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COMPENSATED INTERPOLATING SEQUENCES

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Abstract

This paper deals with two interpolation problems in the unit disc $\mathbb{D}$ of the complex plane for sequences in a Stolz angle. In the first one, bounded sequences are interpolated by the product of a function in the Lipschitz class on $\mathbb{D}$ and other one in the space A^{-1}, thus generalizing a previous result in which, the function in A^{-1} was taken as the derivative of a Blaschke product. In the second one, bounded sequences are interpolated by a sum of functions of the mentioned type.

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1. Introduction

Let $\psi(z, w) = \dfrac{|z - w|}{|1 - \bar{z}w|}$ be the pseudo-hyperbolic distance between $z, w \in \mathbb{D}$. We write (z_n) for any sequence in $\mathbb{D}$ and $z_{m'}$ for the point of (z_n) nearest to z_m in the Euclidean distance. The Blaschke product in $\mathbb{D}$ with zeros at (z_n) is the bounded analytic function defined by

$$B(z) = \prod_{n \in \mathbb{N}} \frac{|z_n|}{z_n} \frac{z - z_n}{1 - \bar{z}_n z}.$$

We denote by B_m and $B_{m,k}$ the Blaschke products in $\mathbb{D}$ with zeros at $(z_n) \setminus \{z_m\}$ and $(z_n) \setminus \{z_m, z_k\}$, respectively. As usual, we will put c for all positive constants.

We recall that (z_n) is called *uniformly separated*, if

$$|B_m(z_m)| \geq c, \quad \forall m \in \mathbb{N}.$$

The Stolz angle with vertex at the point η in the boundary of $\mathbb{D}$ and aperture $\beta > 0$ is the set

$$S_\beta(\eta) = \{z \in \mathbb{D} / |z - \eta| < (1 + \beta)(1 - |z|)\}.$$

Let l^∞ be the space of all bounded sequences of complex numbers and $Lip_{(z_n)}$ be the space of all complex functions Φ on (z_n) such that

$$|\Phi(z_k) - \Phi(z_m)| \leq c|z_k - z_m|, \quad \forall k, m \in \mathbb{N}.$$

We denote by *Lip*, the Lipschitz class on $\mathbb{D}$, that is, the space of all functions in the disc algebra such that

$$|f(z) - f(w)| \leq c|z - w|, \quad \forall z, w \in \mathbb{D}.$$

Finally, A^{-1} denotes the space of all holomorphic functions f on $\mathbb{D}$ such that

$$|f(z)| \leq \frac{c}{1-|z|^2}, \quad \forall z \in \mathbb{D}.$$

Definition 1. (z_n) is called a *balanced interpolating sequence*, if given any $(w_n) \in l^\infty$, there exists $f \in Lip$ such that $(B'f)(z_n) = w_n$, $\forall n \in \mathbb{N}$.

These interpolating sequences in a Stolz angle were characterized in [6] as the uniformly separated ones.

Taking into account that $B' \in A^{-1}$, we introduce two more general types of interpolating sequences.

Definition 2. (z_n) is called a *compensated interpolating sequence*, if given any $(w_n) \in l^\infty$, there exists $f \in Lip$ and $g \in A^{-1}$ not vanishing on (z_n) such that $(fg)(z_n) = w_n$, $\forall n \in \mathbb{N}$.

Note that we ask for the interpolation by the product of a regular function up to the boundary of D and other one that is not bounded when we approximate to it, but existing a certain compensation between both on the sequence, because their product is bounded there.

Definition 3. (z_n) is called a *compensated interpolating sequence for a sum*, if given any $(w_n) \in l^\infty$, there exists $f_1, f_2 \in Lip$ and $g_1, g_2 \in A^{-1}$ not vanishing on (z_n) such that $(f_1g_1 + f_2g_2)(z_n) = w_n$, $\forall n \in \mathbb{N}$.

It is clear that balanced interpolating sequence $\Rightarrow$ compensated interpolating sequence $\Rightarrow$ compensated interpolating sequence for a sum.

Our results on the last two interpolating sequences are the following ones:

Theorem 1. *If (z_n) in a Stolz angle is a compensated interpolating sequence, then*

(1) *(z_n) is the union of at most two uniformly separated sequences.*

(2) $$\frac{|z_m - z_{m'}|}{1 - |z_m|^2} \geq c, \quad \forall m \in \mathbb{N}.$$

Theorem 2. *If* (z_n) *in a Stolz angle verifies the conditions* (1) *and* (2), *then it is a compensated interpolating sequence for a sum*.

Note that the sequences satisfying the conditions (1) and (2) are intermediate between the uniformly separated ones and those that are the union of at most two of these sequences.

We will need the following estimate:

Lemma 1. *If* $f \in Lip$ *and* $g \in A^{-1}$, *then*

$$|(fg)(z)(1 - |z|^2) - (fg)(w)(1 - |w|^2)| \leq c(|f(z)|\psi(z, w) + |z - w|), \quad \forall z, w \in \mathbb{D}.$$

Proof. By the triangular inequality

$$|(fg)(z)(1 - |z|^2) - (fg)(w)(1 - |w|^2)|$$

$$\leq |f(z)||g(z)(1 - |z|^2) - g(w)(1 - |w|^2)| + |g(w)|(1 - |w|^2)|f(z) - f(w)|.$$

It is proved in [5] that $|g(z)(1 - |z|^2) - g(w)(1 - |w|^2)| \leq c\psi(z, w)$, and using $|g(w)| \leq \frac{c}{1 - |w|^2}$ and $|f(z) - f(w)| \leq c|z - w|$, the result follows.

2. Proof of the Theorem 1

Proof. Suppose that (z_n) is a compensated interpolating sequence in a Stolz angle. For $m \in \mathbb{N}$ fixed, let (w_n) be defined by: $w_m = 1$ and $w_k = 0$, if $k \neq m$. Since $(w_n) \in l^\infty$, there exists $f \in Lip$ and $g \in A^{-1}$ not vanishing on (z_n) such that $(fg)(z_n) = w_n$, $\forall n \in \mathbb{N}$.

Since $f \in Lip$ vanishes on $(z_n) \setminus \{z_m\}$, it is proved in [4] that

$$|f(z)| \leq c|z - z_k||B_{m,k}(z)|, \quad \forall k \in \mathbb{N}, k \neq m.$$

Writing this inequality for $z = z_m$ and taking $z_k = z_{m'}$, we have

$$\frac{1}{|g(z_m)|} \leq c|z_m - z_{m'}||B_{m,m'}(z_m)|, \quad \forall m \in \mathbb{N}.$$

Since $g \in A^{-1}$,

$$1 - |z_m|^2 \leq c|z_m - z_{m'}||B_{m,m'}(z_m)|, \quad \forall m \in \mathbb{N}.$$

Now, since (z_n) is in a Stolz angle $S_\beta(\eta)$

$$c < c\frac{|z_m - \eta|}{|z_m - z_{m'}|} < c\frac{1 - |z_m|}{|z_m - z_{m'}|} < c\frac{1 - |z_m|^2}{|z_m - z_{m'}|} \leq |B_{m,m'}(z_m)|, \quad \forall m \in \mathbb{N},$$

and it is proved in [2] that

$$|B_{m,m'}(z_m)| \geq c, \quad \forall m \in \mathbb{N},$$

if and only if (z_n) is the union of at most two uniformly separated sequences.

Since $f \in Lip$ vanishes on $(z_n) \setminus \{z_m\}$, it follows from the Lemma 1 that

$$|(fg)(z_m)(1 - |z_m|^2)| \leq c|z_m - z_{m'}|.$$

Then,

$$c(1 - |z_m|^2) \leq |z_m - z_{m'}|.$$

□

3. Proof of the Theorem 2

Proof. For (z_n) being the union of at most two uniformly separated sequences, it is proved in [1] and [4] that (z_n) is either uniformly separated or it can be rearranged $(z_n) = (a_n) \cup (b_n)$, where (a_n) and (b_n) are uniformly separated.

In case that (z_n) is uniformly separated, then it is balanced and there exists h with the looked form such that $h(z_n) = w_n$, $\forall n \in \mathbb{N}$.

In another case, we write (u_n) for the terms of (w_n) corresponding to the sequence (a_n), and (v_n) for the terms corresponding to (b_n).

We take h such that $h(a_n) = u_n$, $\forall n \in \mathbb{N}$, with the looked form.

For a sequence (z_n) in a Stolz angle, it is proved in [3] that given any $\Phi \in Lip_{(z_n)}$, there exists $f \in Lip$ such that $f(z_n) = \Phi(z_n)$, $\forall n \in \mathbb{N}$, if and only if (z_n) is the union of at most two uniformly separated sequences.

Let Φ be defined by: $\Phi(a_n) = 0$ and $\Phi(b_n) = 1 - |b_n|^2$, $\forall n \in \mathbb{N}$. We have

$$|\Phi(b_n) - \Phi(b_m)| = ||b_m|^2 - |b_n|^2| \leq 2|b_n - b_m|, \quad \forall m, n \in \mathbb{N},$$

and, taking into account (2)

$$|\Phi(a_m) - \Phi(b_n)| = 1 - |b_n|^2 \leq c|a_m - b_n|, \quad \forall m, n \in \mathbb{N}.$$

Then $\Phi \in Lip_{(z_n)}$ and there exists $f \in Lip$ such that $f(z_n) = \Phi(z_n)$, $\forall n \in \mathbb{N}$.

We write

$$\lambda_n = \frac{v_n - h(b_n)}{f(b_n)}, \quad \forall n \in \mathbb{N}.$$

We have $|\lambda_n| \leq \dfrac{c}{1 - |b_n|^2}$. Since (b_n) is uniformly separated, it is proved in [5] that there exists $g \in A^{-1}$ such that $g(b_n) = \lambda_n$, $\forall n \in \mathbb{N}$.

It is immediate now that the function $h + fg$ performs the desired interpolation. □

There remains opened the problem of characterizing the compensate interpolating sequences and also the compensate interpolating sequences for a sum.

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