Journal of Mathematical Sciences: Advances and Applications
Volume 9, Number 1/2, 2011, Pages 39-48

THE WELL-POSEDNESS OF SOLUTIONS FOR A NONLINEAR GENERALIZED ASSET PRICING MODEL

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Abstract

This paper provides the derivation of the nonlinear integral equation for a generalized asset pricing model, which yields an analytic price-dividend function of one state variable. Under some assumptions and using the fixed point theorem, the existence and uniqueness of the price-dividend function, which is the solution of the integral equation, are investigated.

1. Introduction

The classical asset pricing theory (consumption based capital asset pricing model) has been investigated by many researchers in finance and management sciences. Lucas [4] and Breeden [2] proposed a consumption asset pricing model, in which the risk of an asset can be measured by the

2010 Mathematics Subject Classification: 91B50, 91B52, 91B60.

Keywords and phrases: nonlinearity, asset pricing model, well-posedness.

This work is supported by the SWUFE's third period key construction item funds of the 211 project (211D3T06).

Received June 12, 2011

covariance of its return with per capita consumption growth. Mehra and Prescott [5] raised the equity premium puzzle, which refers to the phenomenon that returns on stocks over the past century are higher, by approximately 6 percentage points, than returns on government bonds. They solved the equilibrium asset pricing model with a representative agent of constant relative risk aversion preferences, where the growth rate of the endowment followed a simple two-state first order Markov process. Abel [1] constructed an asset pricing model under habit formation and explained the equity premium puzzle.

In this paper, we will focus on the derivation of a nonlinear integral equation for a generalized asset pricing model, which has a degree n with $n \geq 1$. By using the fixed point theorem, the existence and uniqueness of the solution to the integral equation are proved under certain assumptions. The analytic properties of the price-dividend function for the asset pricing model are investigated. Comparing with the work made in Calin et al. [3], three different aspects should be emphasized. Firstly, we put no restrictions on the marginal utility $1 - A_2 e^{A_1 x}$ for the generalized asset pricing model, while $A_2 = 0$ in the works of Calin et al. [3]. Secondly, the integral equation in our model is nonlinear $(n \geq 1)$, while Calin et al. [3] considered the linear case $n = 1$. Thirdly, the existence and uniqueness of the price-dividend function in this paper are proved by using the fixed point theorem. However, the method adopted in Calin et al. [3] is the Picard series technique.

This paper is organized as follows. We describe the asset pricing model briefly and derive the nonlinear integral equation in Section 2. In Section 3, we set up a space, in which the price-dividend function is a well defined function. The existence and uniqueness of a continuous equilibrium price-dividend function solution of the integral equation are shown to be true under certain assumptions. Section 4 closes the paper with some conclusions and further works.

2. The Asset Pricing Model

This section describes the theoretical framework of Abel's model and it is detailed parameterizations. The nonlinear integral equation for a generalized asset pricing model is derived.

In Abel's asset pricing model, the lifetime utility of the representative investor is given by

$$U_t = \sum_{j=0}^{\infty} \beta^j \frac{[c_{t+j} / v_{t+j}]^{1-\gamma}}{1-\gamma}, \tag{1}$$

where $\beta > 0$ is a subjective discount factor, the parameter γ denotes the utility curvature parameter and has restriction $1 > \gamma > 0$, and v_{t+j} is a preference parameter. Suppose that the preference parameter v_t is specified as

$$v_t = [c_{t-1}^h C_{t-1}^{(1-h)}]^{\alpha}, \quad \alpha \geq 0 \quad \text{and} \quad h \geq 0, \tag{2}$$

where c_{t-1} is the consumer's own consumption in period $t-1$, which represents internal habit and C_{t-1} is an aggregate per capita consumption in period $t-1$, which captures external habit. Noting that when $\alpha = 0$, we are in the case of Mehra and Prescott [5], in which the utility does not depend on habit. When $\alpha = 1$ and $h = 0$, we are in the external habit case in which the habit is external to the individual's choice, or called that we are in the case of relative consumption of catching up with the Joneses. When $\alpha = 1$ and $h = 1$, we are in the internal habit case and individuals' own lagged consumption affects their choice of consumption.

Calin et al. [3] generalized Abel's asset pricing model [1] by allowing α and h to vary between 0 and 1, thus creating intermediate combinations of internal and external habits. The dividend growth to follow an AR(1) process with Gaussian shocks is allowed in [3]. As stated in [3], we also let $c_t = C_t = D_t$ and the dividend process for the risky security in the form

$$D_{t+1} = D_t e^{x_0 + \phi x_t + u_{t+1}}, \tag{3}$$

where x_t is the current continuously compounded growth rate of the dividend and follows an AR(1) process $x_{t+1} = x_0 + \phi x_t + u_{t+1}$ with $|\phi| < 1$, in which x_0 is the constant growth rate per period and $u \sim \text{NIID}(0, \sigma^2)$.

Abel derived that the Euler equation for equity is

$$\frac{P_t}{D_t} = E_t[M_{t+1}(\frac{D_{t+1}}{D_t})(1 + (\frac{P_{t+1}}{D_{t+1}}))], \tag{4}$$

where the one-period stochastic discount factor (SDF) M_{t+1}, which is often called the *inter-temporal rate* of substitution in business cycle models, is given by the form

$$M_{t+1} = \beta(\frac{D_{t+1}}{D_t})^{-\gamma}(\frac{u_{t+1}}{u_t})^{\gamma-1}(\frac{H_{t+2}}{E_t(H_{t+1})}), \tag{5}$$

where $u_{t+1} = D_t^{\alpha}$ and

$$H_{t+2} = 1 - \alpha\beta h(\frac{D_{t+2}}{D_{t+1}})^{1-\gamma}(\frac{D_{t+1}}{D_t})^{\alpha(\gamma-1)}. \tag{6}$$

Here, we propose the following generalized nonlinear Euler equation for equity:

$$\frac{P_t}{D_t} = E_t[M_{t+1}(\frac{D_{t+1}}{D_t})(1 + (\frac{P_{t+1}}{D_{t+1}})^n)], \quad n \geq 1. \tag{7}$$

It is noted that the Euler equation (4) together with Equation (5) is degree 1 in the stock price, current dividends, and next period's dividends. However, Equation (7) with Equation (5) is degree n. We shall discuss the dynamic properties of the solution to the generalized price-dividend function, which satisfies Equation (7). Here, we define the price-dividend function in economy as $p_t = \frac{P_t}{D_t}$.

As in Calin et al. [3], we drop the subscript for time since we seek a function of current dividend growth. Substituting dividend process Equations (3) and (5) into the Euler equation (7), we derive the nonlinear integral equation of the generalized asset pricing model in the form

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma}\frac{A_0 e^{A_1 x}}{1 - A_2 e^{A_1 x}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[u-\sigma^2(1-\gamma)]^2} \times \left(1 - A_2 e^{A_1(x_0+\phi x+u)}\right)\left(1 + p^n(x_0 + \phi x + u)\right)du, \quad (8)$$

where $A_0 = \beta e^{x_0(1-\gamma)+\frac{\sigma^2}{2}(1-\gamma)^2}$, $A_1 = (1-\gamma)(\phi - \alpha)$, and $A_2 = \alpha h A_0$. Now, we give a detailed derivation of Equation (8).

3. Existence and Uniqueness of Price-Dividend Function

Definition 1. Let G denote the real-valued space that consists of all continuous functions f such that, there are positive constants M and k with $|f(x)| < Me^{k|x|}$ for all $x \in R$ and the norm on G defined by

$$\| f(x) \| = \sup_{x \in R} |f(x)e^{-k|x|}| \leq M.$$

We will use the fixed point theorem to prove that the nonlinear equation (8) exists a unique solution in space G. Modifying the integral equation (8) by making the change of variable $t = x_0 + \phi x + u$ and $\psi(x) = x_0 + \phi x + \sigma^2(1-\gamma)$, we get

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma}\frac{A_0 e^{A_1 x}}{1 - A_2 e^{A_1 x}}\int_{-\infty}^{+\infty}\{e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} - A_2 e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} e^{A_1 t} + e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} p^n(t) - A_2 e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} e^{A_1 t} p^n(t)\}dt. \quad (9)$$

In fact, there exists a positive constant l such that $\left|\frac{e^{A_1 x}}{1 - A_2 e^{A_1 x}}\right| < l$, for $x \in (-\infty, +\infty)$.

Theorem 1. *If* $A_0 = \beta e^{x_0(1-\gamma)+\frac{\sigma^2}{2}(1-\gamma)^2}$ *is suitable small, k and M satisfy*

$$A_0(1 + A_2 e^{A_1(x_0+\sigma^2(1-\gamma))+\frac{A_1^2\sigma^2}{2}}) < \frac{M}{2l}, \quad k > |A_1\phi| + |kn\phi|, \tag{10}$$

$$A_0 n M^{n-1} e^{\frac{(kn\sigma)^2}{2}} \{e^{-kn(x_0+\sigma^2(1-\gamma))} + e^{kn(x_0+\sigma^2(1-\gamma))} + A_2 e^{A_1(x_0+\sigma^2(1-\gamma))}$$

$$\times e^{\frac{A_1^2\sigma^2}{2}} (e^{-kn(x_0+\sigma^2(1-\gamma)+A_1\sigma^2)} + e^{kn(x_0+\sigma^2(1-\gamma)+A_1\sigma^2)})\} < \frac{1}{2l}, \tag{11}$$

then there is a unique solution to satisfy integral equation (9) *exists in space G.*

Proof. Let operator T satisfy

$$(Tp)(x) = \frac{1}{\sqrt{2\pi}\sigma} \frac{A_0 e^{A_1 x}}{1 - A_2 e^{A_1 x}} \int_{-\infty}^{+\infty} \{e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} - A_2 e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} e^{A_1 t}$$

$$+ e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} p^n(t) - A_2 e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} e^{A_1 t} p^n(t)\} dt. \tag{12}$$

For arbitrary $p(x)$ in space G, we derive that

$$|(Tp)(x)| = |\frac{1}{\sqrt{2\pi}\sigma} \frac{A_0 e^{A_1 x}}{1 - A_2 e^{A_1 x}} \int_{-\infty}^{+\infty} \{e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} - A_2 e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} e^{A_1 t}$$

$$+ e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} p^n(t) - A_2 e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} e^{A_1 t} p^n(t)\} dt|$$

$$\le |A_0 l\{1 - A_2 e^{A_1\psi(x)+\frac{A_1^2\sigma^2}{2}}$$

$$+ \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} p^n(t) dt$$

$$- \frac{A_2 e^{A_1\psi(x)+\frac{A_1^2\sigma^2}{2}}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-\frac{1}{2\sigma^2}[t-(\psi(x)+A_1\sigma^2)]^2} p^n(t)dt\}|$$

$$\leq |A_0 l\{1 - A_2 e^{A_1\psi(x)+\frac{A_1^2\sigma^2}{2}}$$

$$+ \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} e^{kn|t|} \frac{p^n(t)}{e^{kn|t|}} dt$$

$$- \frac{A_2 e^{A_1\psi(x)+\frac{A_1^2\sigma^2}{2}}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-\frac{1}{2\sigma^2}[t-(\psi(x)+A_1\sigma^2)]^2} e^{kn|t|} \frac{p^n(t)}{e^{kn|t|}} dt\}|$$

$$\leq |A_0 l\{1 - A_2 e^{A_1\psi(x)+\frac{A_1^2\sigma^2}{2}}$$

$$+ \frac{M^n}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} e^{kn|t|} dt$$

$$- \frac{A_2 M^n e^{A_1\psi(x)+\frac{A_1^2\sigma^2}{2}}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-\frac{1}{2\sigma^2}[t-(\psi(x)+A_1\sigma^2)]^2} e^{kn|t|} dt\}|$$

$$\leq |A_0 l\{1 + A_2 e^{A_1\psi(x)+\frac{A_1^2\sigma^2}{2}} + M^n e^{\frac{(kn\sigma)^2}{2}} (e^{-kn\psi(x)} + e^{kn\psi(x)})$$

$$+ A_2 M^n e^{\frac{(kn\sigma)^2}{2}+\frac{A_1^2\sigma^2}{2}+A_1\psi(x)} e^{-kn(\psi(x)+A_1\sigma^2)}$$

$$+ A_2 M^n e^{\frac{(kn\sigma)^2}{2}+\frac{A_1^2\sigma^2}{2}+A_1\psi(x)} e^{kn(\psi(x)+A_1\sigma^2)}\}|. \tag{13}$$

Applying assumptions (10), (11), and $\psi(x) = x_0 + \phi x + \sigma^2(1 - \gamma)$ to (13), we have

$$|Tp(x)e^{-k|x|}| \leq |A_0 l\{1 + A_2 e^{A_1(x_0+\sigma^2(1-\gamma))+\frac{A_1^2\sigma^2}{2}}$$

$$+ M^n e^{\frac{(kn\sigma)^2}{2}} \left(e^{-kn(x_0+\sigma^2(1-\gamma))} + e^{kn(x_0+\sigma^2(1-\gamma))}\right)$$

$$+ A_2 M^n e^{\frac{(kn\sigma)^2}{2}+\frac{A_1^2\sigma^2}{2}+A_1(x_0+\sigma^2(1-\gamma))} e^{-kn((x_0+\sigma^2(1-\gamma))+A_1\sigma^2)}$$

$$+ A_2 M^n e^{\frac{(kn\sigma)^2}{2}+\frac{A_1^2\sigma^2}{2}+A_1(x_0+\sigma^2(1-\gamma))} e^{kn((x_0+\sigma^2(1-\gamma))+A_1\sigma^2)}\}\Big|$$

$$\leq \frac{M}{2} + \frac{M}{2} \leq M, \tag{14}$$

which results in

$$\| Tp \| \leq M. \tag{15}$$

For arbitrary $p_1(x)$ and $p_2(x)$ in space G, we have

$$|Tp_1(x) - Tp_2(x)|$$

$$= \left|\frac{1}{\sqrt{2\pi}\sigma} \frac{A_0 e^{A_1 x}}{1 - A_2 e^{A_1 x}} \{\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} e^{kn|t|} \frac{p_1(t) - p_2(t)}{e^{k|t|}}\right.$$

$$\times \sum_{j=0}^{n-1} \frac{p_1(t)^{n-1-j} p_2(t)^j}{e^{(n-1)k|t|}} dt - A_2 e^{A_1\psi(x)+\frac{A_1^2\sigma^2}{2}}$$

$$\left.\times \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t-(\psi(x)+A_1\sigma^2))^2} e^{kn|t|} \frac{p_1(t) - p_2(t)}{e^{k|t|}} \sum_{j=0}^{n-1} \frac{p_1(t)^{n-1-j} p_2(t)^j}{e^{(n-1)k|t|}} dt\}\right|$$

$$\leq \|p_1 - p_2\| \frac{|A_0 ln M^{n-1}|}{\sqrt{2\pi}\sigma} |\{\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} e^{kn|t|} dt$$

$$+ A_2 e^{A_1\psi(x)+\frac{A_1^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t-(\psi(x)+A_1\sigma^2))^2} e^{kn|t|} dt\}|$$

$$\leq \|p_1 - p_2\| \left|A_0 ln M^{n-1} e^{\frac{(kn\sigma)^2}{2}} \{e^{kn\psi(x)} + e^{-kn\psi(x)}\right.$$

$$+ A_2 e^{\frac{A_1^2\sigma^2}{2}+A_1\psi(x)}\left(e^{-kn(\psi(x)+A_1\sigma^2)} + e^{kn(\psi(x)+A_1\sigma^2)}\right)\}\Big|, \tag{16}$$

which derives from (11) that

$$\begin{aligned}
&|Tp_1(x) - Tp_2(x)|\, e^{-k|x|} \\
&\le \|p_1 - p_2\| \Big|A_0 lnM^{n-1} e^{\frac{(kn\sigma)^2}{2}}\{e^{kn\psi(x)} + e^{-kn\psi(x)} \\
&\quad + A_2 e^{\frac{A_1^2\sigma^2}{2}+A_1\psi(x)}\left(e^{-kn(\psi(x)+A_1\sigma^2)} + e^{kn(\psi(x)+A_1\sigma^2)}\right)\}\Big| e^{-k|x|} \\
&\le \|p_1 - p_2\| \Big|A_0 lnM^{n-1} e^{\frac{(kn\sigma)^2}{2}}\{e^{-kn(x_0+\sigma^2(1-\gamma))} + e^{kn(x_0+\sigma^2(1-\gamma))} \\
&\quad + A_2 e^{A_1(x_0+\sigma^2(1-\gamma))+\frac{A_1^2\sigma^2}{2}}\left(e^{-kn(x_0+\sigma^2(1-\gamma)+A_1\sigma^2)} + e^{kn(x_0+\sigma^2(1-\gamma)+A_1\sigma^2)}\right)\}\Big| \\
&\le \frac{1}{2}\|p_1 - p_2\|.
\end{aligned} \tag{17}$$

From (17), we have

$$\| Tp_1 - Tp_2 \| \le \frac{1}{2}\|p_1 - p_2\|. \tag{18}$$

Using the fixed point theorem, from (15) and (18), we know that there exists a unique solution in space G to satisfy integral equation (9).

4. Conclusion

In this paper, we have investigated two aspects for the generalized asset pricing model. Firstly, the components of the asset pricing model are incorporated into an integral equation that maps the unknown future price-dividend function into the current price-dividend function. Secondly, under some conditions, the generalized price-dividend function is the solution to a nonlinear integral equation, which exists a unique solution. Comparing the solution with numerical approximation of the price-dividend function also needs to be done in a suitable classical space, we leave this task for future research.

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