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ON THE HIGHER RANK NUMERICAL RANGE OF THE SHIFT OPERATOR

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Abstract

For any n-by-n complex matrix T and any $1 \leqslant k \leqslant n$, let $\Lambda_k(T)$ be the set of all $\lambda \in \mathbb{C}$ such that $PTP = \lambda P$ for some rank-k orthogonal projection P be its higher rank-k numerical range. It is shown that if S_n is the n-dimensional shift on $\mathbb{C}^n$, then its rank-k numerical range is the circular disc centered in zero and with radius $\cos(\frac{k\pi}{n+1})$ if $1 < k \leqslant [\frac{n+1}{2}]$ and the empty set if $[\frac{n+1}{2}] < k \leqslant n$, where $[x]$ denote the integer part of x. This extends and refines previous results of Haagerup and de la Harpe [11] on the classical numerical range of the n-dimensional shift on $\mathbb{C}^n$. An interesting result for higher rank-k numerical range of nilpotent operator is also established.

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1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{B}(\mathcal{H})$ be the collection of all bounded linear operator on $\mathcal{H}$. The numerical range of an operator T in $\mathcal{B}(\mathcal{H})$ is the subset

$$W(T) = \{< Tx, x > \in \mathbb{C};\ x \in \mathcal{H},\ \|x\| = 1\},$$

of the plane, where $< ., . >$ denotes the inner product in $\mathcal{H}$ and the numerical radius of T is defined by

$$\omega_2(T) = \sup\{|z|;\ z \in W(T)\}.$$

We denote by S the unilateral shift acting on the Hardy space $\mathbb{H}^2$ of the square summable analytic functions

$$\begin{aligned} S \quad : \quad \mathbb{H}^2 \quad &\to \quad \mathbb{H}^2, \\ f \quad &\mapsto \quad zf(z). \end{aligned}$$

Beurling's theorem implies that the nonzero invariant subspaces of S are of the form $\phi\mathbb{H}^2$, where ϕ is some inner function. Let $S(\phi)$ denote the compressed of S to the subspace $H(\phi) = \mathbb{H}^2 \Theta \phi\mathbb{H}^2$:

$$S(\phi)f(z) = P(zf(z)),$$

where P denotes the orthogonal projection from $\mathbb{H}^2$ onto $H(\phi)$. We denote by $S^*(\phi)$ the adjoint of $S(\phi)$:

$$S^*(\phi) = S(\phi)^* = S^*|_{H(\phi)} = S^*_{|\mathrm{Ker}(\phi(S)^*)}.$$

The space $H(\phi)$ is a finite-dimensional exactly when ϕ is a finite Blaschke product. The numerical radius and numerical range of the model operator $S(\phi)$ seems to be important and have many applications. In [1], Badea and Cassier showed that, there is a relationship between numerical radius of $S(\phi)$ and Taylor coefficients of positive rational

functions on the torus and more recently in [8], the author gave an extension of this result. However, the evaluation of the numerical radius of $S(\phi)$ under an explicit form is always an open problem. The readers may consult [8] for an estimate of $S(\phi)$, where ϕ is a finite Blaschke product with unique zero. In the particular case, where $\phi(z) = z^n$, $S(\phi)$ is unitarily equivalent to S_n, where

$$S_n = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

In [11], it is proved that $W(S_n)$ is the closed disc $D_n = \{z \in \mathbb{C};\ |z| \leqslant \cos(\frac{\pi}{n+1})\}$ and $\omega_2(S_n) = \cos(\frac{\pi}{n+1})$ and more general

Theorem 1.1 ([11])**.** *Let T be an operator on $\mathcal{H}$ such that $T^n = 0$ for some $n \geq 2$. One has*

$$\omega_2(T) \leqslant \|T\| \cos(\frac{\pi}{n+1}),$$

and $\omega_2(T) = \|T\| \cos(\frac{\pi}{n+1})$ when T is unitarily equivalent to $\|T\| S_n$.

In this mathematical note, we extend this result to the higher rank-k numerical range of the shift. The notion of the higher rank-k numerical range of an operator T acting on a Hilbert space $\mathcal{H}$ of dimension at least k is introduced in [5] and it is denoted by

$$\Lambda_k(T) = \{\lambda \in \mathbb{C} : PTP = \lambda P, \text{ for some rank-}k \text{ orthogonal projection } P\}.$$

The introduction of this notion was motivated by a problem in quantum error correction; see [4]. Note that if P is a rank-1 orthogonal projection, then $P = x \otimes x$ for some unit $x \in \mathbb{C}^n$ and $PTP = < Tx, x > P$. Thus when $k = 1$, this concept is reduces to the classical numerical range $W(T)$, which is well known to be convex by the Toeplitz-Hausdorff

theorem. We refer to [14] for a simple proof. In [7], it is conjectured that $\Lambda_k(T)$ is convex, and reduced the convexity problem to the problem of showing that $0 \in \Lambda_k(T')$, where

$$T' = \begin{pmatrix} I_k & X \\ Y & -I_k \end{pmatrix},$$

for arbitrary $X, Y \in \mathcal{M}_k$ (the algebra of $k \times k$ complex matrix). They further reduced this problem to the existence of a Hermitian matrix H satisfying the matrix equation

$$I_k + MH + HM^* - HRH = H, \tag{1.1}$$

for arbitrary $M \in \mathcal{M}_k$ and a positive definite $R \in \mathcal{M}_k$. In [21], Woerdeman proved that Equation (1.1) is equivalent to Ricatti equation

$$HRH - H(M^* - I_k/2) - (M - I_k/2)H - I_k = 0_k, \tag{1.2}$$

and using the theory of Ricatti equations (see [13], Theorem 4), the Equation (1.2) is solvable, which prove the convexity of $\Lambda_k(T)$. In [5], the authors showed that if $\dim \mathcal{H} < \infty$ and $T \in \mathcal{B}(\mathcal{H})$ is a Hermitian matrix with eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$, then the rank-$k$ numerical range $\Lambda_k(T)$ coincides with $[\lambda_k, \lambda_{n+1-k}]$, which is a non-degenerate closed interval if $\lambda_k < \lambda_{n+1-k}$, a singleton set if $\lambda_k = \lambda_{n+1-k}$, and an empty set if $\lambda_k > \lambda_{n+1-k}$. In [17], the authors proved that if $\dim \mathcal{H} = n$, then

$$\Lambda_k(T) = \bigcap_{\theta \in [0, 2\pi[} \left\{ \mu \in \mathbb{C} : e^{i\theta}\mu + e^{-i\theta}\overline{\mu} \leqslant \lambda_k\left(e^{i\theta}T + e^{-i\theta}T^*\right) \right\},$$

for $1 \leqslant k \leqslant n$, where $\lambda_k(H)$ denote the k-th largest eigenvalue of the Hermitian matrix $H \in \mathcal{M}_n$. Thanks to this result, they establish that if $\dim \mathcal{H} = n$ and $T \in \mathcal{B}(\mathcal{H})$ is a normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\Lambda_k(T) = \bigcap_{1 \leqslant j_1 < \cdots < j_{n-k+1} \leqslant n} \operatorname{conv}\{\lambda_{j_1}, \ldots, \lambda_{j_{n+1-k}}\}.$$

We close this section by the following properties, which are easily checked. The readers may consult [4-7], [9], and [16].

(P1) For any a and $b \in \mathbb{C}$, $\Lambda_k(aT + bI) = a\Lambda_k(T) + b$.

(P2) $\Lambda_k(T^*) = \overline{\Lambda_k(T)}$.

(P3) $\Lambda_k(T \oplus S) \supseteq \Lambda_k(T) \bigcup \Lambda_k(S)$.

(P4) For any unitary $U \in \mathcal{B}(\mathcal{H})$, $\Lambda_k(U^*TU) = \Lambda_k(T)$.

(P5) If T_0 is a compression of T on a subspace $\mathcal{H}_0$ of $\mathcal{H}$ such that $\dim \mathcal{H}_0 \geq k$, then $\Lambda_k(T_0) \subseteq \Lambda_k(T)$.

(P6) $W(T) \supseteq \Lambda_2(T) \supseteq \Lambda_3(T) \supseteq \ldots$.

Some results from [1] will be also developed in this context in a forthcoming paper.

2. Main Theorem

In the following theorem, we give the higher rank-k numerical range of the n-dimensional shift on $\mathbb{C}^n$.

Theorem 2.1. *For any $n \geq 2$ and $1 \leqslant k \leqslant n$, $\Lambda_k(S_n)$ coincides with the circular disc $\{z \in \mathbb{C} : |z| \leqslant \cos(\frac{k\pi}{n+1})\}$ if $1 \leqslant k \leqslant [\frac{n+1}{2}]$ and the empty set if $[\frac{n+1}{2}] < k \leqslant n$.*

Proof. First, observe that for $1 \leqslant k \leqslant n$,

$$\begin{aligned}\Lambda_k(S_n) &= \bigcap_{\theta \in [0, 2\pi[} \left\{\mu \in \mathbb{C} : e^{i\theta}\mu + e^{-i\theta}\overline{\mu} \leqslant \lambda_k(e^{i\theta}S_n + e^{-i\theta}S_n^*)\right\} \\ &= \bigcap_{\theta \in [0, 2\pi[} \left\{\mu \in \mathbb{C} : \mathcal{R}e(e^{i\theta}\mu) \leqslant \frac{1}{2}\lambda_k(e^{i\theta}S_n + e^{-i\theta}S_n^*)\right\} \\ &= \bigcap_{\theta \in [0, 2\pi[} e^{i\theta}\left\{z \in \mathbb{C} : \mathcal{R}e(z) \leqslant \frac{1}{2}\lambda_k(e^{i\theta}S_n + e^{-i\theta}S_n^*)\right\} \qquad (2.1)\end{aligned}$$

$$= \bigcap_{\theta\in[0,\,2\pi[} e^{i\theta}\Big\{z \in \mathbb{C} : \mathcal{R}e(z) \leqslant \frac{1}{2}\lambda_k(S_n + S_n^*)\Big\} \tag{2.2}$$

$$= \bigcap_{\theta\in[0,\,2\pi[} e^{i\theta}\Big\{z \in \mathbb{C} : \mathcal{R}e(z) \leqslant \cos\big(\frac{k\pi}{n+1}\big)\Big\}. \tag{2.3}$$

Equations (2.1) and (2.2) are due to the fact that firstly $S_n + S_n^* = D(\theta)^*(e^{i\theta}S_n + e^{-i\theta}S_n^*)D(\theta)$, where $D(\theta)$ denotes the unitary diagonal matrix with entries $e^{i\theta}$, $e^{2i\theta}$, …, $e^{ni\theta}$, which implies that $S_n + S_n^*$ and $e^{i\theta}S_n + e^{-i\theta}S_n^*$ are unitarily equivalent for each θ in $[0, 2\pi[$ (see [11], page 373) and secondly, the eigenvalues of $S_n^* + S_n$ are $\Big(2\cos\big(\frac{m\pi}{n+1}\big)\Big)_{1\leqslant m\leqslant n}$. (We refer the readers to ([3], page 35) or ([10], page 67).) Now from (2.3), it follows that $\Lambda_k(S_n)$ is the intersection of closed half planes. We note that $\cos\frac{k\pi}{n+1}$ is positive, if and only if $1 \leqslant k \leqslant [\frac{n+1}{2}]$. We consider the following two cases:

Case 1. If $1 \leqslant k \leqslant [\frac{n+1}{2}]$, then $\Lambda_k(S_n)$ is the circular disc $\{z \in \mathbb{C} : |z| \leqslant \cos(\frac{k\pi}{n+1})\}$.

Case 2. If $[\frac{n+1}{2}] < k \leqslant n$, using (2.3), it follows that

$$\Lambda_k(S_n) \subseteq \Big\{z \in \mathbb{C} : \mathcal{R}e(z) \leqslant \cos\big(\frac{k\pi}{n+1}\big)\Big\} \bigcap e^{i\pi}\Big\{z \in \mathbb{C} : \mathcal{R}e(z) \leqslant \cos\big(\frac{k\pi}{n+1}\big)\Big\}$$

$$= \emptyset.$$

This completes the proof. □

Theorem 2.2. *For any integer* $k \geq 1$,

$$\Lambda_k(S) = D(0, 1).$$

Proof. Let a fixed $k \geq 1$, from (P5) and Theorem 2.1, we deduce that for all $n \geq k$, $\overline{D(0, \cos(\frac{k\pi}{n+1}))} = \Lambda_k(S_n) \subseteq \Lambda_k(S)$. Let n tends to $+\infty$, and we have $D(0, 1) \subseteq \Lambda_k(S)$. Now let λ in $\Lambda_k(S)$, then there exists a rank-k orthogonal projection P such that $PSP = \lambda P$. Let denote by U_θ the unitary operator on $\mathbb{H}^2$ defined by $U_\theta(f)(z) = f(ze^{-i\theta})$, where $\theta \in [0, 2\pi[$. For every f and g in $\mathbb{H}^2$, we have

$$\begin{aligned} < e^{i\theta}Sf, g > &= \int_0^{2\pi} e^{i(\theta+t)} f(e^{it})\overline{g(e^{it})}\frac{dt}{2\pi} \\ &= \int_0^{2\pi} e^{is} f(e^{i(s-\theta)})\overline{g(e^{i(s-\theta)})}\frac{ds}{2\pi} \\ &= < SU_\theta f, U_\theta g > . \end{aligned}$$

This shows that

$$e^{i\theta}S = U_\theta^* S U_\theta, \text{ for each } \theta \in [0, 2\pi[.$$

Let denote by Q the rank-k orthogonal projection $U_\theta P U_\theta^*$, we can easily check that $QSQ = \lambda e^{i\theta}Q$, thus $\Lambda_k(S)$ is a circular disc centered in 0. On the other hand, if $1 \in \Lambda_k(S)$, then from (P6), $1 \in W(S)$. Thus, there exists a unitary $f \in \mathbb{H}^2$ such that $< Sf, f >= 1$. The known facts about when the Schwartz inequality becomes an equation implies that 1 is an eigenvalue for S (cf. [12], Solution 212), which is absurd. The proof is now complete. □

One easy consequence of Theorem 2.1 is the following:

Corollary 2.3. *One has*

$$W(S) = D(0, 1).$$

On the sequel of this paper, let denote by

$$\rho(k, d) = \begin{cases} k/d, & \text{if } k/d \text{ is an integer,} \\ [k/d] + 1, & \text{unless,} \end{cases}$$

where k and d are arbitrary integers. We convient that $\rho(k,\, d) = 1$ if $d = +\infty$. $\delta(k,\, d)$ denotes the remainder in the Euclidean division of k by d. For $\zeta \geq 0$, $D(0,\, \zeta)$ denotes the open circular disc $\{z \in \mathbb{C} : |z| < \zeta\}$ and $\overline{D(0,\, \zeta)}$ its closure.

Proposition 2.4. *Consider $I_{\mathcal{H}} \otimes S_n^*$ be the operator acting on the Hilbertian tensor product space $\mathcal{H} \otimes \mathbb{C}^n$ and $d = \dim \mathcal{H}$ (we convient that $d = +\infty$ if $\dim \mathcal{H} = +\infty$). Then for each $1 \leqslant k \leqslant nd$, we have*

$$\Lambda_k(I_{\mathcal{H}} \otimes S_n^*) = \begin{cases} \overline{D(0,\, \cos\dfrac{\rho(k,\, d)\pi}{n+1})}, & \text{if } 1 \leqslant \rho(k,\, d) \leqslant [\,\dfrac{n+1}{2}\,], \\ \emptyset, & \text{if } [\,\dfrac{n+1}{2}\,] < \rho(k,\, d) \leqslant n. \end{cases}$$

Proof. Firstly, assume that $1 \leqslant d < +\infty$, then

$$\begin{aligned} &\Lambda_k(I_{\mathcal{H}} \otimes S_n^*) \\ &= \bigcap_{\theta \in [0,\, 2\pi[} \left\{ \mu \in \mathbb{C} : e^{i\theta}\mu + e^{-i\theta}\overline{\mu} \leqslant \lambda_k(e^{i\theta}(I_d \otimes S_n^*) + e^{-i\theta}(I_d \otimes S_n^*)^*) \right\} \\ &= \bigcap_{\theta \in [0,\, 2\pi[} \left\{ \mu \in \mathbb{C} : e^{i\theta}\mu + e^{-i\theta}\overline{\mu} \leqslant \lambda_k(e^{i\theta}(I_d \otimes S_n^*) + e^{-i\theta}(I_d \otimes S_n)) \right\} \\ &= \bigcap_{\theta \in [0,\, 2\pi[} \left\{ \mu \in \mathbb{C} : e^{i\theta}\mu + e^{-i\theta}\overline{\mu} \leqslant \lambda_k(I_d \otimes (e^{i\theta}S_n + e^{-i\theta}S_n^*)) \right\} \\ &= \bigcap_{\theta \in [0,\, 2\pi[} \left\{ \mu \in \mathbb{C} : e^{i\theta}\mu + e^{-i\theta}\overline{\mu} \leqslant \lambda_k(\oplus_{l=1}^{d} (e^{i\theta}S_n + e^{-i\theta}S_n^*)) \right\} \\ &= \bigcap_{\theta \in [0,\, 2\pi[} e^{i\theta} \left\{ z \in \mathbb{C} : 2\mathcal{R}e(z) \leqslant \lambda_k(\oplus_{l=1}^{d} (S_n + S_n^*)) \right\} \\ &= \bigcap_{\theta \in [0,\, 2\pi[} e^{i\theta} \left\{ z \in \mathbb{C} : \mathcal{R}e(z) \leqslant \lambda_k(A_n) \right\}, \end{aligned} \tag{2.4}$$

where A is the block matrix defined by

$$A_n = \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_n \end{pmatrix} \in \mathcal{M}_{nd}(\mathbb{C}),$$

with

$$M_\mu = \begin{pmatrix} \cos(\frac{\mu\pi}{n+1}) & & \\ & \ddots & \\ & & \cos(\frac{\mu\pi}{n+1}) \end{pmatrix} \in \mathcal{M}_d(\mathbb{C}), \text{ for } 1 \leqslant \mu \leqslant n.$$

Thus, $\Lambda_k(I_{\mathcal{H}} \otimes S_n^*)$ is the intersection of closed half plane and it is easy to check from (2.4) that $\Lambda_k(I_{\mathcal{H}} \otimes S_n^*) = \overline{D(0, \lambda_k(A_n))}$, if $\lambda_k(A_n) \geq 0$ and empty set unless. Note that the eigenvalues of A_n are arranged in the decreasing order. To find the k-th largest eigenvalue of A_n, we will consider the Euclidean division of k by d as the following $k = \beta d + r$ with $0 \leqslant r \leqslant d-1$ and $\beta \geq 0$. We consider the following two cases:

Case 1. If $r = 0$. We have $\lambda_k(A) = \cos(\frac{\beta\pi}{n+1}) = \cos\frac{\rho(k,d)\pi}{n+1}$.

Case 2. If $1 \leqslant r \leqslant d-1$. Then $\lambda_k(A) = \cos\frac{(\beta+1)\pi}{n+1} = \cos\frac{\rho(k,d)\pi}{n+1}$.

Thus,

$$\Lambda_k(I_{\mathcal{H}} \otimes S_n^*) = \begin{cases} \overline{D(0, \cos\frac{\rho(k,d)\pi}{n+1})}, & \text{if } 1 \leqslant \rho(k,d) \leqslant [\frac{n+1}{2}], \\ \emptyset, & \text{if } [\frac{n+1}{2}] < \rho(k,d) \leqslant n. \end{cases} \quad (2.5)$$

Now assume that $d = +\infty$. To complete the proof of this proposition, it is enough to prove that

$$\Lambda_k(I_{\mathcal{H}} \otimes S_n^*) = \overline{D(0, \cos(\frac{\pi}{n+1}))}, \text{ for all } k \geq 1. \quad (2.6)$$

Firstly, note that (2.6) yields for $k = 1$, that is,

$$W(I_{\mathcal{H}} \otimes S_n^*) = \overline{D(0, \cos(\frac{\pi}{n+1})}. \tag{2.7}$$

Indeed, it is clear from (P5) that $W(I_{\mathcal{H}} \otimes S_n^*) \supseteq W(S_n^*) = \overline{D(0, \cos(\frac{\pi}{n+1}))}$. For the proof of the second inclusion, let denote by $(\varepsilon_l)_{l\geq 0}$ an orthonormal basis of $\mathcal{H}$ and $(x_l)_{l\geq 0}$ a sequence in $\mathbb{C}$ such that $x = \sum_{l\geq 0} \varepsilon_l \otimes x_l$ is a unit vector in $\mathcal{H} \otimes \mathbb{C}^n$ (i.e., $\|x\|^2 = \sum_{l\geq 0} \|x_l\|^2 = 1$). Then

$$\begin{aligned} < (I_{\mathcal{H}} \otimes S_n^*)x, x > &= < (I_{\mathcal{H}} \otimes S_n^*) \sum_{l\geq 0} \varepsilon_l \otimes x_l, \sum_{m\geq 0} \varepsilon_m \otimes x_m > \\ &= < \sum_{l\geq 0} \varepsilon_l \otimes S_n^* x_l, \sum_{m\geq 0} \varepsilon_m \otimes x_m > \\ &= \sum_{l,m\geq 0} < \varepsilon_l, \varepsilon_m >< S_n^* x_l, x_m > \\ &= \sum_{l\geq 0} < S_n^* \frac{x_l}{\|x_l\|}, \frac{x_l}{\|x_l\|} > \|x_l\|^2. \end{aligned} \tag{2.8}$$

Since $W(S_n^*)$ is a convex compact set, then (2.8) implies that $< (I_{\mathcal{H}} \otimes S_n^*)x, x > \in W(S_n^*) = \overline{D(0, \cos(\frac{\pi}{n+1}))}$. Thus $W(I_{\mathcal{H}} \otimes S_n^*) \subseteq \overline{D(0, \cos(\frac{\pi}{n+1}))}$ and (2.7) holds. Now, from, respectively, (2.7), (P6), (P5), and (2.5), we infer that for each $p > k$, we have

$$\begin{aligned} \overline{D(0, \cos(\frac{\pi}{n+1}))} = W(I_{\mathcal{H}} \otimes S_n^*) &\supseteq \Lambda_k(I_{\mathcal{H}} \otimes S_n^*) \supseteq \Lambda_k(I_p \otimes S_n^*) \\ &= \overline{D(0, \cos(\frac{\pi}{n+1}))}. \end{aligned}$$

This completes the proof. □

Let $D_T = (I_N - T^*T)^{1/2}$ be the defect operator of T and $\mathcal{D}_T$ be the closed range of D_T. Let denote by $d = \dim \mathcal{D}_T$.

Theorem 2.5. *Consider* $T \in \mathcal{B}(\mathcal{H})$ *such that* $\|T\| \leqslant 1$ *and* $T^n = 0$. *Then* $\Lambda_k(T)$ *is contained in the circular disc* $\{z \in \mathbb{C} : |z| \leqslant \cos \frac{\rho(k,d)\pi}{n+1}\}$ *if* $1 \leqslant \rho(k, d) \leqslant [\frac{n+1}{2}]$ *and an empty set if* $\rho(k, d) > [\frac{n+1}{2}]$.

Proof. If T is a contraction with $T^n = 0$, then T can be viewed as a compression of $I_{\mathcal{D}_T} \otimes S_n^*$ acting on the Hilbert space $\mathcal{D}_T \otimes \mathbb{C}^n$ (see [18], [2], [20], Theorem 1.2 in Chapter II). Consider the isometry $V : \mathcal{H} \to \mathcal{D}_T \otimes \mathbb{C}^n$, defined by

$$V(x) = \sum_{t=1}^{n} D_T T^{t-1} x \otimes e_t,$$

where $\{e_l\}_{l=1}^n$ is the canonical basis of $\mathbb{C}^n$. Note that

$$VTx = \sum_{t=1}^{n} D_T T^t x \otimes e_t = \sum_{t=1}^{n-1} D_T T^t x \otimes e_t = (I_{\mathcal{D}_T} \otimes S_n^*)Vx.$$

It follows that

$$T = V^*(I_{\mathcal{D}_T} \otimes S_n^*)V,$$

and from (P5),

$$\Lambda_k(T) = \Lambda_k(V^*(I_{\mathcal{D}_T} \otimes S_n^*)V) \subseteq \Lambda_k(I_{\mathcal{D}_T} \otimes S_n^*), \text{ for any } 1 \leqslant k \leqslant nd. \tag{2.9}$$

Therefore, if $1 \leqslant k \leqslant nd$, (2.9) and Proposition 2.4 implies that $\Lambda_k(T) \subseteq \overline{D(0, \cos \frac{\rho(k,d)\pi}{n+1})}$ if $1 \leqslant \rho(k, d) \leqslant [\frac{n+1}{2}]$ and an empty set if $[\frac{n+1}{2}] < \rho(k, d) \leqslant n$. Finally, if $k > nd$, $\Lambda_k(T) = \emptyset$ from (P6). □

The previous theorem establishes the connection between the higher rank-k numerical range of nilpotent operators with nilpotency degree n and the n-dimensional shift on $\mathbb{C}^n$. In the particular case, where $k = 1$ and using the fact that $\rho(1, d) = 1$ for all $1 \leqslant d \leqslant +\infty$, it becomes obvious to see

Corollary 2.6 (Haagerup and de la Harpe [11]). *Consider* $T \in \mathcal{B}(\mathcal{H})$ *such that* $\|T\| \leqslant 1$ *and* $T^n = 0$. *Then we have* $\omega_2(T) \leqslant \cos(\frac{\pi}{n+1})$.

In the next result, we will show that Theorem 2.5 generalizes to operator of class C_0. Recall that a completely non-unitary operator T is said to be of class C_0 [20] if $u(T) = 0$ for some nonzero function u in $\mathbb{H}^\infty$. Among the functions u with this property, there is an inner one which divides all others. This function is denoted by m_T, and is uniquely determined up to a nonstant factor of absolute value one. Note also that for the class C_0, the defect numbers of T and T^* are always the same (see [20], Theorem 5.2, page 267).

Theorem 2.7. *If T is an operator of class* C_0, *then*

$$\Lambda_k(T) \subseteq \Lambda_k(I_{\mathcal{D}_{T^*}} \otimes S(m_T)),$$

for each $k \geq 1$.

Proof. Let us consider the isometry $V : \mathcal{H} \to \mathcal{D}_{T^*} \otimes \mathbb{H}^2$, defined by

$$V(x) = \sum_{j=0}^{\infty} D_{T^*} T^{*j} x \otimes f_j, \tag{2.10}$$

where $(f_j)_{j\geq 0}$ is the orthonormal basis of $\mathbb{H}^2$ defined by $f_j(z) = z^j$ for each $j \geq 0$. Note that (2.10) is a consequence of the fact that T^* tend strongly to zero. On the other hand, we have

$$VT^*(x) = \sum_{j=0}^{\infty} D_{T^*} T^{*j}(x) \otimes S^*(f_j) = (I_{\mathcal{D}_{T^*}} \otimes S^*)V(x).$$

So $VT^* = (I_{\mathcal{D}_{T^*}} \otimes S^*)V$ and $VT^{*2} = (I_{\mathcal{D}_{T^*}} \otimes S^*)^2 V$. By induction on n, we obtain that $VT^{*n} = (I_{\mathcal{D}_{T^*}} \otimes S^*)^n V$ for each $n \geq 0$.

Consequently,

$$Vh(T^*) = h(I_{\mathcal{D}_{T^*}} \otimes S^*)V, \text{ for each } h \in \mathbb{H}^\infty. \tag{2.11}$$

Now $m_T(T) = 0$ thus $\overline{m_T}(T^*) = 0$ and from (2.11), we have

$$0 = \overline{m_T}(I_{\mathcal{D}_{T^*}} \otimes S^*)V = (I_{\mathcal{D}_{T^*}} \otimes m_T(S)^*)V.$$

So,

$$\mathrm{Im}(V) \subseteq \mathrm{Ker}(I_{\mathcal{D}_{T^*}} \otimes m_T(S)^*) = \mathcal{D}_{T^*} \otimes H(m_T),$$

and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{V} & \mathcal{D}_{T^*} \otimes H(m_T) \\ \Big\downarrow{\scriptstyle T^*} & & \Big\downarrow{\scriptstyle I_{\mathcal{D}_{T^*}} \otimes m_T(S)^*} \\ \mathcal{H} & \xrightarrow{V} & \mathcal{D}_{T^*} \otimes H(m_T) \end{array}$$

It follows that

$$VT^* = (I_{\mathcal{D}_{T^*}} \otimes m_T(S)^*)_{|\mathcal{D}_{T^*} \otimes H(m_T)}V = (I_{\mathcal{D}_{T^*}} \otimes S^*(m_T))V. \tag{2.12}$$

In other words, T^* is the restriction of the sum of a number of copies of $S^*(m_T)$, the number of copies being equal to the defect number of T. An application of (P2) and (2.12) ends the proof. □

Corollary 2.8 ([2], Proposition 1)**.** *If T is an operator of class C_0, then $W(T) \subseteq W(S(m_T))$, and therefore, $\omega_2(T) \leqslant \omega_2(S(m_T))$.*

Theorem 2.9. *Let n, k, and q be arbitrary integers such that $n \geq 2$, $1 \leqslant k \leqslant n$, and $2 \leqslant q \leqslant n-1$.*

(1) *If* $\delta(n, q) = 0$, $S_n^q = \oplus_{i=1}^{q} S_{\rho(n,q)}$.

(2) *If* $1 \leqslant \delta(n, q) \leqslant q-1$, $S_n^q = (\oplus_{i=1}^{\delta(n,q)} S_{\rho(n,q)}) \oplus (\oplus_{i=1}^{q-\delta(n,q)} S_{\rho(n,q)-1})$.

(3) *If* $\delta(n, q) = 0$,

$$\Lambda_k(S_n^q) = \begin{cases} \overline{D(0, \cos\frac{\rho(k,q)\pi}{\rho(n,q)+1})}, & \text{if } 1 \leqslant \rho(k, q) \leqslant [\frac{\rho(n,q)+1}{2}], \\ \emptyset, & \text{unless.} \end{cases}$$

(4) *If* $1 \leqslant \delta(n, q) \leqslant q-1$ *and* $1 \leqslant \delta(k, q) \leqslant \delta(n, q)$,

$$\Lambda_k(S_n^q) = \begin{cases} \overline{D(0, \cos\frac{\rho(k,q)\pi}{\rho(n,q)+1})}, & \text{if } \rho(k, q) \leqslant [\frac{\rho(n,q)+1}{2}], \\ \emptyset, & \text{unless.} \end{cases}$$

(5) *If* $1 \leqslant \delta(n, q) \leqslant q-1$ *and* $\delta(k, q) = 0$ *or* $\delta(n, q)+1 \leqslant \delta(k, q) \leqslant q-1$,

$$\Lambda_k(S_n^q) = \begin{cases} \overline{D(0, \cos\frac{\rho(k,q)\pi}{\rho(n,q)})}, & \text{if } \rho(k, q) \leqslant [\frac{\rho(n,q)}{2}], \\ \emptyset, & \text{unless.} \end{cases}$$

Proof. Firstly, note that for $1 \leqslant s \leqslant n$, we have

$$S_n^q(e_s) = \begin{cases} e_{s+q}, & \text{if } 1 \leqslant s \leqslant n-q, \\ 0, & \text{if } n-q < s \leqslant n. \end{cases} \tag{2.13}$$

Consider the Euclidean division of n by q. Then, there exists $\alpha \geq 1$ and $0 \leqslant r \leqslant q-1$ such that $n = \alpha q + r$.

Assume that $r = 0$, then $n = \alpha q$ and necessarily $\alpha \geq 2$. For $1 \leqslant i \leqslant q$, let denote by

$$\mathcal{F}_i = \{e_{i+jq};\ 0 \leqslant j \leqslant \alpha - 1\} \text{ and } \widehat{\mathcal{F}}_i = \text{spam } \mathcal{F}_i.$$

We claim that $\mathbb{C}^n = \oplus_{i=1}^q \widehat{\mathcal{F}}_i$ and $S_{n|\widehat{\mathcal{F}}_i}^q = S_\alpha$. To prove the claim, note that $\sum_{i=1}^q \text{Card}\,\widehat{\mathcal{F}}_i = n$. On the other hand, for $1 \leqslant i \neq i' \leqslant q$, $\mathcal{F}_i \bigcap \mathcal{F}_{i'} = \emptyset$. Otherwise, there exist $0 \leqslant j,\ j' \leqslant \alpha - 1$ such that $i + jq = i' + j'q$, that is, $i' - i = q(j - j')$. We can assume that $i' > i$ and $j > j'$, thus $q = \dfrac{i' - i}{j - j'} \leqslant q - 1$, which is absurd. Thus $\mathbb{C}^n$ is the direct summand of $\widehat{\mathcal{F}}_i$ for $1 \leqslant i \leqslant q$. Now from (2.13), for all $1 \leqslant i \leqslant q$ and $0 \leqslant j \leqslant \alpha - 1$, the compressed of S_n^q to $\widehat{\mathcal{F}}_i$ is given by

$$S_n^q(e_{i+jq}) = e_{i+(j+1)q},\ \text{if}\ 0 \leqslant j \leqslant \alpha - 2,$$

$$S_n^q(e_{i+\alpha q}) = 0,$$

which implies that $S_{n|\widehat{\mathcal{F}}_i}^q$ is unitarily equivalent to S_α and evidently S_n^q is unitarily equivalent to $\oplus_{i=1}^q S_\alpha = I_q \otimes S_\alpha$. Then Proposition 2.4 implies that

$$\Lambda_k(S_n^q) = \begin{cases} \overline{D(0,\ \cos\dfrac{\rho(k,q)\pi}{\alpha+1})}, & \text{if}\ 1 \leqslant \rho(k,\ q) \leqslant [\ \dfrac{\alpha+1}{2}\ ], \\ \emptyset, & \text{unless.} \end{cases}$$

For the sequel of the proof, assume that $1 \leqslant r \leqslant q - 1$. For $1 \leqslant i \leqslant q$, let denote by

$$\begin{cases} \mathcal{F}_i = \{\, e_{i+jq};\ 0 \leqslant j \leqslant \alpha \,\} & \text{and}\ \widehat{\mathcal{F}}_i = \text{spam}\,\mathcal{F}_i, \quad \text{for}\ 1 \leqslant i \leqslant r, \\ \mathcal{G}_i = \{\, e_{i+jq};\ 0 \leqslant j \leqslant \alpha - 1 \,\} & \text{and}\ \widehat{\mathcal{G}}_i = \text{spam}\,\mathcal{G}_i, \quad \text{for}\ r+1 \leqslant i \leqslant q, \end{cases}$$

using the same scheme as before, we easily prove that

$$\mathbb{C}^n = \left(\oplus_{i=1}^r \widehat{\mathcal{F}}_i\right) \oplus \left(\oplus_{i=r+1}^q \widehat{\mathcal{G}}_i\right), \tag{2.14}$$

$$S_{n|\widehat{\mathcal{F}}_i}^q = S_{\alpha+1}, \tag{2.15}$$

$$S_n^q\big|_{\widehat{\mathcal{G}_i}} = S_\alpha, \tag{2.16}$$

which implies that

$$S_n^q = \left(\oplus_{i=1}^{r}\ S_{\alpha+1}\right) \oplus \left(\oplus_{i=1}^{q-r}\ S_\alpha\right).$$

With respect to the basis $\{\mathcal{F}_i;\ 1 \leqslant i \leqslant r\} \bigcup \{\mathcal{G}_i;\ r+1 \leqslant i \leqslant q\}$ of $\mathbb{C}^n$, S_n^q have this representation matrix

$$\begin{pmatrix} S_{\alpha+1} & & & & & & \\ & \ddots & & & & & \\ & & S_{\alpha+1} & & & & \\ & & & S_\alpha & & & \\ & & & & \ddots & & \\ & & & & & S_\alpha & \end{pmatrix},$$

which implies that

$$\Lambda_k(S_n^q) = \Lambda_k\Big((I_r \otimes S_{\alpha+1}) \oplus (I_{q-r} \otimes S_\alpha)\Big).$$

On the other hand,

$$\begin{aligned} & e^{i\theta}\Big((I_r \otimes S_{\alpha+1}) \oplus (I_{q-r} \otimes S_\alpha)\Big) + e^{-i\theta}\Big((I_r \otimes S_{\alpha+1}) \oplus (I_{q-r} \otimes S_\alpha)\Big)^* \\ &= e^{i\theta}\Big((I_r \otimes S_{\alpha+1}) \oplus (I_{q-r} \otimes S_\alpha)\Big) + e^{-i\theta}\Big((I_r \otimes S_{\alpha+1}^*) \oplus (I_{q-r} \otimes S_\alpha^*)\Big) \\ &= \Big(I_r \otimes (e^{i\theta}S_{\alpha+1} + e^{-i\theta}S_{\alpha+1}^*)\Big) \oplus \Big(I_{q-r} \otimes (e^{i\theta}S_\alpha + e^{-i\theta}S_\alpha^*)\Big) \\ &= \Big(\oplus_{i=1}^{r}\ (e^{i\theta}S_{\alpha+1} + e^{-i\theta}S_{\alpha+1}^*)\Big) \oplus \Big(\oplus_{i=1}^{q-r}\ (e^{i\theta}S_\alpha + e^{-i\theta}S_\alpha^*)\Big). \end{aligned}$$

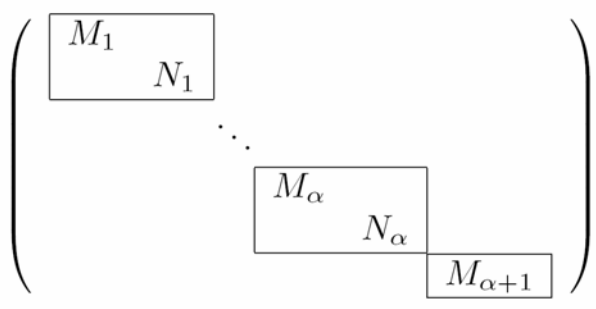
The eigenvalues of $e^{i\theta}S_{\alpha+1} + e^{-i\theta}S_{\alpha+1}^*$ and $e^{i\theta}S_\alpha + e^{-i\theta}S_\alpha^*$ are, respectively,

$$\left(2\cos\left(\frac{\mu\pi}{\alpha+2}\right)\right)_{1 \leqslant \mu \leqslant \alpha+1} \text{ and } \left(2\cos\left(\frac{\nu\pi}{\alpha+1}\right)\right)_{1 \leqslant \nu \leqslant \alpha}.$$

Thus,

$$\Lambda_k(S_n^q) = \bigcap_{\theta\in[0,\,2\pi[} e^{i\theta}\Big\{ z \in \mathbb{C} : \mathcal{R}e(z) \leqslant \lambda_k(B)\Big\},$$

where B denotes this matrix representation



with

$$M_\mu = \begin{pmatrix} \cos(\frac{\mu\pi}{\alpha+2}) & & \\ & \ddots & \\ & & \cos(\frac{\mu\pi}{\alpha+2}) \end{pmatrix} \in \mathcal{M}_r(\mathbb{C}),\ 1 \leqslant \mu \leqslant \alpha+1,$$

and

$$N_\nu = \begin{pmatrix} \cos(\frac{\nu\pi}{\alpha+1}) & & \\ & \ddots & \\ & & \cos(\frac{\nu\pi}{\alpha+1}) \end{pmatrix} \in \mathcal{M}_{q-r}(\mathbb{C}),\ 1 \leqslant \nu \leqslant \alpha.$$

Note that

$$\cos\frac{\pi}{\alpha+2} > \cos\frac{\pi}{\alpha+1} > \cos\frac{2\pi}{\alpha+2} > \cos\frac{2\pi}{\alpha+1} > \cdots > \cos\frac{\alpha\pi}{\alpha+1} > \cos\frac{(\alpha+1)\pi}{\alpha+2},$$

which implies that the eigenvalues in the last representation are arranged in the decreasing order. Now, consider the Euclidean division of k by q as the following $k = \alpha' q + r'$ with $0 \leqslant r' \leqslant q-1$ and $\alpha' \geqslant 0$. We distinguish the following three cases:

Case 1. If $r' = 0$, $\lambda_k = \cos(\frac{\alpha'\pi}{\alpha+1})$, and

$$\Lambda_k(S_n^q) = \begin{cases} \overline{D(0,\ \cos(\frac{\alpha'\pi}{\alpha+1}))}, & \text{if } \alpha' \leqslant [\frac{\alpha+1}{2}], \\ \emptyset, & \text{unless.} \end{cases}$$

Case 2. If $1 \leqslant r' \leqslant r$, $\lambda_k = \cos\frac{(\alpha'+1)\pi}{\alpha+2}$, and

$$\Lambda_k(S_n^q) = \begin{cases} \overline{D(0,\ \cos\frac{(\alpha'+1)\pi}{\alpha+2})}, & \text{if } (\alpha'+1) \leqslant [\frac{\alpha+2}{2}], \\ \emptyset, & \text{unless.} \end{cases}$$

Case 3. If $r+1 \leqslant r' \leqslant q-1$, $\lambda_k = \cos\frac{(\alpha'+1)\pi}{\alpha+1}$, and

$$\Lambda_k(S_n^q) = \begin{cases} \overline{D(0,\ \cos\frac{(\alpha'+1)\pi}{\alpha+1})}, & \text{if } (\alpha'+1) \leqslant [\frac{\alpha+1}{2}], \\ \emptyset, & \text{unless.} \end{cases}$$

□

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