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COMMON FIXED POINT FOR GENERALIZED (ψ, ϕ)-WEAK CONTRACTIONS SATISFYING A GENERALIZED CONTRACTIVE CONDITION

MASOUD POURHASAN and VAHID REZA HOSSEINI

Department of Mathematics
Faculty of Mathematics
Islamic Azad University
Shoushtar Branch
Iran
e-mail: masoudpourhasan@yahoo.com

Department of Mathematics
Faculty of Science
Imam Khomeini International University
Qazvin
Iran

Abstract

In this paper, we introduce the class of generalized (ψ, ϕ)-weak contractions mappings condition. We establish that these mappings necessarily have a unique common fixed point in complete metric spaces. This result generalizes an existing result in metric spaces.

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1. Introduction

A mapping $T : X \to X,$ where (X, d) is a metric space, is said to be *contraction*, if there exists $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y). \tag{1.1}$$

If the metric space (X, d) is complete, then the mapping satisfying (1.1) has a unique fixed point (Banach contraction mapping principle). Inequality (1.1) also implies the continuity of T. Generalization of the above contraction mapping has been a very active field of research during recent years. Weakly contractive mappings have been dealt with in a number of papers weakly [1, 4, 6]. Rhoades [4] assumed a weakly contractive mapping $T : X \to X,$ which satisfies the condition

$$d(Tx, Ty) \leq kd(x, y) - \phi(d(x, y)), \tag{1.2}$$

where $x, y \in X$ and $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\phi(t) = 0,$ if and only if $t = 0.$ If one takes $\phi(t) = kt,$ where $0 < k < 1,$ then (1.2) reduces to (1.1). Rhoades obtained the following result. Zhang and Song [6] used generalized (ϕ, ψ)-weak contraction, which is defined for two mappings and gave conditions for existence of a common fixed point.

Let $A \in [0, \infty),\ R_A^+ = [0, \infty) \to R.$ Let $\Im$ satisfy that

(i) $F(0) = 0$ and $F(t) > 0$ for each $t \in (0, A);$

(ii) F is non-decreasing on $R_A^+;$

(iii) F is continuous.

Define $\Im[0, A) = \{F \mid F \text{ satisfied (i)-(iii)}\}.$

Theorem 1.1 ([6]). *Let (X, d) be a complete metric space and $T, S : X \to X$ be two mappings such that for all $x, y \in X$*

$$d(Tx, Sy) \leq \psi(M(x, y)) - \phi(M(x, y)), \tag{1.3}$$

where $\phi : [0, \infty) \to [0, \infty)$ *is a lower semi-continuous function with* $\phi(t) > 0$ *for* $t \in (0, \infty)$ *and* $\phi(0) = 0$

$$M(x, y) = \text{Max}\left\{d(x, y),\ d(Tx, x),\ (Sy, y),\ \frac{1}{2}[d(y, Tx) + d(x, Sy)]\right\}. \quad (1.4)$$

Then, there exists the unique point $u \in X$ *such that* $u = Tu = Su$.

2. Main Result

Theorem 2.1. *Let* (X, d) *be a complete metric space and* $T, S : X \to X$ *be two mappings such that for all* $x, y \in X$

$$\psi(\Im(d(Tx, Sy))) \leq \psi(\Im(d(M(x, y)))) - \phi(\Im(d(M(x, y)))), \quad (2.1)$$

where

(a) $\psi : [0, \infty) \to [0, \infty)$ *is a continuous monotone non-decreasing function with* $\psi(t) = 0$, *if and only if* $t = 0$;

(b) $\phi : [0, \infty) \to [0, \infty)$ *is a lower semi-continuous function with* $\phi(t) = 0$, *if and only if* $t = 0$;

(c) *M is defined by* (1.4);

(d) *where* $\phi : R^+ \to R^+$ *is a Lebesgue integrable mapping, which is sumable, non-negative and such that for each.*

Then, there exists the unique point $u \in X$ *such that* $u = Tu = Su$.

Proof. Let $x_0 \in X$ be an arbitrary point in *X*, then

$$Tx_{2n+1} = x_{2n}; \qquad Sx_{2n} = x_{2n+1},$$

and prove that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Suppose now that n is an odd number. Substituting $x \in x_n$ and $y \in x_{n-1}$ in (2.1), and using properties of functions and $\psi, \phi,$ we obtain

$$\begin{aligned}\psi(\Im(d(x_n, x_{n+1}))) &= \psi(\Im(d(Tx_n, Sx_{n-1}))) \\ &\leq \psi(\Im(M(x_{n-1}, x_n))) - \phi(\Im(M(x_{n-1}, x_n))) \\ &\leq \psi(\Im(M(x_{n-1}, x_n))), \qquad (2.2)\end{aligned}$$

which implies that

$$\Im(d(x_n, x_{n+1})) \leq \Im(M(x_{n-1}, x_n)). \qquad (2.3)$$

Then

$$d(x_n, x_{n+1}) \leq M(x_{n-1}, x_n). \qquad (2.4)$$

Now from triangle inequality for d, we have

$$\begin{aligned}&M(x_n, x_{n-1}) \\ &= \text{Max}\left\{d(x_{n+1}, x_n), d(x_{n+1}, x_n), d(x_{n-1}, x_n), \frac{1}{2}[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]\right\} \\ &= \text{Max}\left\{d(x_{n+1}, x_n), d(x_{n-1}, x_n), \frac{1}{2}[d(x_{n+1}, x_{n-1})]\right\} \\ &\leq \text{Max}\left\{d(x_{n+1}, x_n), d(x_{n-1}, x_n), \frac{1}{2}[d(x_{n+1}, x_n) + d(x_n, x_{n-1})]\right\}, \qquad (2.5)\end{aligned}$$

if $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$, (2.6)

then

$$M(x_{n-1}, x_n) = d(x_n, x_{n+1}) > 0. \qquad (2.7)$$

It furthermore implies that

$$\psi(\Im(d(x_n, x_{n+1}))) \leq \psi(\Im(d(x_n, x_{n+1}))) - \phi(\Im(d(x_n, x_{n+1}))), \qquad (2.8)$$

which is a contradiction.

So, we have

$$d(x_{n+1}, x_n) \leq M(x_n, x_{n-1}) \leq d(x_{n+1}, x_n). \qquad (2.9)$$

Similarly, we can obtain inequalities (2.9) also in the case when n is an even number. Therefore, the sequence $d(x_{n+1}, x_n)$ is monotone non-increasing and bounded.

So,

$$\lim_{n \to \infty} d(x_{n+1}, x_n) = \lim_{n \to \infty} M(x_n, x_{n-1}) = r \geq 0. \tag{2.10}$$

Letting $n \to \infty$ in inequality

$$\psi(\Im(d(x_n, x_n + 1))) \leq \psi(\Im(M(x_n, x_n + 1))) - \phi(\Im(d(x_n, x_n + 1))). \tag{2.11}$$

Then

$$\psi(\Im(r)) \leq \psi(\Im(r)) - \phi(\Im(r)), \tag{2.12}$$

which is a contradiction unless $r = 0$. Hence,

$$\lim_{n \to \infty} d(x_{n+1}, x_n) = 0. \tag{2.13}$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence. Because of (2.13), it is sufficient to show that the subsequence $\{x_{2n}\}$ is a Cauchy sequence. Suppose opposite, that $\{x_n\}$ is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ for which, we can find subsequence $\{x_n\}$ and $\{x_{2m(k)}\}$ of $\{x_{2m(k)}\}$ such that $n(k)$ is smallest index for which

$$n(k) > m(k) > k; \quad d(x_{2m(k)}, x_{2n(k)}) \geq \varepsilon.$$

This means that

$$d(x_{2m(k)}, x_{2n(k)-2}) > \varepsilon. \tag{2.14}$$

We obtained

$$\varepsilon \leq d(x_{2m(k)}, x_{2n(k)}) \tag{2.15}$$

$$\leq d(x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) \tag{2.16}$$

$$< \varepsilon + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}). \tag{2.17}$$

Letting $k \to \infty$ and using (2.13), we can conclude that

$$\lim_{k \to \infty} d(x_{2m(k)}, x_{2n(k)}) = \varepsilon, \tag{2.18}$$

moreover, from

$$\left|d(x_{2m(k)}, x_{2n(k)+1}) - d(x_{2m(k)}, x_{2n(k)})\right| \leq d(x_{2n(k)}, x_{2n(k)+1}),$$

$$\left|d(x_{2m(k)-1}, x_{2n(k)}) - d(x_{2m(k)}, x_{2n(k)})\right| \leq d(x_{2m(k)}, x_{2m(k)+1}). \quad (2.19)$$

Using (2.9) and (2.13), we get

$$\lim_{n\to\infty} d(x_{2m(k)}, x_{2n(k)+1}) = \lim_{n\to\infty} d(x_{2m(k)-1}, x_{2n(k)}) = \varepsilon, \quad (2.20)$$

and from

$$\left|d(x_{2m(k)-1}, x_{2n(k)+1}) - d(x_{2m(k)-1}, x_{2n(k)})\right| \leq d(x_{2n(k)}, x_{2n(k)+1}). \quad (2.21)$$

Also, from the definition of M and from (2.9), (2.20), and (2.21), we have

$$M(x_{2m(k)-1}, x_{2n(k)}) = \varepsilon,$$

therefore, from the definition of M and from (2.9) and (2.21), we have

$$\lim_{n\to\infty} M(x_{2m(k)-1}, x_{2n(k)}) = \varepsilon,$$

$$\begin{aligned}\psi(\Im(d(x_{2m(k)-1}, x_{2n(k)}))) &= \psi(\Im(d(Tx_{2m(k)-1}, Sx_{2n(k)}))) \\ &\leq \psi(\Im(M(x_{2m(k)-1}, x_{2n(k)}))) \\ &\quad -\phi(\Im(M(x_{2m(k)-1}, x_{2n(k)}))).\end{aligned}$$

Letting $k \to \infty$ and using (2.1) and (1.3), we get

$$\{\psi(\Im(\varepsilon)) \leq \psi(\Im(\varepsilon)) - \phi(\Im(\varepsilon))\}, \quad (2.22)$$

which is a contradiction with $\varepsilon > 0$. Thus, x_{2n} is a Cauchy sequence and hence x_{2n} is a Cauchy sequence. In complete metric space X, there exists u such that $x_n \to \infty$ as $n \to \infty$. Now prove that u is fixed point for T and S. Let $Tu \neq u$, therefore $d(Tu, u) > 0$, then $\exists N_1 \in N$ such that $n > N_1$:

$$d(x_{2n+1}, u) < \frac{1}{2}; \quad d(x_{2n}, u) < \frac{1}{2} d(u, Tu); \quad d(x_{2n+1}, x_{2n}) < \frac{1}{2} d(u, Tu),$$

$$d(u, Tu) \leq M(u, x_{2n})$$

$$\leq \text{Max}\left\{d(u, x_{2n}), d(u, Tu), d(x_{2n}, x_{2n+1}), \frac{1}{2}[d(u, x_{2n+1}) + d(x_{2n}, Tu)]\right\}$$

$$\leq \text{Max}\left\{\frac{1}{2}d(u, Tu), d(u, Tu), d(u, Tu), \frac{1}{2}[d(u, Tu) + d(u, Tu)]\right\} = d(u, Tu).$$

Then $M(u, x_{2n}) = d(u, Tu)$,

$$\psi(\Im(d(Tu, x_{2n}))) \leq \psi(\Im(M(Tu, x_{2n}))) - \phi(\Im(M(Tu, x_{2n}))). \tag{2.23}$$

Letting $n \to \infty$, we obtain

$$\psi(\Im(d(Tu, u))) \leq \psi(\Im(M(Tu, u))) - \phi(\Im(M(Tu, u))), \tag{2.24}$$

which is a contradiction. Since $\psi(\Im(d(Tu, u))) = 0$, then $\Im(d(Tu, u)) = 0$. Therefore $d(Tu, u) = 0$, using that u is fixed point for T, we have

$$\begin{aligned}\psi(\Im(d(Tu, u))) &= \psi(\Im(M(Tu, Su))) \\ &\leq \psi(\Im(M(u, u))) - \phi(\Im(M(u, u))) \\ &= \psi(\Im(M(u, Su))) - \phi(\Im(M(u, Su))).\end{aligned}$$

Using an argument similar to the above, we conclude that $d(u, Su) = 0$ or $u = Su$. If there exists another fixed point $v \in X$, then from

$$\begin{aligned}\psi(\Im(d(\nu, u))) &= \psi(\Im(M(Tu, Su))) \\ &\leq \psi(\Im(M(u, \nu))) - \phi(\Im(M(u, \nu))) \\ &= \psi(\Im(d(u, \nu))) - \phi(\Im(d(u, \nu))),\end{aligned} \tag{2.25}$$

then $u = v$. □

Corollary 2.1. Let (X, d) *be a complete metric space and* $T, S : X \to X$ *be self-mappings satisfying the inequality and* $\phi(t) = 1$, *then obtain* [1].

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