Journal of Pure and Applied Mathematics: Advances and Applications
Volume 6, Number 1, 2011, Pages 31-49

A LAPLACE TYPE PROBLEM FOR IRREGULAR LATTICE WITH THE CELL COMPOSED OF n TRAPEZIUMS AND $4n$ OBSTACLES (1, 2, 3)

G. CARISTI and MARIUS STOKA

Department of S. E. A.
University of Messina
98122, Messina
Italy
email: gcaristi@unime.it

Accademia delle Scienze di Torino
via Maria Vittoria, 3
Turin
Italy

Abstract

In this paper, we consider a lattice with the cell composed of n trapeziums and $4n$ obstacles (1, 2, 3), as in the Figures 1, 10, and 12, we compute the probability that a segment of random position and of constant length intersects a side of lattice. In particular, we obtain the Laplace probability.

1. Cell Composed of a Trapezium

Let $\Re_1(a, b, \alpha; m)$ be a lattice with fundamental cell $C_0^{(1)}$ an isosceles trapezium is represented in Figure 1;

2010 Mathematics Subject Classification: 60D05, 52A22.

Keywords and phrases: geometric probability, stochastic geometry, random sets, random convex sets, integral geometry.

Received July 21, 2011

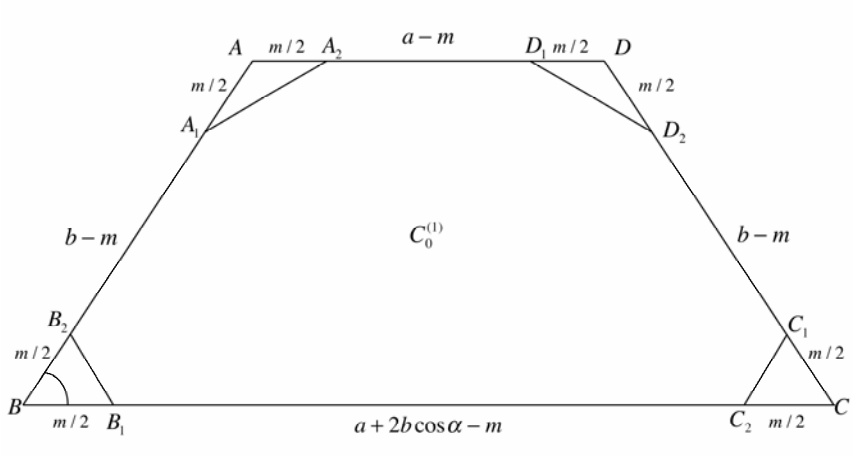

Figure 1.

where $0 < \alpha \leq \frac{\pi}{2}$ is an angle and $0 \leq m < \min(a, b)$.

We have

$$\text{area } AA_1A_2 = \text{area } BB_1B_2 = \text{area } CC_1C_2$$

$$= \text{area } DD_1D_2 = \frac{m^2}{8} \sin \alpha, \tag{1}$$

and consequently,

$$\text{area } C_0^{(1)} = (a + b \cos \alpha) b \sin \alpha - \frac{m^2}{2} \sin \alpha. \tag{2}$$

We consider a segment s of random position and of constant length $l < \min(a - m, b - m)$ and we compute the probability that s intersects a side of the lattice R_1, that is the same as the probability $P_{\text{int}}^{(1)}$ that s intersects the side of the fundamental cell $C_0^{(1)}$.

The position of the segment s is determinated by its middle point O and by the angle φ that it forms with the side BC of the cell $C_0^{(1)}$.

To compute the probability $P_{\text{int}}^{(1)}$, we consider the limit positions of the segment s in the trapezium $C_0^{(1)}$ for an assigned value of φ and let $\hat{C}_0^{(1)}(\varphi)$ be the polygon determinated by these positions (Figure 2).

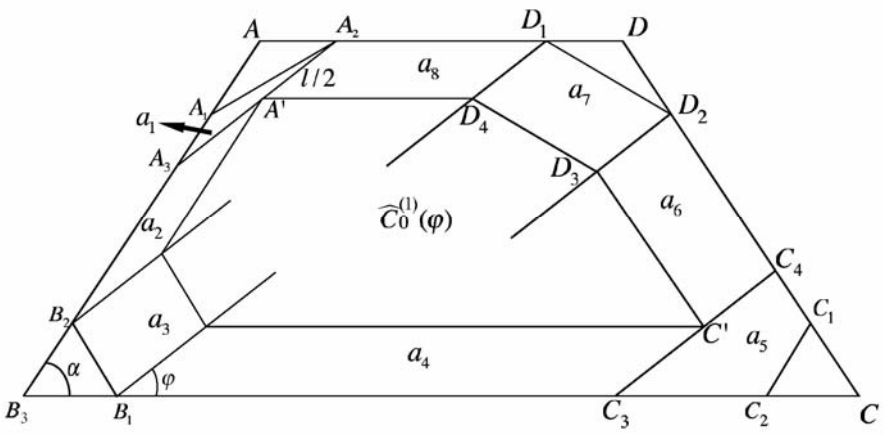

Figure 2.

Considering this figure, we can write

$$\text{area } \widehat{C}_0^{(1)}(\varphi) = \text{area } C_0^{(1)} - [\text{area } a_1(\varphi) + \text{area } a_2(\varphi) + \ldots + \text{area } a_8(\varphi)]. \quad (3)$$

To compute area $a_1(\varphi)$, we consider the figure

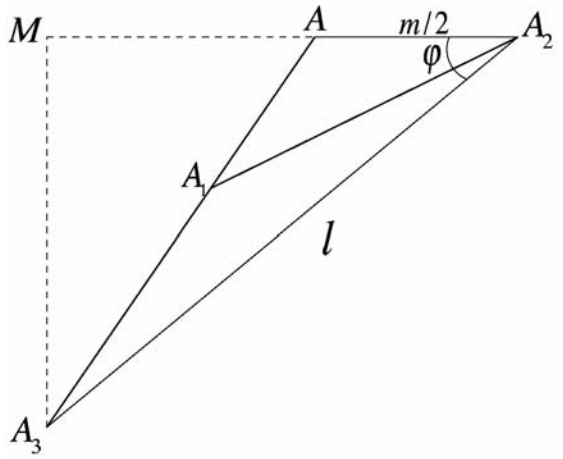

Figure 3.

From the figure, we obtain

$$\text{area } a_1(\varphi) = \text{area } AA_2A_3 - \text{area } AA_1A_2.$$

We have

$$|A_3M| = l \sin \varphi,$$

therefore,

$$\text{area } AA_2A_3 = \frac{ml \sin \varphi}{4},$$

and considering the (1) follows

$$\text{area } \alpha_1(\varphi) = \frac{ml \sin \varphi}{4} - \frac{m^2 \sin \varphi}{8}. \tag{4}$$

In the triangle AA_2A_3, we have $\widehat{AA_3A_2} = \alpha - \varphi$, therefore,

$$\frac{|AA_3|}{\sin \varphi} = \frac{l}{\sin \alpha},$$

that is,

$$|AA_3| = \frac{l \sin \varphi}{\sin \alpha}. \tag{5}$$

The figure

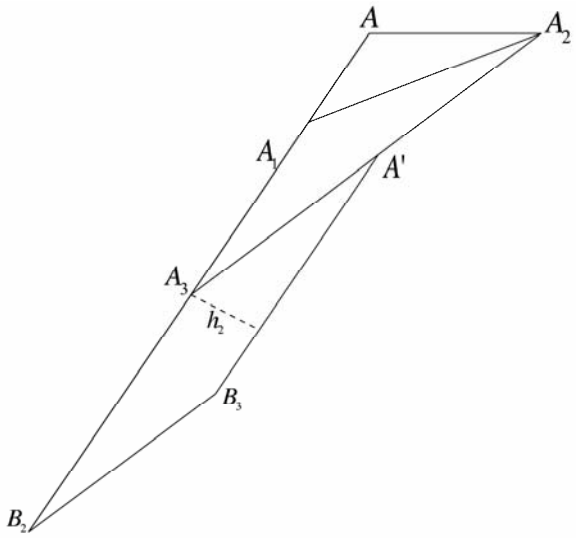

Figure 4.

give us

$$h_2 = \frac{l}{2}\sin \widehat{A_3B_2B_3} = \frac{l}{2}\cos(\alpha - \varphi),$$

and the (5)

$$|B_2A_3| = |AB_2| - |AA_3| = b - \frac{m}{2} - \frac{l\sin\varphi}{\sin\alpha}.$$

Therefore,

$$\text{area } a_2(\varphi) = \frac{l}{2}\sin(\alpha - \varphi)\cdot\left(b - \frac{m}{2} - \frac{l\sin\varphi}{\sin\alpha}\right). \tag{6}$$

To compute area $a_3(\varphi)$, we consider the figure

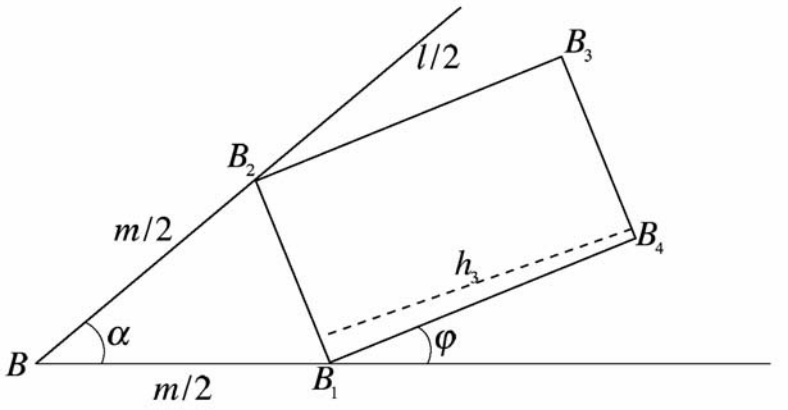

Figure 5.

From here, follow

$$\widehat{B_4B_1B_2} = \pi - \left(\varphi + \frac{\pi}{2} - \alpha\right) = \frac{\pi}{2} - \varphi + \frac{\alpha}{2},$$

$$\widehat{B_1B_2B_3} = \pi - \widehat{B_4B_1B_2} = \frac{\pi}{2} + \varphi - \frac{\alpha}{2},$$

$$|B_1B_2| = m\sin\frac{\alpha}{2}, \quad h_3 = \frac{l}{2}\sin\widehat{B_4B_1B_2} = \frac{l}{2}\cos\left(\varphi - \frac{\alpha}{2}\right),$$

therefore,

$$\text{area } a_3(\varphi) = \frac{ml}{2} \sin\frac{\alpha}{2} \cos\left(\varphi - \frac{\alpha}{2}\right). \tag{7}$$

If in this formula, we put $\pi - \alpha$ instead of α, we obtain

$$\text{area } a_7(\varphi) = \frac{ml}{2} \cos\frac{\alpha}{2} \sin\left(\varphi + \frac{\alpha}{2}\right). \tag{8}$$

Considering now the figure

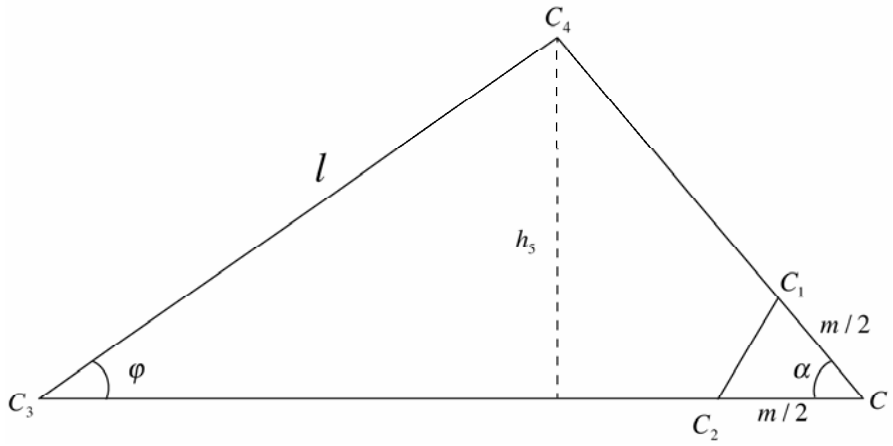

Figure 6.

The triangle CC_3C_4 give us

$$\frac{|CC_3|}{\sin(\varphi + \alpha)} = \frac{|CC_4|}{\sin\varphi} = \frac{l}{\sin\alpha},$$

hence,

$$|CC_3| = \frac{l\sin(\varphi + \alpha)}{\sin\alpha}, \quad |CC_4| = \frac{l\sin\varphi}{\sin\alpha}. \tag{9}$$

Moreover,

$$h_5 = l\sin\varphi,$$

therefore, considering the relation (1),

$$\text{area } a_5(\varphi) = \frac{l^2 \sin\varphi \sin(\varphi + \alpha)}{2 \sin\alpha} - \frac{m^2}{8} \sin\alpha. \tag{10}$$

From the figure,

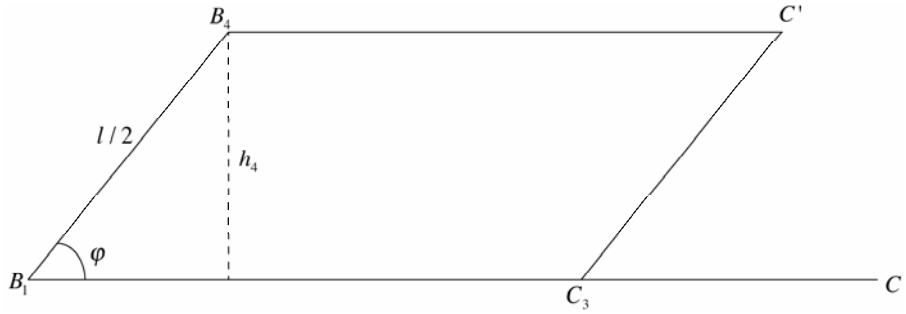

Figure 7.

follow

$$h_4 = \frac{l}{2} \sin\varphi.$$

Moreover, because

$$|BC| = a + 2b\cos\alpha,$$

the relation (9) give us

$$|B_1C_3| = |BC| - |BB_1| - |CC_3| = a + 2b\cos\alpha - \frac{m}{2} - \frac{l\sin(\varphi + \alpha)}{\sin\alpha}.$$

Therefore,

$$\text{area } a_4(\varphi) = \left[a + 2b\cos\alpha - \frac{m}{2} - \frac{l\sin(\varphi + \alpha)}{\sin\alpha}\right] \cdot \frac{l}{2} \sin\varphi. \tag{11}$$

Considering now the figure

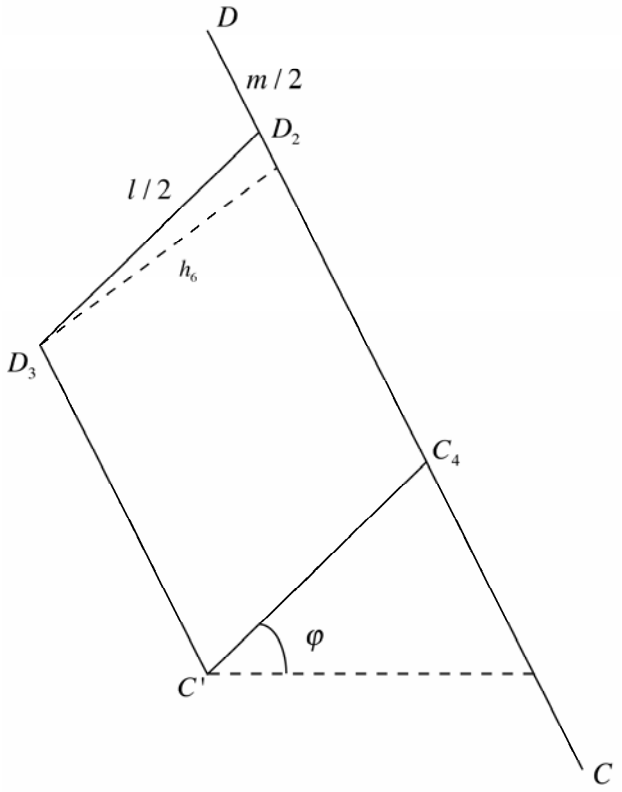

Figure 8.

The relation (9) give us

$$|D_2C_4| = |CD| - \frac{m}{2} - |CC_4| = b - \frac{m}{2} - \frac{l \sin \varphi}{\sin \alpha}.$$

Then, from the triangle CC_3C_4 (see Figure 6), we have

$$\widehat{CC_4C_3} = \pi - (\varphi + \alpha),$$

and consequently,

$$h_6 = \frac{l}{2} \sin (\varphi + \alpha).$$

Hence,

$$\text{area } a_6(\varphi) = \left(b - \frac{m}{2} - \frac{l \sin \varphi}{\sin \alpha} \right) \cdot \frac{l}{2} \sin (\varphi + \alpha). \tag{12}$$

At the end, from the figure,

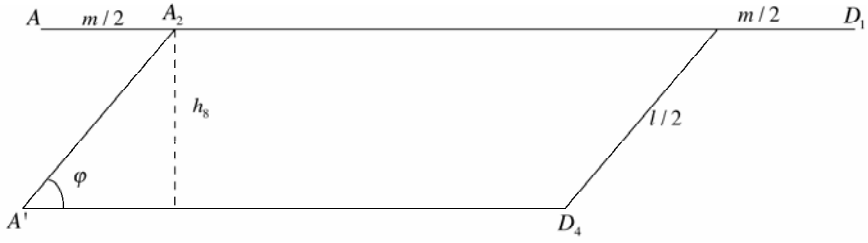

Figure 9.

follow

$$|A_2 D_1| = a - m; \quad h_8 = h_4 = \frac{l}{2} \sin \varphi,$$

therefore,

$$\text{area } a_8(\varphi) = \frac{(a - m) l \sin \varphi}{2}. \tag{13}$$

Replacing in the (3), the expressions (4), (6), (7), (8), (10), (11), (12), and (13), we obtain

$$\text{area } \widehat{C}_0^{(1)}(\varphi) = \text{area } C_0^{(1)} - \{ [b \sin \alpha \cos \varphi + (a + 2b \cos \alpha) \sin \varphi] l$$

$$- \frac{l^2}{2} \sin 2\varphi - \frac{m^2}{4} \sin \alpha \}. \tag{14}$$

Denoting with M_1, the set of segments s, which have the middle point in the trapezium $C_0^{(1)}$ and with N_1, the set of segments s completely located in the trapezium, we have [3]

$$P_{\text{int}}^{(1)} = 1 - \frac{\mu(N_1)}{\mu(M_1)}, \tag{15}$$

where μ is the Lebesgue measure in the Euclidean plane.

We compute the measures $\mu(M_1)$ and $\mu(N_1)$ with the Poincaré kinematic measure [2]

$$dK = dx \wedge dy \wedge d\varphi,$$

where x, y are the coordinates of O and φ is the defined angle.

We have $\varphi \in [0, \alpha]$, therefore,

$$\mu(M_1) = \int_0^{\alpha} d\varphi \iint_{\{(x,\, y)\in C_0^{(1)}\}} dxdy = \int_0^{\alpha} \left(\text{area } C_0^{(1)}\right) d\varphi = \alpha \text{ area } C_0^{(1)}, \qquad (16)$$

and considering the relation (14),

$$\mu(N_1) = \int_0^{\alpha} d\varphi \iint_{\{(x,y)\in \hat{C}_0^{(1)}(\varphi)\}} dxdy$$

$$= \int_0^{\alpha} \left[\text{area } \hat{C}_0^{(1)}(\varphi)\right] d\varphi$$

$$= \alpha \text{ area } C_0^{(1)} - \{ [\, b \sin^2 \alpha + (a + 2b \cos \alpha)(1 - \cos \alpha)\,]\, l$$

$$- (1 - \cos 2\alpha) \frac{l^2}{4} - \frac{m^2}{4} \alpha \sin \alpha \,\}. \qquad (17)$$

The formulas (15), (16), and (17) give us

$$P_{\text{int}}^{(1)} = \frac{1}{\alpha \left[(a + b \cos \alpha) \cdot b \sin \alpha - \frac{m^2}{2} \sin \alpha \right]}$$

$$\times \{[b \sin^2 \alpha + (a + 2b \cos \alpha)(1 - \cos \alpha)]l$$

$$- (1 - \cos 2\alpha) \frac{l^2}{4} - \frac{m^2}{4} \alpha \sin \alpha \}. \qquad (18)$$

For $m = 0$ and $\alpha = \frac{\pi}{2}$, the cell $C_0^{(1)}$ becomes a rectangle with sides a and b and the probability (18) becomes

$$P = \frac{2(a+b)l - l^2}{\pi ab},$$

that is the probability of Laplace.

2. Cell Composed of Two Trapeziums

Let $\Re_2(a, b, c, \alpha; m)$ be the lattice with fundamental cell $C_0^{(2)}$ composed of two isosceles trapeziums $C_{01}^{(2)}$ and $C_{02}^{(2)}$ as in the Figure 10;

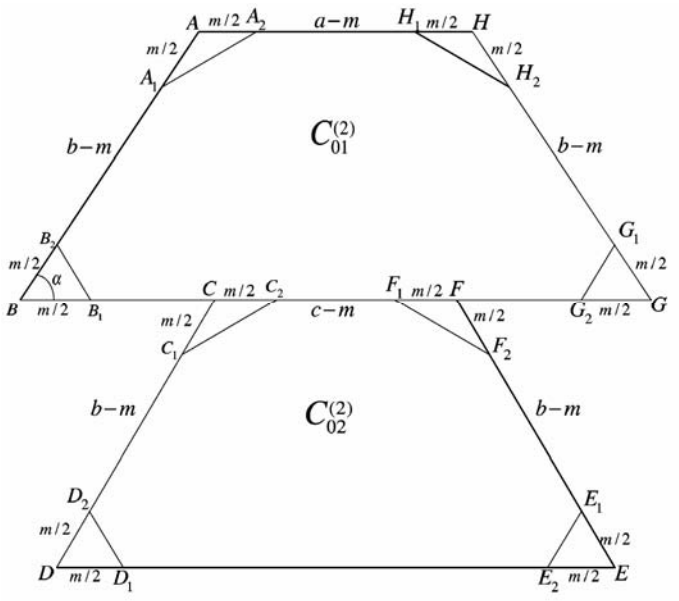

Figure 10.

where $0 < \alpha \le \frac{\pi}{2}$ is an angle.

We have

$$|AB| = |CD| = |EF| = |GH| = b,$$

$$|AH| = a, \quad |BG| = a + 2b\cos\alpha,$$

$$|CF| = c, \quad |DE| = c + 2b\cos\alpha, \tag{19}$$

and

$$\text{area } AA_1A_2 = \text{area } CC_1C_2 = \text{area } FF_1F_2$$
$$= \text{area } HH_1H_2 = \text{area } BB_1B_2 = \text{area } DD_1D_2$$
$$= \text{area } EE_1E_2 = \text{area } GG_1G_2 = \frac{m^2}{8}\sin\alpha, \tag{20}$$

with the expressions (19) and (20) follow

$$\text{area } C_{01}^{(2)} = (a + b\cos\alpha)\cdot b\sin\alpha - \frac{m^2}{2}\sin\alpha,$$
$$\text{area } C_{02}^{(2)} = (c + b\cos\alpha)\cdot b\sin\alpha - \frac{m^2}{2}\sin\alpha. \tag{21}$$

Let s be a segment of random position and of constant length $l < \min(b - m, c - m)$ and we want to compute the probability that s intersects a side of the lattice R_2, that is the same as the probability $P_{\text{int}}^{(2)}$ that s intersects a side of the fundamental cell $C_0^{(2)}$.

As in the previous section, the position of the segment s is determinated by its middle point O and by the angle φ that it forms with the side BG (or DE) of the cell $C_0^{(2)}$.

To compute the probability $P_{\text{int}}^{(2)}$, we consider the limit positions of the segment s in the trapezium $C_{0i}^{(2)}$, $(i = 1, 2)$ for an assigned value of φ and let $\widehat{C}_{0i}^{(2)}(\varphi)$ be the polygon determinated by these positions (Figure 11).

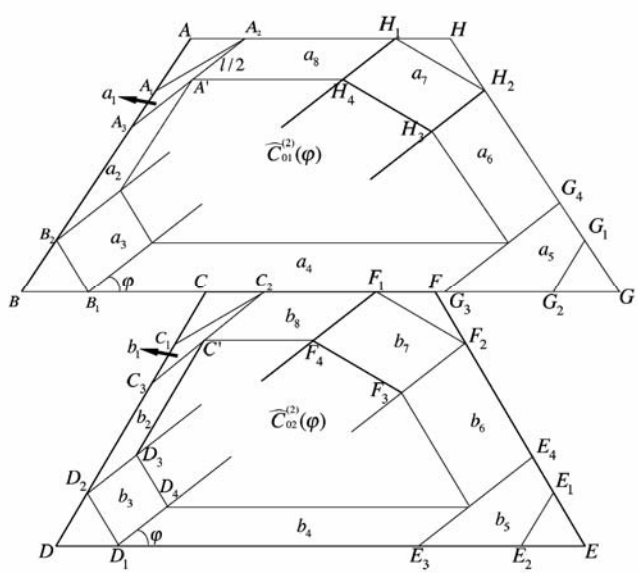

Figure 11.

Denoting with $M_i^{(2)}(i = 1, 2)$, the set of segments s, which have the middle point in the trapezium $C_{0i}^{(2)}$ and with $N_i^{(2)}$, the set of segments s completely contained in the trapezium, we have [3]

$$P_{\text{int}}^{(2)} = 1 - \frac{\mu\left(N_1^{(2)}\right) + \mu\left(N_2^{(2)}\right)}{\mu\left(M_1^{(2)}\right) + \mu\left(M_2^{(2)}\right)}. \tag{22}$$

As in the Section 1, we have

$$\mu\left(M_1^{(2)}\right) = \int_0^{\alpha} d\varphi \iint\limits_{\{(x,y)\in C_{01}^{(2)}\}} dxdy = \int_0^{\alpha} \left[\text{area } C_{01}^{(2)}\right] d\varphi = \alpha \text{ area } C_{01}^{(2)},$$

$$\mu\left(M_2^{(2)}\right) = \int_0^{\alpha} d\varphi \iint\limits_{\{(x,y)\in C_{02}^{(2)}\}} dxdy = \int_0^{\alpha} \left[\text{area } C_{02}^{(2)}\right] d\varphi = \alpha \text{ area } C_{02}^{(2)}. \tag{23}$$

Then, considering the formula (17), we can write

$$\mu\left(N_1^{(2)}\right) = \alpha \text{ area } C_{01}^{(2)} - \{[b \sin^2 \alpha + (a + 2b \cos \alpha)(1 - \cos \alpha)]l$$

$$- (1 - \cos 2\alpha)\frac{l^2}{4} - \frac{m^2}{4}\alpha \sin \alpha\}.$$

$$\mu\left(N_2^{(2)}\right) = \alpha \text{ area } C_{02}^{(2)} - \{[b \sin^2 \alpha + (c + 2b \cos \alpha)(1 - \cos \alpha)]l$$

$$- (1 - \cos 2\alpha)\frac{l^2}{4} - \frac{m^2}{4}\alpha \sin \alpha\}. \quad (24)$$

The relations (22), (23), and (24) give us

$$P_{\text{int}}^{(2)} = \frac{1}{\alpha[(a + c + 2b \cos \alpha) \cdot b \sin \alpha - m^2 \sin \alpha]}$$

$$\times \{[2b \sin^2 \alpha + (a + c + 4b \cos \alpha)(1 - \cos \alpha)]l$$

$$- (1 - \cos 2\alpha)\frac{l^2}{2} - \frac{m^2}{2}\alpha \sin \alpha\}. \quad (25)$$

For $m = 0$ and $\alpha = \frac{\pi}{2}$, the trapeziums $C_{01}^{(2)}$ and $C_{02}^{(2)}$ become a rectangle with sides a, b and c, b and the probability (25) becomes

$$P = \frac{2\left(\frac{a + c}{2} + b\right)l - l^2}{\pi b\left(\frac{a + c}{2}\right)},$$

which represented the probability of Laplace for a rectangle with sides $\frac{a + c}{2}$, b, that is, the media of the rectangle with sides a, b respectively c, b.

3. Cell Composed of Three Trapeziums

Let $\Re_3(a, b, c, d, \alpha; m)$ be the lattice with fundamental cell $C_0^{(3)}$ composed of three isosceles trapeziums $C_{01}^{(3)}$, $C_{02}^{(3)}$, and $C_{03}^{(3)}$ as in the Figure 12;

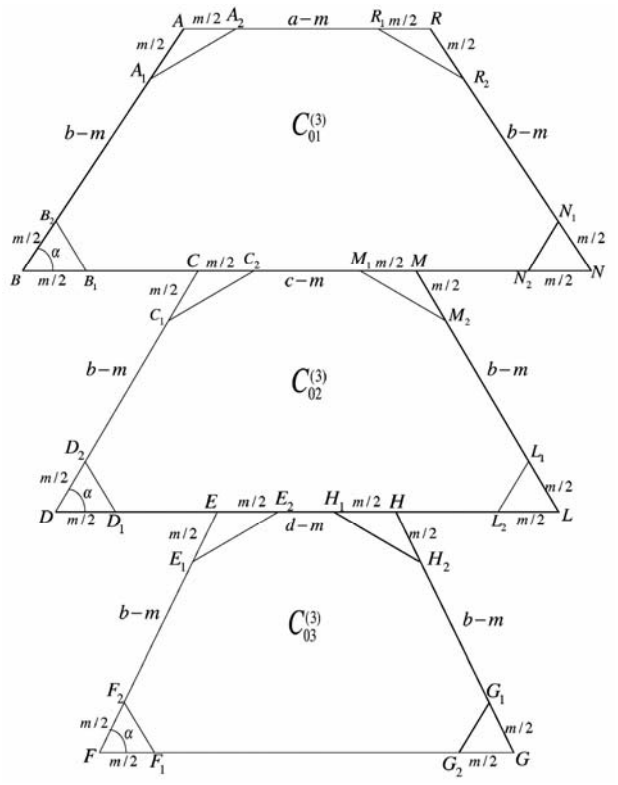

Figure 12.

where $0 < \alpha \leq \dfrac{\pi}{2}$ is an angle.

We have

$$|AB| = |CD| = |EF| = |GH| = |LM| = |NR| = b,$$

$$|AR| = a, \quad |NB| = a + 2b\cos\alpha,$$

$$|CM| = c, \quad |DL| = c + 2b\cos\alpha,$$

$$|EH| = d, \quad |FG| = d + 2b\cos\alpha, \tag{26}$$

and

$$\text{area } AA_1A_2 = \text{area } CC_1C_2 = \text{area } EE_1E_2 = \text{area } HH_1H_2$$

$$= \text{area } MM_1M_2 = \text{area } RR_1R_2 = \text{area } BB_1B_2$$

$$= \text{area } DD_1D_2 = \text{area } FF_1F_2 = \text{area } GG_1G_2$$

$$= \text{area } LL_1L_2 = \text{area } NN_1N_2 = \frac{m^2}{8}\sin\alpha. \tag{27}$$

From the relations (26) and (27) follow

$$\text{area } C_{01}^{(3)} = (a + b\cos\alpha)\cdot b\sin\alpha - \frac{m^2}{2}\sin\alpha,$$

$$\text{area } C_{02}^{(3)} = (c + b\cos\alpha)\cdot b\sin\alpha - \frac{m^2}{2}\sin\alpha,$$

$$\text{area } C_{03}^{(3)} = (d + b\cos\alpha)\cdot b\sin\alpha - \frac{m^2}{2}\sin\alpha. \tag{28}$$

Let s be a segment of random position and of constant length $l < \min(b - m,\ d - m)$ and we want to compute the probability $P_{\text{int}}^{(3)}$ that s intersects a side of the fundamental cell $C_0^{(3)}$.

As in the previous section, we consider $\widehat{C}_{0i}^{(3)}(\varphi)(i = 1,\ 2,\ 3)$, the polygon determined by the limit positions of the segment s in the trapezium $C_{0i}^{(3)}$ for an assigned value of φ.

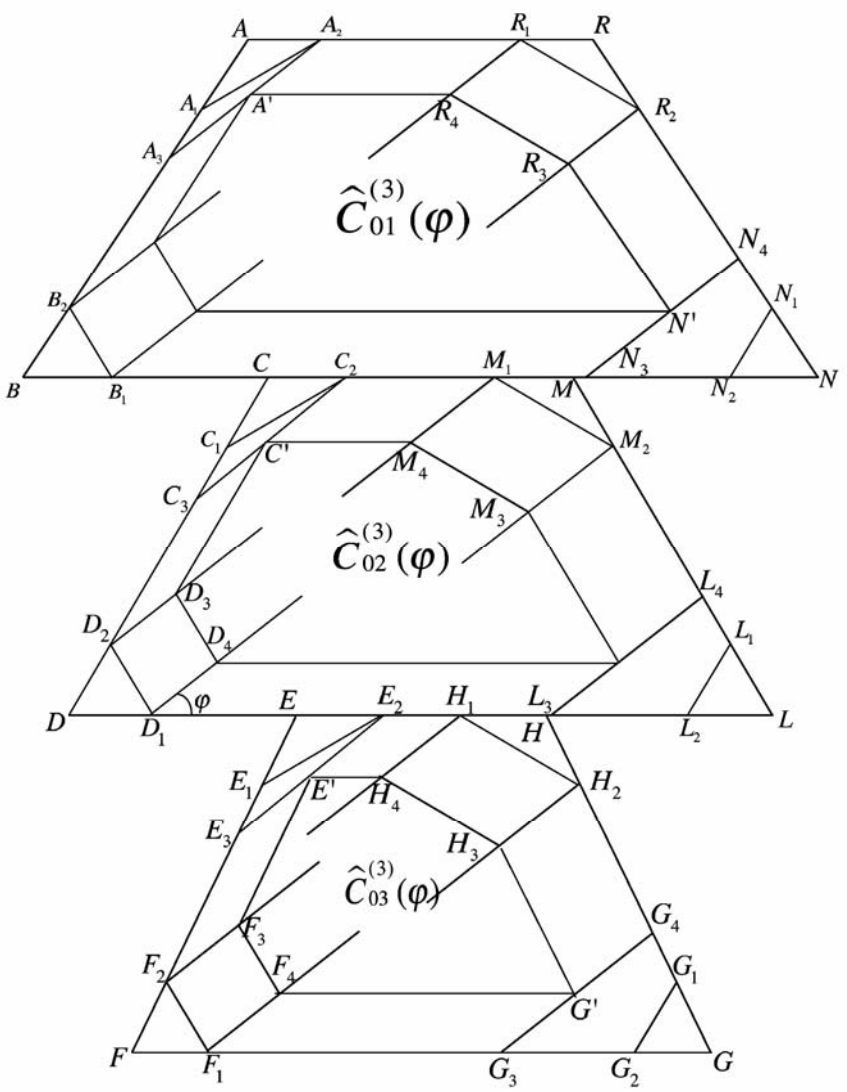

Figure 13.

Denoting with $M_i^{(3)}(i = 1, 2, 3)$, the set of segments s, which have the middle point in the trapezium $C_{0i}^{(3)}$ and with $N_i^{(3)}$, the set of segments s completely contained in the trapezium, we have [3]

$$P_{\text{int}}^{(3)} = 1 - \frac{\mu\left(N_1^{(3)}\right) + \mu\left(N_2^{(3)}\right) + \mu\left(N_3^{(3)}\right)}{\mu\left(M_1^{(3)}\right) + \mu\left(M_2^{(3)}\right) + \mu\left(M_3^{(3)}\right)}. \tag{29}$$

We have

$$\mu\left(M_1^{(3)}\right) = \alpha \text{ area } C_{0i}^{(3)}, \quad (i = 1, 2, 3). \tag{30}$$

Considering the formula (17), we can write

$$\mu\left(N_1^{(3)}\right) = \alpha \text{ area } C_{01}^{(3)} - \{[b \sin^2 \alpha + (a + 2b \cos \alpha)(1 - \cos \alpha)]l$$

$$- (1 - \cos 2\alpha)\frac{l^2}{4} - \frac{m^2}{4}\alpha \sin \alpha\}, \tag{31}$$

$$\mu\left(N_2^{(3)}\right) = \alpha \text{ area } C_{02}^{(3)} - \{[b \sin^2 \alpha + (c + 2b \cos \alpha)(1 - \cos \alpha)]l$$

$$- (1 - \cos 2\alpha)\frac{l^2}{4} - \frac{m^2}{4}\alpha \sin \alpha\}, \tag{32}$$

$$\mu\left(N_3^{(3)}\right) = \alpha \text{ area } C_{03}^{(3)} - \{[b \sin^2 \alpha + (d + 2b \cos \alpha)(1 - \cos \alpha)]l$$

$$- (1 - \cos 2\alpha)\frac{l^2}{4} - \frac{m^2}{4}\alpha \sin \alpha\}. \tag{33}$$

The relations (29), (30), and (31) give us

$$P_{\text{int}}^{(3)} = \frac{1}{\alpha\left[(a + c + d + 3b \cos \alpha) \cdot b \sin \alpha - \frac{3m^2}{2} \sin \alpha\right]}$$

$$\times \{[3b \sin^2 \alpha + (a + c + d + 6b \cos \alpha)(1 - \cos \alpha)]l$$

$$- (1 - \cos 2\alpha)\frac{3l^2}{4} - \frac{3m^2}{4}\alpha \sin \alpha\}. \tag{34}$$

For $m = 0$ and $\alpha = \frac{\pi}{2}$, the trapezium $C_{01}^{(3)}$ becomes a rectangle with sides a, b, the trapezium $C_{02}^{(3)}$ becomes a rectangle with sides c, b, and the trapezium $C_{03}^{(3)}$ becomes a rectangle with sides d, b; while the probability (34) becomes

$$P = \frac{2\left(\frac{a+c+d}{3}+b\right)l - l^2}{\pi b\left(\frac{a+c+d}{3}\right)},$$

which represented the probability of Laplace for a rectangle with sides $\frac{a+c+d}{3}$, b, that is, the media of the rectangle with sides a and b, c and b, d and b.

References

[1] G. Caristi and M. Stoka, A Laplace type problem for a lattice with the cell with axial symmetry and different type of obstacles (I), (to appear).

[2] H. Poincaré, Calcul des Probabilitiés, Ed. 2, Carré, Paris, 1912.

[3] M. Stoka, Probabilitiés géométriques de type Buffon dans le plan Euclidien, Atti Acc. Sci. Torino T. 110 (1975-1976), 53-59.

■