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MIKHLIN-HÖRMANDER THEOREM FOR HIGHER ORDER DIFFERENTIAL OPERATORS ON RIEMANNIAN MANIFOLDS

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Abstract

Let M be a Riemannian manifold, which satisfies the doubling volume property. Let L be a $2m$-order differential operator on M and $m(\lambda)$, $\lambda \in \mathbb{R}$, a multiplier satisfying the Mikhlin-Hörmander condition. We also assume that the associated heat kernel satisfies a certain upper Gaussian estimate, and we prove that the spectral multiplier $m(L)$ is bounded on L^p, $p \in (1, \infty)$ and from L^1 to L^1 week.

1. Introduction and Statement of the Result

Let $m(\xi)$ be a bounded measurable function in $\mathbb{R}^n$ and T_m be the operator defined by $\widehat{T_m f}(\xi) = m(\xi)\hat{f}(\xi)$, where $\hat{f}$ denotes the Fourier transform of f.

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The classical Mikhlin-Hörmander multiplier theorem [17, 20], asserts that if the multiplier $m(\lambda)$ satisfies the condition:

$$\sup_{\lambda\in\mathbb{R}^n} |\lambda|^{\alpha}|\partial^{\alpha} m(\lambda)| < \infty,$$

for any multi-index α, with $|\alpha| \le \left[\frac{n}{2}\right] + 1$, then T_m is bounded on L^p, $p \in (1, \infty)$ and from L^1 to L^1 week. In the present paper, we generalize this theorem for multipliers of higher order differential operators on Riemannian manifolds.

1.1. The manifolds of our interest. Let M be a complete and non-compact Riemannian manifold. We denote by $d(.,.)$ the Riemannian distance, by dx the Riemannian measure, by $B(x, r)$ the ball centered at $x \in M$ with radius $r > 0$, and by $|B(x, r)|$ its volume. We assume that M satisfies the doubling volume property, i.e., there is a constant $c > 0$, such that

$$|B(x, 2r)| \le c|B(x, r)|, \text{ for all } x \in M, r > 0. \tag{1.1}$$

From (1.1), it follows that there exist constants $c, D > 0$, such that

$$\frac{|B(x, r)|}{|B(x, t)|} \le c\left(\frac{r}{t}\right)^{D}, \text{ for all } x \in M, r \ge t > 0. \tag{1.2}$$

Note that $\mathbb{R}^D$, Lie groups of polynomial volume growth and manifolds with $RicM \ge 0$ are examples of such a manifold.

1.2. The operators. Let us denote by L a formally symmetric and non-negative higher order differential operator of order $2m$, $m \ge 1$, i.e.,

$$L = \sum_{|\alpha|\le 2m} a_{\alpha}(x)\partial_x^{\alpha}.$$

Let $p_t(x, y)$, $t > 0$, $x, y \in M$, be the associated heat kernel, i.e., the kernel of the heat semigroup e^{-tL}. We assume that there is a constant $c > 0$ such that

$$|p_t(x,\ y)| \le c \exp\left(-c\,\frac{d(x,\ y)^{2m/(2m-1)}}{t^{1/(2m-1)}}\right)|B(x,\ t^{1/2m})|^{-1},\ x,\ y \in M,\ t > 0. \tag{1.3}$$

One interesting example of such operator is $L = (-\Delta)^m$, i.e., the m-th power of the Laplacian on $\mathbb{R}^D$, for $D < 2m$. A large class of homogeneous operators satisfying our assumptions is in [3]. There is a lot of interest for higher order differential operators as one can see also in [7, 8, 10, 11, 12, 23] and in their references.

1.3. Statement of the result. We denote by dE_λ the spectral measure of L. By the spectral theorem,

$$L = \int_0^\infty \lambda dE_\lambda.$$

If $m : \mathbb{R} \to \mathbb{R}$ is a bounded Borel function, by the spectral theorem, we can define the spectral multiplier

$$m(L) = \int_0^\infty m(\lambda) dE_\lambda,$$

which is a bounded operator on $L^2(M)$, with $\|m(L)\|_{2\to 2} \le \|m\|_\infty$.

Let us denote by $\mathcal{C}^A(\mathbb{R})$ the Lipschitz space of order $A > 0$, and let us fix a function $0 \le \phi \in \mathcal{C}^\infty(\mathbb{R})$, with $\phi(t) = 1$, for $t \in [1, 2]$, $\phi(t) = 0$, $t \in (2^{-1}, 4)^c$. In the present work, we prove the following:

Theorem 1. *Let M be a Riemannian manifold as above and $m(\lambda)$, $\lambda \in \mathbb{R}$, be a multiplier satisfying for some $A > D/2$, the condition*

$$\sup_{t>0} \|\phi(.)m(t.)\|_{\mathcal{C}^A} < \infty. \tag{1.4}$$

Then, the operator $m(L)$ is bounded on L^p, $p \in (1, \infty)$ and from L^1 to L^1 week.

For generalizations of the Mikhlin-Hörmander multiplier theorem to abstract contexts, we refer the reader, for example, to [1, 2, 4, 9, 14, 16, 18, 19, 21].

In our case, the wave operator $cost\sqrt{L}$ does not have the finite propagation speed property (cf. [26]). So, our proof is based on very good kernel estimates and a Calderon-Zygmund decomposition. For our techniques, see also [6, 14].

Throughout this article, the different constants will always be denoted by the same letter c. When their dependence or independence is significant, it will be clearly stated.

2. Multipliers with Compact Support

2.1. An approximation lemma

Lemma 1 (cf. [22]). *For $f \in C_0^n(\mathbb{R})$ and $A = n + \alpha$ with $\alpha \in (0, 1]$, $n \in \mathbb{N}$, we set*

$$\|f\|_{\mathcal{C}^A} = \sum_{m=0}^{n} \|f^{(m)}\|_\infty + \sup\left\{\frac{|f^{(n)}(x+t) - f^{(n)}(x)|}{t^\alpha};\ t > 0,\ x \in \mathbb{R}\right\}.$$

Then, for all $\lambda > 0$, there is a continuous and integrable function ψ_λ and a constant $c > 0$, independent of λ and f, such that

$$\|\hat{\psi}_\lambda\|_\infty \le c,\ \mathrm{supp}(\hat{\psi}_\lambda) \subset [-\lambda, \lambda],\ \text{and}\ \|f - f * \psi_\lambda\|_\infty \le c\|f\|_{\mathcal{C}^A} \lambda^{-A}.$$

2.2. Auxiliary estimates. We set $A_q(y) = B(y, 2^{(q+1)/2m}) \setminus B(y, 2^{q/2m})$, $q \in \mathbb{Z}$, $y \in M$.

Lemma 2. *There are constants $c > 0$ and $\delta \in (0, 1)$, such that for all $q, j \in \mathbb{Z}$, with $q \ge j$, all $x \in A_q(y)$ and $|t| \le \delta 2^{(q-j)/2(2m-1)}$*

$$\left|e^{ite^{-2^j L}} p_{2^j}(x, y)\right| \le \frac{ce^{-c2^{(q-j)/2(2m-1)}}}{|B(y, 2^{j/2m})|}. \tag{2.1}$$

Proof. We have $e^{ite^{-2^j L}} = \sum_{n\geq 0} \frac{(it)^n}{n!} (e^{-2^j L})^n = \sum_{n\geq 0} \frac{(it)^n}{n!} e^{-n2^j L}$.

So

$$\left| e^{ite^{-2^j L}} p_{2^j}(x, y) \right| \leq \sum_{n\geq 0} \frac{|t|^n}{n!} |e^{-n2^j L} p_{2^j}(x, y)|$$

$$= \sum_{n\leq 2^{(q-j)/2}} \frac{|t|^n}{n!} |p_{(n+1)2^j}(x, y)|$$

$$+ \sum_{n>2^{(q-j)/2}} \frac{|t|^n}{n!} |p_{(n+1)2^j}(x, y)|$$

$$:= S_1 + S_2. \tag{2.2}$$

Estimation of S_1. Using (1.3), we have for every $x \in A_q(y)$, $n \leq 2^{(q-j)/2}$.

$$S_1 \leq \sum_{n\leq 2^{(q-j)/2}} \frac{|t|^n}{n!} c \exp\left(\frac{-cd(x, y)^{2m/(2m-1)}}{((n+1)2^j)^{1/(2m-1)}} \right) |B(y, ((n+1)2^j)^{1/2m})|^{-1}$$

$$\leq c \exp\left(-c \frac{2^{q/(2m-1)}}{(2 \cdot 2^{(q-j)/2} 2^j)^{1/(2m-1)}} \right) |B(y, 2^{j/2m})|^{-1} \sum_{n=0}^{\infty} \frac{|t|^n}{n!}$$

$$\leq c \exp\left(-c 2^{(q-j)/(2m-1)} 2^{-(q-j)/2(2m-1)} \right) |B(y, 2^{j/2m})|^{-1} e^{|t|}$$

$$\leq c |B(y, 2^{j/2m})|^{-1} e^{-c2^{(q-j)/2(2m-1)}} e^{\delta 2^{(q-j)/2(2m-1)}}$$

$$\leq c |B(y, 2^{j/2m})|^{-1} e^{-c2^{(q-j)/2(2m-1)}}. \tag{2.3}$$

Estimation of S_2. Using (1.3) and that $\frac{1}{n!} \leq c\left(\frac{e}{n}\right)^n$, we take

$$S_2 \leq c |B(y, 2^{j/2m})|^{-1} \sum_{n>2^{(q-j)/2}} \frac{|t|^n}{n!}$$

$$\leq c|B(y,\, 2^{j/2m})|^{-1} \sum_{n>2^{(q-j)/2}} \left(\frac{e\delta 2^{(q-j)/2(2m-1)}}{2^{(q-j)/2}} \right)^n$$

$$\leq c|B(y,\, 2^{j/2m})|^{-1} \sum_{n>2^{(q-j)/2}} (e\delta)^n$$

$$\leq c|B(y,\, 2^{j/2m})|^{-1} \sum_{n>2^{(q-j)/2}} e^{-cn}$$

$$\leq c|B(y,\, 2^{j/2m})|^{-1} e^{-c2^{(q-j)/2(2m-1)}}, \tag{2.4}$$

choosing δ small enough. Then (2.1) follows from (2.2), (2.3), and (2.4).

□

Lemma 3. *Let $A > 0$ and $f \in \mathcal{C}_0^A(0, \infty)$. Then, there is a constant $c > 0$ such that for all $j \in \mathbb{Z}$ and $y \in M$, we have*

(i) $\|f(e^{-2^j L})p_{2^j}(., y)\|_2 \leq c\|f\|_{\mathcal{C}^A}|B(y, 2^{j/2m})|^{-1/2}$.

(ii) $\|f(e^{-2^j L})p_{2^j}(., y)\|_{L^1(B(y,2^{j/2m}))} \leq c\|f\|_{\mathcal{C}^A}$. (2.5)

Proof. (i) Integrating (1.3), we get that

$$\|p_{2^j}(., y)\|_2 \leq c|B(y, 2^{j/2m})|^{-1/2}.$$

By the spectral theorem, it is

$$\|f(e^{-2^j L})p_{2^j}(., y)\|_2 \leq \|f\|_\infty \|p_{2^j}\|_2 \leq c\|f\|_{\mathcal{C}^A}|B(y, 2^{j/2m})|^{-1/2}.$$

(ii) By Cauchy-Schwartz and claim (i), we get (2.5). □

Lemma 4. *Let $A > 0$ and $f \in \mathcal{C}_0^A(0, \infty)$. Then, there is a constant $c > 0$ such that for all $q, j \in \mathbb{Z}$, $q \geq j$, and $y \in M$, we have*

(i) $\|f(e^{-2^j L})p_{2^j}(., y)\|_{L^2(A_q(y))} \leq c\|f\|_{\mathcal{C}^A} 2^{-A(q-j)/2(2m-1)}|B(y, 2^{j/2m})|^{-1/2}$.

(2.6)

(ii) $\|f(e^{-2^j L})p_{2^j}(.,y)\|_{L^1(A_q(y))} \le c\|f\|_{\mathcal{C}^A} 2^{-\left(A-\frac{D}{2}\right)(q-j)/2(2m-1)}$. (2.7)

Proof. (i) We consider the function $\psi_{j,q}$, given by Lemma 1 satisfying

$$\|\hat{\psi}_{j,q}\|_\infty < c \quad \text{and} \quad \|f - f * \psi_{j,q}\|_\infty \le c\|f\|_{\mathcal{C}^A} 2^{-A(q-j)/2(2m-1)}, \tag{2.8}$$

and $\operatorname{supp}(\hat{\psi}_{j,q}) \subseteq [-\delta 2^{(q-j)/2(2m-1)}, +\delta 2^{(q-j)/2(2m-1)}]$. From (2.8), we take

$$\begin{aligned}
&\|(f - f * \psi_{j,q})(e^{-2^j L})p_{2^j}(.,y)\|_{L^2(A_q(y))} \\
&\qquad \le \|f - f * \psi_{j,q}\|_\infty \|p_{2^j}(.,y)\|_2 \\
&\qquad \le c\|f\|_{\mathcal{C}^A} 2^{-A(q-j)/2(2m-1)} |B(y, 2^{j/2m})|^{-1/2}.
\end{aligned} \tag{2.9}$$

For every $x \in A_q(y)$ by (2.1) and (2.8), we get

$$\begin{aligned}
&(f * \psi_{j,q})(e^{-2^j L})_{p_{2^j}}(x, y)| \\
&\qquad \le c\int_{|t|\le \delta 2^{(q-j)/2(2m-1)}} |\hat{f}(t)||\widehat{\psi_{j,q}}(t)| \left| e^{ite^{-2^j L}} p_{2^j}(x, y) \right| dt \\
&\qquad \le c\|\hat{f}\|_\infty 2^{(q-j)/2(2m-1)} e^{-c2^{(q-j)/2(2m-1)}} |B(y, 2^{j/2m})|^{-1} \\
&\qquad \le c\|f\|_{\mathcal{C}^A} e^{-c2^{(q-j)/2(2m-1)}} |B(y, 2^{j/2m})|^{-1}.
\end{aligned}$$

Then by (1.2),

$$\begin{aligned}
&\| (f * \psi_{j,q})(e^{-2^j L}) \, p_{2^j}(.,y)\|_{L^2(A_q(y))} \\
&\qquad \le |A_q(y)|^{\frac{1}{2}} \|(f * \psi_{j,q})(e^{-2^j L})_{p_{2^j}}(.,y)\|_\infty \\
&\qquad \le c\|f\|_{\mathcal{C}^A} e^{-c2^{(q-j)/2(2m-1)}} |B(y, 2^{j/2m})|^{-1} |B(y, 2^{(q+1)/2m})|^{1/2} \\
&\qquad \le c\|f\|_{\mathcal{C}^A} e^{-c2^{(q-j)/2(2m-1)}} 2^{D(q-j)/2m} |B(y, 2^{j/2m})|^{-1/2}
\end{aligned}$$

$$\leq c\|f\|_{\mathcal{C}^A} 2^{-A(q-j)/2(2m-1)} |B(y, 2^{j/2m})|^{-1/2}. \tag{2.10}$$

The relation (2.6) follows by (2.9) and (2.10).

(ii) From (2.6), Cauchy- Schwartz, and (1.2), we have

$$\begin{aligned}
&\|f(e^{-2^j L})p_{2^j}(., y)\|_{L^1(A_q(y))} \\
&\leq \|f(e^{-2^j L})p_{2^j}(., y)\|_{L^2(A_q(y))} |A_q(y)|^{1/2} \\
&\leq c\|f\|_{\mathcal{C}^A} 2^{-A(q-j)/2(2m-1)} |B(y, 2^{j/2m})|^{-1/2} |B(y, 2^{(q+1)/2m})|^{1/2} \\
&\leq c\|f\|_{\mathcal{C}^A} 2^{-A(q-j)/2(2m-1)} 2^{\frac{D}{2}(q-j)/2m} \\
&\leq c\|f\|_{\mathcal{C}^A} 2^{-\left(A-\frac{D}{2}\right)(q-j)/2(2m-1)},
\end{aligned}$$

and the proof is complete. □

2.3. L^p **- boundedness.** We are now ready to prove the following:

Proposition 1. *Let* $A > \dfrac{D}{2}$ *and* $m \in \mathcal{C}_0^A(0, \infty)$. *Then* $m(L)$ *is bounded on* L^p, *for all* $p \in [1, \infty]$.

Proof. Let $f(s) = m(-\log s)s^{-1}$. Then, we take $m(s) = f(e^{-s})e^{-s}$. Because $m \in \mathcal{C}_0^A(0, \infty)$, then so does f. In terms of L, it is

$$m(L) = f(e^{-L})e^{-L},\ K(x, y) = m(L)\delta_x(y) = f(e^{-L})p_1(x, y),$$

where $\delta_x(y)$ is the Dirac mass at x and $K(x, y)$ is the kernel of $m(L)$. It suffices to show that the Kernel $K(x, y)$ is integrable. Then, we have the L^∞ - boundedness and by interpolation and symmetry the L^p, boundedness for all $p \in [1, \infty]$.

$$\| K(.,y)\|_1 = \|f(e^{-L})p_1(.,y)\|_{L^1(B(y,1))} + \sum_{q=0}^{\infty}\|f(e^{-L})p_1(.,y)\|_{L^1(A_q(y))}.$$

Using (2.5) and (2.7), we take because $A > D/2$,

$$\| K(.,y)\|_1 \le c\|f\|_{\mathcal{C}^A} + c\|f\|_{\mathcal{C}^A}\sum_{q=0}^{\infty} 2^{-\left(A-\frac{D}{2}\right)q/2(2m-1)} < c,$$

and the proof is complete. □

3. Preparation for the Proof

3.1. Calderon-Zygmund decomposition. Following ([5], p. 73-75), we consider bounded supported $0 \le f \in L^1 \cap L^2$ and $a > 0$. There are constants $c = c(M) > 0$, $k = k(M) \in \mathbb{N}^*$, and a sequence of balls $B_i = B(x_i, r_i)$ such that

(1) $f(x) \le ca$, a.e. in $M - \bigcup_i B_i$;

(2) $|B_i|^{-1}\int_{B_i} f(x)dx \le ca$;

(3) $\sum_i |B_i| \le (c/a)\|f\|_1$;

(4) each point belongs to at most k balls B_i.

Now following [13], we consider the functions

$$\eta_i(x) = \frac{\mathbf{1}_{B_i}(x)}{\sum_j \mathbf{1}_{B_j}(x)},\ x \in B_i,\ \eta_i(x) = 0, \text{ and } x \in M - B_i.$$

Also, we set $w_i = \eta_i f$ and

$$b_i = e^{-r_i^{2m}L}w_i,\ \theta_i = w_i - b_i, \text{ and } g = \mathbf{1}_{M-\bigcup_i B_i}.$$

We can write the function f in the form

$$f = g + \sum_i b_i + \sum_i \theta_i.$$

We continue with a lemma, whose analogue in second order one can see in [6].

Lemma 5. *Let* $b \in L^1$ *supported in* $B(z, r)$, $z \in M$, $r \le t^{1/2m}$. *Then, there is a constant* $c > 0$ *independent of* b *such that for all* $u \in L^2$,

$$\langle u, e^{-tL} b \rangle \le \frac{c \| b \|_1}{|B(z, r)|} \langle Mu, \mathbf{1}_{B(z,r)} \rangle, \tag{3.1}$$

where $Mu(y) = \sup_{r>0} |B(y, r)|^{-1} \int_{B(y,r)} |u(x)|\, dx$, *is the Hardy-Littlewood maximal function.*

Proof. Using (1.3), we take

$$e^{-tL} b(y) = \int_M p_t(x, y) b(y) dy$$

$$\le c \int_{B(z,r)} \exp\left(- c \frac{d(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}} \right) |B(y, t^{1/2m})|^{-1} |b(y)| dy.$$

Because $d(y, z) < r \le t^{1/2m}$, we have that

$$\exp\left(- c \frac{d(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}} \right) \le c \exp\left(- c' \frac{d(x, z)^{2m/(2m-1)}}{t^{1/(2m-1)}} \right),$$

and by (1.1), we have also $|B(y, t^{1/2m})|^{-1} \le c |B(z, t^{1/2m})|^{-1}$. So

$$e^{-tL} b(y) \le c \exp\left(- c \frac{d(x, z)^{2m/(2m-1)}}{t^{1/(2m-1)}} \right) |B(z, t^{1/2m})|^{-1} \int_{B(z,r)} |b(y)| dy$$

$$= c \exp\left(\frac{- c' d(x, z)^{2m/(2m-1)}}{t^{1/(2m-1)}} \right) |B(z, t^{1/2m})|^{-1} \|b\|_1 |B(z, r)|^{-1} \int_M \mathbf{1}_{B(z,r)}(y) dy$$

$$\leq \frac{c\|b\|_1}{|B(z,\ r)|}\int_{B(z,r)}\exp\left(-c''\frac{d(x,\ y)^{2m/(2m-1)}}{t^{1/(2m-1)}}\right)|B(y,\ t^{1/2m})|^{-1}\mathbf{1}_{B(z,r)}(y)dy.$$

We set $D_q(y,t)=B(y,2^{(q+1)}t^{1/2m})-B(y,2^qt^{1/2m})$, then

$$M=B\ (y,\ t^{1/2m})\cup\bigcup_{q\geq 0}D_q(y,\ t).$$

We take that

$$\int_{B(y,\ t^{1/2m})}\exp\left(-c\frac{d(x,\ y)^{2m/(2m-1)}}{t^{1/(2m-1)}}\right)|u(x)||B(y,\ t^{1/2m})|^{-1}dx\ \leq\ Mu(y).$$

$$\int_{D_q(y,t)}\frac{\exp\left(-c\dfrac{d(x,\ y)^{2m/(2m-1)}}{t^{1/(2m-1)}}\right)|u(x)|}{|B(y,\ t^{1/2m})|}dx$$

$$\leq \frac{ce^{-c2^{q2m/(2m-1)}}}{|B(y,\ t^{1/2m})|}\int_{D_q(y,t)}|u(x)|dx$$

$$\leq \frac{ce^{-c2^{q2m/(2m-1)}}|B(y,\ 2^{(q+1)}t^{1/2m})|}{|B(y,\ t^{1/2m})|}Mu(y)$$

$$\leq ce^{-c2^{q2m/(2m-1)}}2^{q^D}Mu(y)$$

$$\leq ce^{-c2^{q2m/(2m-1)}}Mu(y).$$

So, we take

$$\int_M\exp\left(-c\frac{d(x,\ y)^{2m/(2m-1)}}{t^{1/(2m-1)}}\right)|u(x)||B(y,\ t^{1/2m})|^{-1}dx$$

$$\leq\ Mu(y)+c\sum_{q\geq 0}e^{-c2^{q2m/(2m-1)}}Mu(y)\leq cMu(y),$$

and finally,

$$\langle u,\ e^{-tL}b\rangle\leq\frac{c\|b\|_1}{|B(z,\ r)|}\int_{B(z,r)}Mu(y)\mathbf{1}_{B(z,r)}(y)dy$$

$$\leq \frac{c\|b\|_1}{|B(z, r)|}\langle Mu(y), \mathbf{1}_{B(z,r)}\rangle.$$

□

Corollary 1. *There is a constant* $c > 0$ *independent of f such that*

$$\left\|\sum_i b_i\right\|_2^2 \leq ca\|f\|_1. \tag{3.2}$$

Proof. Let $u \in L^2$. We recall that $b_i = e^{-r_i^{2m}L}w_i$ and $\text{supp}(w_i) \subseteq B(x_i, r_i)$. Then by (3.1) and part (2) of Calderon-Zygmund decomposition, we have

$$\begin{aligned}\langle u, \sum_i b_i\rangle &\leq \sum_i \langle u, e^{-r_i^{2m}L}w_i\rangle \\ &\leq c\sum_i \frac{\|w_i\|_1}{|B(x_i, r_i)|}\langle Mu, \mathbf{1}_{B(x_i,r_i)}\rangle \\ &\leq ca\langle Mu, \sum_i \mathbf{1}_{B(x_i,r_i)}\rangle \\ &\leq ca\|u\|_2\|\sum_i \mathbf{1}_{B(x_i,r_i)}\|_2. \end{aligned} \tag{3.3}$$

Now by parts (3) and (4) of Calderon-Zygmund decomposition,

$$\left\|\sum_i \mathbf{1}_{B(x_i,r_i)}\right\|_2^2 \leq c\sum_i |B_i| \leq \frac{c}{a}\|f\|_1. \tag{3.4}$$

The relation (3.2) follows from (3.3) and (3.4) by setting $u = \sum_i b_i$.

3.2. Kernel estimates. Following [4], since by (1.3) $\lim_{t\to\infty}\|e^{-tL}f\|_2 = 0$, we conclude that $\lim_{t\to\infty}\|\int_0^{1/t} dE_\lambda(f)\|_2 = 0$. So, the point $\lambda = 0$ may be neglected in the spectral resolution of L and we can consider $m(\lambda)$ defined in $(0, \infty)$.

Following [17, 14], we consider a $\mathcal{C}^\infty$ function ϕ, satisfying

$$\operatorname{supp}(\phi) \subset (2^{-1}, 2) \text{ and } \sum_{j\in\mathbb{Z}} \phi(2^j t) = 1,\ t > 0.$$

We divide the multiplier m into pieces m_j, $j \in \mathbb{Z}$, by setting

$$m_j(s) = m(s)\phi(2^j s), \quad j \in \mathbb{Z}. \tag{3.5}$$

We have

$$\operatorname{supp}(m_j) \subset (2^{-j-1}, 2^{-j+1}) \text{ and } m(s) = \sum_{j\in\mathbb{Z}} m_j(s). \tag{3.6}$$

We denote by $K(x, y)$ and $K_j(x, y)$ the kernels of the m and m_j, respectively. Then $K(x, y) = \sum_{j\in\mathbb{Z}} K_j(x, y)$. We set $h_j(s) = m_j(-2^{-j}\log s)s^{-1}$, $j \in \mathbb{Z}$. Then, the functions h_j are compactly supported and $\|h_j\|_{\mathcal{C}^A} \le c$. Note that

$$m_j(L) = h_j(e^{-2^j L})e^{-2^j L} \text{ and } K_j(x, y) = h_j(e^{-2^j L})p_{2^j}(x, y),\ j \in \mathbb{Z}. \tag{3.7}$$

Lemma 6. *There is a $c > 0$ such that for every $q, j \in \mathbb{Z}$, $q \ge j$, $y \in M$,*

$$\|K_j(., y)\|_{L^1(A_q(y))} \le c2^{-\left(A-\frac{D}{2}\right)(q-j)/2(2m-1)}. \tag{3.8}$$

Proof. By (3.7) and (2.7) follows (3.8). □

We consider the functions $g_{j,\tau}(s) = h_j(s)s^{2^{\tau-j}}$, $\tau \ge j$. Let $S_{j,\tau}(x, y)$ be the kernel of $m_j(L)e^{-2^\tau L}$. Then, we obtain that

$$\|g_{j,\tau}\|_{\mathcal{C}^A} \le ce^{-c2^{\tau-j}} \text{ and } S_{j,\tau}(x, y) = g_{j,\tau}(e^{-2^j L})p_{2^j}(x, y). \tag{3.9}$$

Lemma 7. *There is a $c > 0$ such that for every $\tau, j \in \mathbb{Z}$, $\tau \ge j$, $y \in M$,*

$$\|S_{j,\tau}(.,y)\|_1 \le ce^{-c2^{\tau-j}}. \tag{3.10}$$

Proof. By (2.5) and (3.9), it is $\|S_{j,\tau}(.,y)\|_{L^1(B(y,2^{j/2m}))} \le ce^{-c2^{\tau-j}}$. Also by (2.7), we take because $A > \frac{D}{2}$,

$$\|S_{j,\tau}(.,y)\|_{L^1(B(y,2^{j/2m})^c)} \le ce^{-c2^{\tau-j}} \sum_{q\ge j} 2^{-\left(A-\frac{D}{2}\right)(q-j)/2(2m-1)} \le ce^{-c2^{\tau-j}}.$$

□

Let now $\Lambda_{j,\tau}(x,\ y)$ be the kernels of $(I - e^{-2^\tau L})m_j(L)$.

Lemma 8. *There is a* $c > 0$ *such that for every* $\tau,\ j \in \mathbb{Z},\ \tau \ge j,$ $y \in M$,

$$\|\Lambda_{j,\tau}(.,y)\|_{L^1(B(y,2^{\tau/2m})^c)} \le c2^{-\left(A-\frac{D}{2}\right)(\tau-j)/2(2m-1)}. \tag{3.11}$$

Proof. We have that

$$\Lambda_{j,\tau}(x,\ y) = (I - e^{-2^\tau L})m_j(L)\delta_x(y) = K_j(x,\ y) - S_{j,\tau}(x,\ y). \tag{3.12}$$

By (3.8), we have because $A > \frac{D}{2}$,

$$\begin{aligned}\|K_j(.,y)\|_{L^1(B(y,2^{\tau/2m})^c} &= \sum_{q\ge\tau}\|K_j(.,y)\|_{L^1(A_q(y))}\\ &\le c\sum_{q\ge\tau} 2^{-\left(A-\frac{D}{2}\right)(q-j)/2(2m-1)}\\ &\le c2^{-\left(A-\frac{D}{2}\right)(\tau-j)/2(2m-1)}.\end{aligned} \tag{3.13}$$

Now (3.11) follows from (3.13), (3.12), and (3.10). □

We consider the functions $f_{j,\tau}(s) = (1 - s^{2^{\tau-j}})h_j(s)$, $\tau \le j$. We observe that

$$\|f_{j,\tau}\|_{\mathcal{C}^A} \le c2^{\tau-j},\ \Lambda_{j,\tau}(x,\ y) = f_{j,\tau}(e^{-2^j L})p_{2^j}(x,\ y),\ \tau \le j. \tag{3.14}$$

Lemma 9. *There is a* $c > 0$ *such that for every* $\tau,\ j \in \mathbb{Z},\ \tau \le j,\ y \in M,$

$$\|\Lambda_{j,\tau}(.,\ y)\|_1 \le c2^{\tau-j}. \tag{3.15}$$

Proof. By (2.5), (2.7), and (3.14), we have because $A > \frac{D}{2}$,

$$\|\Lambda_{j,\tau}(.,\ y)\|_1 \le \|\Lambda_{j,\tau}(.,\ y)\|_{B(y,\,2^{j/2m})} + \sum_{q\ge j}\|\Lambda_{j,\tau}(.,\ y)\|_{L^1(A_q(y))}$$

$$\le c2^{\tau-j} + c2^{\tau-j}\sum_{q\ge j} 2^{-\left(A-\frac{D}{2}\right)(q-j)/2(2m-1)} \le c2^{\tau-j}.$$

□

Let $\Lambda_\tau(x,\ y)$ be the kernel of $(I - e^{-2^\tau L})m(L)$.

Lemma 10. *There is a* $c > 0$ *such that for every* $\tau \in \mathbb{Z},\ y \in M,$

$$\|\Lambda_\tau(.,\ y)\|_{L^1(B(y,\,2^{\tau/2m})^c)} \le c. \tag{3.16}$$

Proof. It is $\Lambda_\tau(x,\ y) = \sum_{j\le\tau}\Lambda_{j,\tau}(x,\ y) + \sum_{j>\tau}\Lambda_{j,\tau}(x,\ y)$. By (3.11) and (3.15), we take by using that $A > \frac{D}{2}$,

$$\|\Lambda_\tau(.,y)\|_{L^1(B(y,\,2^{\tau/2m})^c)} \le \sum_{j\le\tau} c2^{-\left(A-\frac{D}{2}\right)(\tau-j)/2(2m-1)} + \sum_{j>\tau} c2^{\tau-j} \le c.$$

□

4. Proof of Theorem 1

We will prove that there is a constant $c > 0$, such that for every $f \in L^1$,

$$|\{|m(L)f| > a\}| \le \frac{c\|f\|_1}{a}. \tag{4.1}$$

We recall that $f = g + \sum_i b_i + \sum_i \theta_i$, so

$$|\{|m(L)f| > a\}|$$

$$\leq |\{|m(L)g| > \frac{a}{3}\}| + |\{|m(L)\sum_i b_i| > \frac{a}{3}\}| + |\{|m(L)\sum_i \theta_i| > \frac{a}{3}\}|. \quad (4.2)$$

By the property (1) of Calderon-Zygmund decomposition, we take

$$|\{|m(L)g| > \frac{a}{3}\}| \leq \frac{9}{a^2}\|m(L)g\|_2^2 \leq \frac{c}{a^2}\|g\|_2^2 \leq \frac{c}{a}\|\frac{g}{a}g\|_1 \leq \frac{c}{a}\|f\|_1. \quad (4.3)$$

By (3.2), we take

$$|\{|m(L)\sum_i b_i| > \frac{a}{3}\}| \leq \frac{c}{a^2}\|\sum_i b_i\|_2^2 \leq \frac{c}{a}\|f\|_1. \quad (4.4)$$

By (3.16), it is $\|m(L)\theta_i\|_{L^1(B(x_i, cr_i)^c)} \leq c\|w_i\|_1$, and then by the property (3) of Calderon-Zygmund decomposition, we take

$$|\{|m(L)\sum_i \theta_i| > \frac{a}{3}\}| \leq |\{x \in M - \bigcup_i B_i : |m(L)\sum_i \theta_i| > \frac{a}{3}\}| + |\bigcup_i B_i| \leq \frac{c}{a}\|f\|_1. \quad (4.5)$$

The theorem follows from (4.1), (4.2), (4.3), (4.4), and (4.5). □

5. Final Remarks

It is also interesting the problem of H^p-boundedness for the multipliers $m(L)$. One has to suppose that M has bounded geometry. Also that the estimate

$$|p_t(x, y) - p_t(x, z)| \leq \frac{c}{|B(x, t^{1/2m})|}\exp\left(-c\frac{d(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}}\right)\left(\frac{d(y, z)}{t^{1/2m}}\right)^{\gamma}, \quad (5.1)$$

holds for every $x, y, z \in M$, $t > 0$ with $d(y, z) \leq t^{1/2m}$, for some $\gamma \in (0, 1]$.

Then by using the strategy of [14] follows that if $m(\lambda)$ satisfies the condition (1.4) for $A > D\left(\frac{1}{p} - \frac{1}{2}\right)$, then $m(L)$ is bounded on H^p, for every $p \in \left(\frac{D}{D+\gamma}, 1\right]$.

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