}^{p^{\alpha_n}}$$

$$= \sum_{\substack{i_1,\dots,i_r,j_1,\dots,j_r\geq 0\\ i_{k+1},\dots,i_n,j_{k+1},\dots,j_n\geq 0}} (y_1\otimes\cdots\otimes y_{\rho_1})^{p^{i_1+j_1-1}}$$

$$\otimes\cdots\otimes(y_{\rho_{r-1}+1}\otimes\cdots\otimes y_k)^{p^{i_2+j_2-1}}\otimes y_{k+1}^{p^{\alpha_{k+1}+i_{k+1}+j_{k+1}}}$$

$$\otimes\cdots\otimes y_n^{p^{\alpha_n+i_n+j_n}}\otimes\xi_{j_1}^{p^{l_1+i_1-1}}\cdots\xi_{j_r}^{p^{l_r+i_r-1}}\xi_{j_{k+1}}^{p^{\alpha_{k+1}+i_{k+1}}}\cdots\xi_{j_n}^{p^{\alpha_n+i_n}}$$

$$\otimes\xi_{i_1}^{p^{l_1-1}}\cdots\xi_{i_r}^{p^{l_{r-1}}}\xi_{i_{k+1}}^{p^{\alpha_{k+1}}}\cdots\xi_{i_n}^{p^{\alpha_n}}.$$

The right side of equality is

$$(1\otimes\Delta)\lambda(z) = \sum_{\substack{m_i,\dots,m_r\geq 0\\ m_{k+1},\dots,m_n\geq 0}} (y_1\otimes\cdots\otimes y_{\rho_1})^{p^{m_1-1}}$$

$$\otimes\cdots\otimes(y_{\rho_{r-1}+1}\otimes\cdots\otimes y_k)^{p^{m_r-1}}\otimes y_{k+1}^{p^{\alpha_{k+1}+m_{k+1}}}$$

$$\otimes\cdots\otimes y_n^{p^{\alpha_n+m_n}}\otimes\Delta(\xi_{m_1}^{p^{l_1-1}}\cdots\xi_{m_r}^{p^{l_r-1}}\xi_{m_{k+1}}^{p^{\alpha_{k+1}}}\cdots\xi_{m_n}^{p^{\alpha_n}}).$$

On the other hand, we have

$$\Delta(\xi_{m_1}^{p^{l_1-1}}\cdots\xi_{m_r}^{p^{l_r-1}}\xi_{m_{k+1}}^{p^{\alpha_{k+1}}}\cdots\xi_{m_n}^{p^{\alpha_n}})$$

$$= \Delta(\xi_{m_1})^{p^{l_1-1}}\cdots\Delta(\xi_{m_r})^{p^{l_r-1}}\Delta(\xi_{m_{k+1}})^{p^{\alpha_{k+1}}}\cdots\Delta(\xi_{m_n})^{p^{\alpha_n}}$$

$$= (\sum_{i_1+j_1=m_1}\xi_{j_1}^{p^{i_1}}\otimes\xi_{i_1})^{p^{l_1-1}}\cdots(\sum_{i_r+j_r=m_r}\xi_{j_r}^{p^{i_r}}\otimes\xi_{i_r})^{p^{l_r-1}}$$

$$(\sum_{i_{k+1}+j_{k+1}=m_{k+1}}\xi_{j_{k+1}}^{p^{i_{k+1}}}\otimes\xi_{i_{k+1}})^{p^{\alpha_{k+1}}}\cdots(\sum_{i_n+j_n=m_n}\xi_{j_n}^{p^{i_n}}\otimes\xi_{i_n})^{p^{\alpha_n}}$$

$$= \sum\xi_{j_1}^{p^{l_1+i_1-1}}\cdots\xi_{j_r}^{p^{l_r+i_r-1}}\xi_{j_{k+1}}^{p^{\alpha_{k+1}+i_{k+1}}}\cdots\xi_{j_n}^{p^{\alpha_n+i_n}}$$

$$\otimes\xi_{i_1}^{p^{l_1-1}}\cdots\xi_{i_r}^{p^{l_{r-1}}}\xi_{i_{k+1}}^{p^{\alpha_{k+1}}}\cdots\xi_{i_n}^{p^{\alpha_n}}.$$

When we compare both sides, the equality holds. Hence M_1 is unstable module. Similarly M_2 can be constructed. Therefore, the theorem follows by induction. □

Let $M_{*,*}$ denote the morphism set in the category $\mathcal{F}$

$$Hom_{\mathcal{F}_w}(S_*, S^*),$$

where S_* and S^* are the graded symmetric invariants and coinvariants, respectively. So $M_{*,*}$ is a bigraded algebra with multiplication induced by the tensor product. Let $\alpha \in Hom_{\mathcal{F}_w}(S_i, S^j)$ and $\beta \in Hom_{\mathcal{F}_w}(S_k, S^l)$. Then $\alpha \bullet \beta$ is the composite

$$S_{i+k}(V) \xrightarrow{\psi} S_i(V) \otimes S_k(V) \xrightarrow{\alpha\otimes\beta} S^j(V^*) \otimes S^l(V^*) \xrightarrow{\rho} S^{j+l}(V^*),$$

where ψ is a coproduct and ρ is the cup product. Hence, $M_{*,*}$ is an $\mathcal{A}_p \times \mathcal{A}_p$-module with the $\mathcal{A}_p \times 1$-action on $M_{*,j}$ induced by the right action on homology and the $1 \times \mathcal{A}$-action on $M_{i,*}$ induced by the left action on cohomology. Using the equivalence between $\mathcal{U}'/Nil'$ and $\mathcal{F}_w$, we see that

$$M_{*,j} = \mathcal{N}(F'(2)^j / \Sigma_j).$$

Let

$$u_{(1,p^k)} = (y_1 \otimes \cdots \otimes y_{p^k})^{p^k} \in M_{1,p^k} \simeq \mathcal{N}(F'(2)^{p^k} / \Sigma_{p^k}),$$

and

$$u_{(p^k,1)} = y^{p^k} \in M_{p^k,1} \simeq \mathcal{N}(F'(2))^{p^k} \simeq F'(2)^{p^k}.$$

Then, we have the following theorem proved by Kuhn in [7]:

Theorem 3.2. *The bigraded algebra* $M_{*,*}$ *is isomorphic to the polynomial algebra generated by the elements* $u_{(1,\,p^k)}$ *and* $u_{(p^k,1)}$ *of bidegree* $(1,\,p^k)$ *and* $(p^k,\,1)$, *respectively.*

Proof. Let $\alpha \in M_{i,\,j}$ and $\beta \in M_{k,\,l}$. Then $\alpha \bullet \beta \in M_{i+k,\,j+l}$, which is image of $\alpha \otimes \beta$ under tensor product

$$\mathcal{N}(F'(2)^{\otimes i} / \Sigma_i)^j \otimes \mathcal{N}(F'(2)^{\otimes k} / \Sigma_k)^l \to \mathcal{N}(F'(2)^{\otimes(i+k)} / \Sigma_{i+k})^{j+l}.$$

By Theorem 3.1, any element in $\mathcal{N}(F'(2)^{\otimes m} / \Sigma_m)$ can be written as

$$(y_1 \otimes \cdots \otimes y_{p^{k_1}})^{p^{-k_1+i_1}} \otimes \cdots \otimes (y_{\rho_{n-1}} \otimes \cdots \otimes y_{\rho_n})^{p^{-k_n+i_n}} \otimes y^{p^{i_n+1}} \otimes \cdots \otimes y_m^{p^{i_m}}$$

$$= u_{(1,\,p^{k_1})}^{i_1} \cdots u_{(1,\,p^{k_n})}^{i_n} u_{(p^{i_n+1},1)} \cdots u_{(p^{i_m},1)}.$$

This completes the proof. □

Let $K'(\cdot) = \mathbb{F}_p[x_i;\, i \in \mathbb{Z}]$ be a polynomial algebra on generators x_k of bidegree $(1,\,p^k)$. Define $g_0 : M_{*,*} \longrightarrow K'(\cdot)$ by $g_0(u_{(p^k,1)}) = x_{-k}^{p^k}$ and $g_0(u_{(1,\,p^k)}) = x_k$, where $u_{(p^k,1)}$ and $u_{(1,\,p^k)}$ have bidegrees $(p^k,\,1)$ and $(1,\,p^k)$, respectively. So g_0 is an $\mathcal{A}_p$-monomorphism.

Define $g_j : M_{*,\,p^j} \longrightarrow K'(\cdot)$ by $g_j(u_{(p^k,1)}) = x_{j-k}^{p^k}$ and $g_j(u_{(1,\,p^k)})$ $= x_{k-j}$, $j \geq 1$. Since each map g_j is mono, so is g. It is epi because by definition of g_j every algebra generator is hit. So, we have just proved the following result:

Theorem 3.2. *The map* $g : \lim M_{*,*p^k} \longrightarrow K'(\cdot)$ *is an isomorphism.*

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