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ON THE CLASSICAL SHARP SOBOLEV INEQUALITY AND RELATED PROBLEMS

EZEQUIEL R. BARBOSA

Departamento de Matemática
Universidade Federal de Minas Gerais
Caixa Postal 702
30123-970, Belo Horizonte, MG, Brazil
e-mail: ezequiel@mat.ufmg.br

Abstract

This article reports the main results in the best constant theory for the classical sharp Sobolev inequality on compact Riemannian manifolds. Moreover, we prove that the second best constant $B_1(p, g)$ depends continuously on g in relation to the C^0 -topology.

1. Introduction

Many works has been devoted to the study of sharp Sobolev-type inequalities. The main motivation is its connection with geometric problems as isoperimetric inequalities and the Yamabe problem. Thus, a theory of best constant has been developed in recent decades. We refer the reader to the following material for sharp Sobolev-type inequalities: [3], [9], [10], [11], [13], [14], [18], [19], [20], [21], [22], [23], [25], [26], [30]. In this work, we will focus on the classical sharp Sobolev one.

In order to fix notations, let us make some definitions. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 2$. We denote

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by $H^{1,p}(M)$, $1 \le p < \infty$, the standard first-order Sobolev space defined as the completion of $C^\infty(M)$ under the norm

$$\|u\|_{H^{1,p(M)}} = \left(\int_M |\nabla_g u|^p dv_g + \int_M |u|^p dv_g\right)^{\frac{1}{p}},$$

where dv_g is the Riemannian volume element of g.

The Sobolev embedding theorem ensures that the inclusion $H^{1,p}(M) \subset L^{p^*}(M)$, with $1 \le p < n$, is continuous for $p^* = \frac{np}{n-p}$. So, there exist constants $A, B \in \mathbb{R}$ such that, for any $u \in H^{1,p}(M)$,

$$\left(\int_M |u|^{p^*} dv_g\right)^{\frac{1}{p^*}} \le A\left(\int_M |\nabla_g u|^p dv_g\right)^{\frac{1}{p}} + B\left(\int_M |u|^p dv_g\right)^{\frac{1}{p}}. \qquad (I_g^p)$$

An equivalent inequality, is that for any $u \in H^{1,p}(M)$,

$$\left(\int_M |u|^{p^*} dv_g\right)^{\frac{p}{p^*}} \le A\int_M |\nabla_g u|^p dv_g + B\int_M |u|^p dv_g, \qquad (II_g^p)$$

where the constants A and B here being distinct than those in (I_g^p).

The first Sobolev best constant associated to (I_g^p) is defined by

$$\alpha(n, p, g) = \inf\{A \in \mathbb{R} : \text{there exists } A \in \mathbb{R} \text{ such that } (I_g^p) \text{ is valid}\},$$

and the second Sobolev best constant associated to (I_g^p) is

$$\beta(n, p, g) = \inf\{B \in \mathbb{R} : \text{there exists } A \in \mathbb{R} \text{ such that } (I_g^p) \text{ is valid}\}.$$

In a parallel manner, the first Sobolev best constant associated to (II_g^p) is

$$\overline{\alpha}(n, p, g) = \inf\{A \in R : \text{there exists } B \in \mathbb{R} \text{ such that } (II_g^p) \text{ is valid}\},$$

and the second Sobolev best constant associated to (II_g^p) is

$\overline{\beta}(n, p, g) = \inf\{B \in \mathbb{R} : \text{there exists } A \in \mathbb{R} \text{ such that } (II_g^p) \text{ is valid}\}.$

The sharp Sobolev inequality on $H^{1,p}(M)$ with respect to $\alpha(n, p, g)$ states that there exists a constant $B > 0$ such that

$$\left(\int_M |u|^{p^*} dv_g\right)^{\frac{1}{p^*}} \leq a(n,p,g)\left(\int_M |\nabla_g u|^p dv_g\right)^{\frac{1}{p}} + B\left(\int_M |u|^p dv_g\right)^{\frac{1}{p}}. \ (I^p_{g,opt}(\alpha))$$

We say that $(I^p_{g,opt}(\alpha))$ is valid if there exists a constant $B \in \mathbb{R}$ such that $(I^p_{g,opt})$ holds for all $u \in H^{1,p}(M)$. Another hand, the sharp Sobolev inequality on $H^{1,p}(M)$ with respect to $\beta(n, p, g)$ states that there exists a constant $A > 0$ such that

$$\left(\int_M |u|^{p^*} dv_g\right)^{\frac{1}{p^*}} \leq A\left(\int_M |\nabla_g u|^p dv_g\right)^{\frac{1}{p}} + \beta(n,p,g)\left(\int_M |u|^p dv_g\right)^{\frac{1}{p}} \ (I^p_{g,\,opt}(\beta))$$

and we say that $(I^p_{g,opt}(\beta))$ is valid if there exists a constant $A \in \mathbb{R}$ such that $(I^p_{g,opt}(\beta))$ holds for all $u \in H^{1,p}(M)$. In a similar way, we define the sharp Sobolev inequalities with respect to best constants $\overline{\alpha}(n, p, g)$ and $\overline{\beta}(n, p, g)$.

Here arise the questions about the validity or not of the sharp Sobolev inequalities and the exact value (or upper and lower bounds) for the best constants. First, Aubin gives in [1] the exact value of the best constant $\alpha(n, p, g)$:

$$\alpha(n, p, g) = \sup_{u \in \mathcal{D}^{1,p}(\mathbb{R}^n), u \neq 0} \frac{\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{\frac{1}{p^*}}}{\left(\int_{\mathbb{R}^n} |\nabla_u|^p \, dx\right)^{\frac{1}{p}}},$$

and $\overline{\alpha}(n, p, g) = \alpha(n, p, g)^p$, where $\mathcal{D}^{1,p}\left(\mathbb{R}^n\right)$ is the completion of $C_0^\infty\left(\mathbb{R}^n\right)$ under the norm

$$\|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

After that, the sharp inequalities $(I^p_{g,opt}(\alpha))$ and $(II^p_{g,opt}(\overline{\alpha}))$ was proved to be valid by Hebey and Vaugon [28], in the $p = 2$ case, by Druet [14] for $p = 1$, by Druet [17], Aubin and Li [4], in the $1 < p < 2$ case. In [15], Druet also proves that sharp inequality $(II^p_{g,opt}(\overline{\alpha}))$ is not valid when $2 < P < \frac{n+2}{3}$ and the scalar curvature is positive in some point $Po \in M$.

When $n \geq 2$ and $p \in [1, n)$, was showed by Hebey [18] that $\beta(n, p, g) = V_g(M)^{-\frac{1}{n}}$ and the sharp inequality $(I^p_{g,opt}(\beta))$ is valid. Moreover, from works of Bakry [5] and Hebey-Druet [25], for any $p \in [1, 2)$ if $n = 2$, and $p \in [1, 2]$ if $n \geq 3$, $\overline{\beta}(n, p, g) = V_g(M)^{-\frac{p}{n}}$ and the sharp inequality $(II^p_{g,opt}(\overline{\beta}))$ is valid. Conversely, if $p > 2$, the sharp inequality $(II^p_{g,opt}(\overline{\beta}))$ is false.

So, for $1 \leq p \leq 2$, consider the second Sobolev best constants

$$B_0(p, g) = \inf\{B \in \mathbb{R} : (II^p_{g,opt}(\overline{\alpha})) \text{ is valid}\}$$

and

$$A_0(p, g) = \inf\{A \in \mathbb{R} : (II^p_{g,opt}(\overline{\beta})) \text{ is valid}\}.$$

Clearly, for any $u \in H^{1,p}(M)$ one has

$$\left(\int_M |u|^{p^*} dv_g\right)^{\frac{p}{p^*}} \leq \alpha(n,p,g)^p \int_M |\nabla_g u|^p dv_g + B_o(p,g)\int_M |u|^p dv_g, \ (J^p_{g,opt}(\overline{\alpha})),$$

and

$$\left(\int_M |u|^{p^*} dv_g\right)^{\frac{p}{p^*}} \leq A_0(p,g)\int_M |\nabla_g u|^p dv_g + V_g(M)^{-\frac{p}{n}}\int_M |u|^p dv_g, \ (J^p_{g,opt}(\overline{\beta})).$$

Note that these inequalities are sharp with respect to both the first and second constants, in the sense that none of them can be lowered. Also, we define de best constants

$$B_1(p, g) = \inf\{B \in \mathbb{R} : (I^p_{g,opt}(\alpha)) \text{ is valid}\}$$

and

$$A_1(p, g) = \inf\{A \in \mathbb{R} : (I^p_{g,opt}(\beta)) \text{ is valid}\}.$$

Ilias [29] showed that when $p = 2$, $n \geq 3$ and $Ric_g \geq (n-1)k^2 g$,

$$\alpha(n, 2, g) \leq A_1(2, g) \leq A_0(2, g)^{\frac{1}{2}} \leq \sqrt{\frac{4}{n(n-2)kV_g{}^{\frac{2}{n}}}}.$$

We point out here that the constants $B_0(p, g)$ and $A_0(p, g)$ are connected with spectral studies. On the other hand, the constants α and β are important for the Yamabe problem, as existence and multiplicity, and for its relation with the isoperimetric inequality. In the next section, we will say something on the best constants $B_0(p, g)$ and $B_1(p, g)$.

2. Extremal Functions

A function $u_0 \in H^{1,p}(M), u_o \neq 0$, is said to be extremal for $(J^p_{g,opt}(\alpha))$ if

$$\left(\int_M |u_0|^{p^*}\right)^{\frac{p}{p^*}} = K(n, p)^p \int_M |\nabla_g u_0|^p dv_g + B_0(p, g)\int_M |u_0|^p dv_g.$$

In a similar way, we define extremal functions for that other sharp inequalities. Note that the constant functions are extremal functions for $(J^p_{g,opt}(\beta))$ and $(J^p_{g,opt}(\overline{\beta}))$. But this is not true for the other cases, since $B_0(p, g) > V_g(M)^{-\frac{p}{n}}$ and $B_1(p, g) > V_g(M)^{-\frac{1}{n}}$ can happen.

Hebey and Druet [18] showed that for all $p \in (1, n)$ and $n \geq 2$, $(J^p_{g,opt}(\alpha))$ possesses extremal functions. In the case $p \in \left(1, \sqrt{n}\right)$ and $p < 2$, Druet and Djadli [12] showed that $(J^p_{g,opt}(\overline{\alpha}))$ always possesses

extremal functions. In the $p = 2$ case, the situation changes. By standard elliptic theory, an extremal function for $(J^2_{g,\,opt}(\overline{\alpha}))$ is smooth, and either positive or negative. Up to replacing u_0 by $-u_0$ and up to a positive constant scale factor, an equivalent definition is that

$$\Delta_g u_0 + \frac{B_0(2,\,g)}{K(n,\,2)^2}\,u_0 = \frac{1}{K(n,\,2)^2}\,u_0^{2^*-1} \text{ in } M$$

and $\int_M u_0^{2^*} dv_g = 1$. Taking $u = 1$ in $(J^2_{g,opt}(\overline{\alpha}))$ gives that $B_0(2,\,g) \geq V_g^{-\frac{2}{n}}$, where V_g is the volume of M with respect to g. When $n \geq 4$, see for instance [25], one has that $B_0(2,\,g) \geq B_0(2,\,g)_{extr}$ where

$$B_0(2,\,g)_{extr} = \frac{n-2}{4(n-1)}\,K(n,\,2)^2 \max_M S_g$$

and S_g is the scalar curvature of g. When $(M, g) = (S^n, g_0)$, the standard unit n-sphere, $B_0(2,\,g_0) = B_0(2,\,g_0)_{extr} = V_{go}^{-\frac{2}{n}}$ and the extremal functions for $(J^2_{g_0,\,opt}(\overline{\alpha}))$ are known. Namely, up to a scale factor, $u_0 = 1$ or

$$u_0 = (\beta - \cos r)^{1-\frac{n}{2}},$$

where $\beta > 1$, and r is the distance to a given point in S^n. See for instance [24]. Moreover, by Hebey [24], if $g \in [g_0]$ is a conformal metric to g_0, and $n \geq 4$, then $B_0(2,\,g) = B_0(2,\,g)_{extr}$ and there exists extremal functions for $(J^2_{g,\,opt}(\overline{\alpha}))$ if and only if, up to a positive constant scale factor, g and g_0 are isometric, in which case all the extremal functions are known. In the general case of arbitrary smooth compact Riemannian n-manifold, Djadli and Druet, in [12], showed that on smooth compact Riemannian manifolds of dimension $n \geq 4$, at least, one of the following assertions holds:

(a) an extremal function for $(J^2_{g,\,opt}(\overline{\alpha}))$ exists, or

(b) $B_0(2,\,g) = B_0(2,\,g)_{extr}$.

Combining the result of [12] with the solution of Yamabe problem, by Aubin [2] and Schoen [31], one easily concludes that there exists extremal functions for $(J^2_{g,opt}(\overline{\alpha}))$ when either $S_g \leq 0$ or S_g is constant.

Conformal metrics to the standard metric on the unit n-sphere provide examples where (b) is true and (a) is false. Independently, quotients of the unit n-sphere provide examples where (b) is false and (a) is true. Finally, the standard unit n-sphere provide an example where both (a) and (b) are true. Then, the following questions arise:

Q.1. Is there a compact Riemannian manifold (M, g), not conformally diffeomorphic to the standard unit n-sphere, for which (b) is true and (a) is false?

Q.2. Is there a compact Riemannian manifold (M, g), not isometric to the standard unit n-sphere, for which both (a) and (b) are true?

Q.3. Is there a compact Riemannian manifold (M, g), for which $S_g > 0$ and nonconstant (or changing signal) such that (a) is true?

Positive answer to the questions Q.1 and Q.2, based on the notion of critical function, was given by Hebey and Vaugon in [27]. Examples of manifolds with nonconstant positive scalar curvatures such that $(J^2_{g,opt}(\overline{\alpha}))$ possesses extremal functions was given by Barbosa and Montenegro in [6], responding the question Q.3.

3. Continuous Dependence of the Best Constants

Aubin [1] and Talenti [32] showed that the first best constant $\alpha(n, p, g)$ is given by

$$K(n, p) = \frac{1}{n}\left(\frac{n(p-1)}{n-p}\right)^{\frac{p-1}{p}}\left(\frac{\Gamma(n+1)}{\Gamma(\frac{n}{p})\,\Gamma(n+1-\frac{n}{p})\,w_{n-1}}\right)^{\frac{1}{n}}$$

for $p > 1$ and

$$K(n, 1) = \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{\frac{1}{n}} = \lim_{p\to 1} K(n, p),$$

where Γ is the Gamma function. Thus, the first best constants $\alpha(n,p,g) = K(n,p)$ and $\overline{\alpha}(n,p,g) = K(n,p)^p$ not depend on the metric g, and they are continuous with respect to the parameter p. So, the following questions on the best constants $B_0(p, g)$ and $B_1(p, g)$ arise:

(a) $B_0(p, g)$ and $B_1(p, g)$ depend continuously on the metric g in some topology?

(b) $B_0(p, g)$ and $B_1(p, g)$ depend continuously on the parameter p?

The questions (a) and (b) was studied by Barbosa and Montenegro in [7] and [8], for $B_0(p, g)$. Precisely, they proved the following results.

Theorem 3.1. *Let M be a compact Riemannian manifold of dimension $n \geq 2$ and $\mathcal{M}$ the space of smooth Riemannian metrics on M. Assume $1 < p < \min\{2, \sqrt{n}\}$. Then, the map $g \in \mathcal{M} \to B_0(p, g)$ is continuous in the C^0 – topology, i.e., if the components of metric g_{ij}^{α} converges to g_{ij} in $C^0(M)$, then $B_0(p, g^{\alpha}) \to B_0(p, g)$ as $\alpha \to +\infty$.*

Theorem 3.2. *Let M be a compact Riemannian manifold of dimension n and $\mathcal{M}$ as in Theorem* 3.1. *Assume $p = 2$ and $n \geq 4$. If (g^{α}) is a sequence in $\mathcal{M}$ such that $g^{\alpha} \to g$ in $C^0(M)$ and $Scal_{g^{\alpha}} \to Scal_g$ pointwise in M, where $Scal_g$ denotes the scalar curvature of the metric g, then $B_0(2, g^{\alpha}) \to B_0(2, g)$ as $\alpha \to +\infty$. In particular, the map $g \in \mathcal{M} \to B_0(2, g)$ is continuous in the C^2-topology. Moreover, the scalar curvature convergence or C^2-convergence assumption is necessary.*

Theorem 3.3. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 4$. Then, the map $p \to B_0(p, g), 1 \leq p < 2$, is continuous.*

With the following result, we answer the question (a) for $B_1(p, g)$.

Theorem 3.4. *Let M be a compact Riemannian manifold of dimension $n \geq 2$ and $\mathcal{M}$ the space of smooth Riemannian metrics on M. Assume $1 < p < n$. Then, the map $g \in \mathcal{M} \to B_1(p, g)$ is continuous in the C^0-topology, i.e., if the components of metric g_{ij}^{α} converges to g_{ij} in $C^0(M)$, then $B_1(p, g^{\alpha}) \to B_1(p, g)$ as $\alpha \to +\infty$.*

The proof of Theorem 3.4 is made by contradiction. In this case, we find two possible alternatives. One of them is directly eliminated according to the definition of best constant $B_1(p, g)$. The other alternative implies the existence of a family of minimizers of certain functionals which concentrate in some point. The proof then consists in finding estimates of these minimizes around a concentration point and in combining them in order to obtain the desired second contradiction.

3.1. Proof of Theorem 3.4

We present a sketch of the proof of Theorem 3.4. Let (g_{α}) be a sequence of metrics on M such that g_{α} converges to a metric g in the C^0-topology. Suppose, by contradiction, that there exists $\varepsilon_0 > 0$ such that $|B_1(p, g_{\alpha}) - B_1(p, g)| > \varepsilon_0$ for infinitely many α. Then, at least, one of the situations holds: $B_1(p, g) - B_1(p, g_{\alpha}) > \varepsilon_0$ or $B_1(p, g_{\alpha}) - B_1(p, g) > \varepsilon_0$ for infinitely many α. If the first situation holds, replacing $B_1(p, g_{\alpha})$ by $B_1(p, g) - \varepsilon_0$ in the sharp inequality associated to the metric g_{α} and letting $\alpha \to +\infty$, we contradict the definition of $B_1(p, g)$.

Suppose then that the second situation holds, i.e., $B_1(p, g) + \varepsilon_0 < B_1(p, g_{\alpha})$ for infinitely many α. For each α, consider the functional

$$J_{\alpha}(u) = \left(\int_M |u|^{p^*} dv_{g_{\alpha}} \right)^{\frac{1}{p^*}} - (B_1(p, g) + \varepsilon_0) \left(\int_M |u|^p dv_{g_{\alpha}} \right)^{\frac{1}{p}}$$

defined on $\Lambda_\alpha = \left\{ u \in H^{1,p}(M) : \int_M |\nabla_{g_\alpha} u|^p \, dv_{g_\alpha} = 1 \right\}$. From the definition of $B_1(p, g_\alpha)$, it follows directly that $S_\alpha := \sup_{\Lambda_\alpha} J_\alpha(u) > K(n, p)$. But this implies the existence of a nonnegative maximizer $u_\alpha \in \Lambda_\alpha$ for S_α. The Euler-Lagrange equation for u_α is then

$$S_\alpha \Delta_{p,g_\alpha} u_\alpha + (B_1(p, g) + \varepsilon_0) \| u_\alpha \|_p^{1-p} u_\alpha^{p-1} = \| u_\alpha \|_{p^*}^{1-p^*} u_\alpha^{p^*-1}, \quad (E_\alpha)$$

where $\Delta_{p, g_\alpha} u = -\mathrm{div}_{g_\alpha}\left(|\nabla_{g_\alpha} u|^{p-2} \nabla_{g_\alpha} u \right)$ is the p-Laplacian of u with respect to the metric g_α. By the standard elliptic theory, u_α belongs to $C^1(M)$ and $u_\alpha > 0$ on M. We set

$$u_\alpha = \| u_\alpha \|_{p^*}^{-1} u_\alpha \text{ and } \lambda_\alpha = S_\alpha^{-1} \| u_\alpha \|_{p^*}^{1-p}.$$

Then, by the equation (E_α),

$$\Delta_{p, g_\alpha} v_\alpha + (B_1(p, g) + \varepsilon_0) \lambda_\alpha \| v_\alpha \|_p^{1-p} v_\alpha^{p-1} = \lambda_\alpha v_\alpha^{p^*-1}, \quad (F_\alpha)$$

and $\int_M v_\alpha^{p^*} dv_{g_\alpha} = 1$. Our goal now is to study the sequence $(v_\alpha)_\alpha$ as $\alpha \to +\infty$. From the convergence $g_\alpha \to g$, it follows that $(v_\alpha)_\alpha$ is bounded in $H^{1,p}(M)$ with respect to the metric g. So, there exists $v \in H^{1,p}(M)$, $v \geq 0$, such that $v_\alpha \to v$ weakly in $H^{1,p}(M)$ as $\alpha \to +\infty$, up to a subsequence. Moreover, by the Sobolev embedding compactness theorem, one easily finds

$$\int_M v_\alpha^q \, dv_{g_\alpha} \to \int_M v^q dv_g \quad (1)$$

for any $1 \leq q < p^*$. So, if $v \neq 0$, letting $\alpha \to +\infty$ in the equation (F_α), one concludes that v satisfies

$$K(n, p) \| \nabla_g v \|_p^{1-p} \Delta_{p,g} v + (B_1(p, g) + \varepsilon_0) \| v \|_p^{1-p} v^{p-1} = v^{p^*-1}. \quad (E)$$

In this case, by $(J^p_{g,\,opt}(\alpha))$ and (E), one has

$$\left(\int_M v^{p^*} dv_g\right)^{\frac{1}{p^*}} < K(n,\,p)\left(\int_M \left|\nabla_g v\right|^p dv_g\right)^{\frac{1}{p}} + (B_1(p,\,g)+\varepsilon_0)\left(\int_M v^p dv_g\right)^{\frac{1}{p}}$$

$$= \int_M v^{p^*} dv_g.$$

This implies that $\int_M v^{p^*} dv_g > 1$. But this inequality contradicts $\int_M v^{p^*} dv_g \le \liminf \int_M v_\alpha^{p^*} dv_{g_\alpha} = 1$. We then assume that $v = 0$ on M and prove also that this assumption leads to a contradiction. We divide the proof into four stages.

We next only mention each one of them.

First stage: We say that $x \in M$ is a point of concentration for (v_α) if for any $\delta > 0$,

$$\limsup_{\alpha\to+\infty} \int_{B_{g_\alpha}(x,\,\delta)} v_\alpha^{p^*} dv_{g_\alpha} > 0.$$

It follows from standard development that, up to a subsequence, (v_α) possesses only one concentration point $x_0 \in M$. In particular,

$$\lim_{\alpha\to+\infty} \int_{B_{g_\alpha}(x_0,\,\delta)} v_\alpha^{p^*} dv_{g_\alpha} = 1,$$

for all $\delta > 0$. By Moser's iterative scheme, we also get that

$$v_\alpha \to 0 \text{ in } C^0_{loc}(M \setminus \{x_0\}). \tag{2}$$

Second stage: Let $x_\alpha \in M$ be a maximum point of v_α, i.e., $v_\alpha(x_\alpha) = \|v_\alpha\|_\infty$. Let $x_0 \in M$ be such that $x_\alpha \to x_0$, up to a subsequence. Thus, for each $R > 0$, we have

$$\lim_{\alpha\to+\infty} \int_{B_{g_\alpha}(x_{\alpha,\,R\mu_\alpha})} v_\alpha^{p^*} dv_{g_\alpha} = 1 - \varepsilon_R \tag{3}$$

where $\mu_\alpha = \|u_\alpha\|_\infty^{-\frac{p^*}{n}}$ and $\varepsilon = \varepsilon(R) \to 0$ as $R \to +\infty$.

Third stage: There exist constants $c, \delta > 0$, independent of α, such that $d_{g_\alpha}(x, x_\alpha)^{n/p^*} u_\alpha(x) \le c$ for all $x \in \overline{B}_{g_\alpha}(x_\alpha, \delta)$, where d_{g_α} stands for the distance with respect to the metric g_α.

Fourth stage: We now argue with the third stage in order to obtain a contradiction. Some possibly different positive constants independent of α will be denoted by c. Now, let $\delta > 0$ small enough and consider a smooth cutoff function $\eta_\alpha \in C_0^\infty(B_{g_\alpha}(x_\alpha, 4\delta))$ such that $0 \le \eta_\alpha \le 1$, $\eta_\alpha = 1$ in $B_{g_\alpha}(x_\alpha, 2\delta)$ and $\left|\nabla_{g_\alpha} \eta_\alpha\right| \le c$. By definition of $K(n, p)$,

$$\left(\int_{B_{g_\alpha}(x_\alpha, 4\delta)} (\eta_\alpha v_\alpha) p^* \, dx \right)^{\frac{p}{p^*}} \le K(n, p)^p \int_{B_{g_\alpha}(x_\alpha, 4\delta)} |\nabla(\eta_\alpha v_\alpha)|^p \, dx.$$

Then, it easily follows, as in Druet [17], that for α large enough

$$\frac{1}{2}(B_1(p, g) + \varepsilon_0) \le \frac{1 - \left(\int_{B_{g_\alpha}(x_\alpha, 2\delta)} (v_\alpha)^{p^*} dx \right)^{\frac{p}{p^*}}}{\left(\int_M (v_\alpha)^p dx \right)^{\frac{1}{p}}} + c \frac{\int_{M \setminus B_{g_\alpha}(x_0, \delta)} v_\alpha |\nabla v_\alpha|^{p-1} dv_{g_\alpha}}{\left(\int_M (v_\alpha)^p dx \right)^{\frac{1}{p}}}$$

$$+ c \frac{\int_{B_{g_\alpha}(x_\alpha, 4\delta)} \eta_\alpha^p |\nabla v_\alpha|^p d_{g_\alpha}(x_\alpha, x)^2 dv_{g_\alpha}}{\left(\int_M (v_\alpha)^p dx \right)^{\frac{1}{p}}} + c \left(\int_M v_\alpha^p dv_{g_\alpha} \right)^{1 - \frac{1}{p}}. \qquad (4)$$

By the first stage and $v = 0$,

$$\lim_{\alpha \to +\infty} \frac{\int_{M \setminus B_{g_\alpha}(x_0, \delta)} v_\alpha |\nabla v_\alpha|^{p-1} dv_{g_\alpha}}{\left(\int_M (v_\alpha)^p dx \right)^{\frac{1}{p}}} = 0$$

and

$$\lim_{\alpha \to +\infty} \left(\int_M v_\alpha^p \, dv_{g_\alpha} \right)^{1 - \frac{1}{p}} = 0.$$

Also, if $p \leq 2$, see Druet [17], it follows from the stages above that

$$1 - \left(\int_{B_{g_\alpha}(x_\alpha, 2\delta)} (v_\alpha)^{p^*} dx \right)^{\frac{p}{p^*}} \leq c \int_M v_\alpha^p dv_{g_\alpha}$$

and

$$\int_{B_{g_\alpha}(x_\alpha, 4\delta)} \eta_\alpha^p |\nabla v_\alpha|^p d_{g_\alpha}(x_\alpha, x)^2 dv_{g_\alpha} \leq c \int_M v_\alpha^p dv_{g_\alpha}.$$

When $p > 2$, see again Druet [17], one obtain

$$1 - \left(\int_{B_{g_\alpha}(x_\alpha, 2\delta)} (v_\alpha)^{p^*} dx \right)^{\frac{p}{p^*}} \leq c \left(\int_M v_\alpha^p \, dv_{g_\alpha} \right)^{\frac{2}{p}}$$

and

$$\int_{B_{g_\alpha}(x_\alpha, 4)} \eta_\alpha^p |\nabla v_\alpha|^p d_{g_\alpha}(x_\alpha, x)^2 dv_{g_\alpha} \leq c \left(\int_M v_\alpha^p dv_{g_\alpha} \right)^{\frac{2}{p}}.$$

Hence, the right hand-side of (4) goes to zero as $\alpha \to +\infty$, a contradiction.

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