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A STUDY OF A POLLUTION PROBLEM IN POROUS MEDIA DURING VAPOR EXTRACTION

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Abstract

Soil Vapor Extraction (SVE) is a common remediation technique for removing volatile organic compounds (VOCs) from unsaturated contaminated soils. A pollution problem is presented to describe the vapor

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transport of a volatile organic contaminant in a porous media with a non-equilibrium kinetics, during vapor extraction. An inverse problem permitting to show that the initial concentration of the pollutant in liquid phase, can be identified, since we can compute its concentration in gaseous phase at final time, is studied. The existence and uniqueness of the direct problem is presented as well as the existence of the inverse problem in bidimensional case in non pondered spaces. In addition, the initial concentration of the pollutant in liquid phase is constructed by reformulating the inverse problem as a problem of optimization in least squares sense.

1. Introduction

Soil Vapor Extraction (SVE), also known as soil venting, vapor stripping, or in situ volatilization, is a widely applied technology for removal of volatile organic compounds (VOCs) from unsaturated zone. SVE procedures are reviewed by [18], while screening tools for the evaluation of SVE suitability are described by [13]. The technique is based on simple physical/chemical principles: By applying a vacuum, contaminated air in connected soil pores is extracted and replaced by clean atmospheric air; this perturbs the local chemical equilibrium, causing the transfer of contaminant from the residual phase, dissolved phase, and sorbed phase into the soil gas, where it can be removed. In theory, this should eventually lead to a complete cleanup, see [5]. The rate of removal is often controlled by the interphase mass transfer of contaminants from the nonaqueous phase liquids (NAPLs), aqueous, or sorbed phases to the gas phase. The gas flow rate and rate of mass transfer to the gas phase are the most important factors controlling the efficiency of contaminant removal using SVE , see [21]. Moreover, the efficiency of this technique depends on various factors including the geological settings of the contaminated area, the absolute permeability of the soil, the distribution of water saturation within the soil, the organic content, and volatility of the compounds to be removed, see, for example, [3]. In high permeable, homogenous soils, the efficiency of SVE is generally high, although long tailing behavior of the concentration in the removed air is often found and attributed to mass transfer limitations, see, for example, [9, 10, 25]. Mathematical models describing transport of VOCs in the gas phase of the soil have been developed by [5], [6], [11],

[12], [16], [20], and [24]. A macroscopic mathematical model is presented to describe the vapor transport of a contaminant organic volatile in a porous media in bidimensional setting. The proposed model assumes that rate-limited vaporization is described with a first-order kinetics. Thus a system of 2D reaction-diffusion is presented. An inverse problem is presented which permits to identify the liquid concentration of the volatile organic contaminant at initial time, since we can compute its concentration in gaseous phase at final time. This inverse problem is derived from a time-dependent pollution problem, modelling the flow vapor and the mass transfer of contaminant between the vapor and liquid phases, in unsaturated soils during vapor extraction. This problem was studied in [1] and [2] for one dimensional case in pondered spaces. Here, we perform the existence and uniqueness of the direct problem by using Galerkin method, as well as the existence of the inverse one by using FredHolm Alternative in nonpondered spaces, in bidimensional case. In addition, the initial concentration of the pollutant is constructed by reformulating the inverse problem as a problem of optimization in least squares sense.

2. Mathematical Model

We are interested here with the flow of the vapor in the unsaturated soil zone considered as a rigid non deformable porous media, and the mass transfer of contaminant between the gas and liquid phases. We neglect infiltration events and assume the soil moisture to be immobile and constant in time. Furthermore, we neglect the NAPL and sorped phases. Darcy's law can be used to describe airflow during vapor extraction due to the applied vacuum of SVE (see [5], [14], [15], [17]). In this article we do not consider the process of advection-diffusion on the microscopic scale of a single pore of the soil matrix, but on the macroscopic scale appropriate for measurement of phenomena described. Under the assumptions of a static moisture distribution and an incompressible air phase, the advective-dispersive transport equation in Cartesian coordinates is (see, for instance, [5])

$$\theta \frac{\partial C_w}{\partial t} + \varepsilon_a \frac{\partial C_a}{\partial t} = -\mathrm{div}(\varepsilon_a \vec{v} C_a - \varepsilon_a D \nabla C_a), \tag{1}$$

where the subscripts a, w designate the air and water phases of the contaminant, C_a and $C_w[ML^{-3}]$ are the corresponding concentrations, ε_a and θ are the volumetric air and water contents (dimensionless) $[L^3L^{-3}]$. The unknowns are the state variables C_a and C_w. The two terms on the left-hand side of (1) represent the changes in contaminant mass stored in the water phase and the vapor phase, respectively. The linear air velocity, $\vec{v}[L/T]$, is determined from the solution of the air continuity equation and the subsequent application of the Darcy equation to pressure. The dispersion tensor $D[L^2T^{-1}]$, is the standard form (see [7]), expressed in terms of longitudinal dispersivity $\alpha_L[L]$, transverse dispersivity $\alpha_T[L]$, and the molecular diffusion $D_d[L^2T^{-1}]$. Boundary conditions for (1) can be of the following types:

Specified concentration (Type 1)

$$C_a = C_0.$$

Specified mass flux (Type 2)

$$v_n C_a - D\frac{\partial C_a}{\partial n} = v_0 C_0.$$

During vapor extraction, the ground surface boundary condition can be represented by a zero concentration gradient if sealed, or by a zeros mass flux (Type 2 boundary condition) if open. Other inlet boundaries for clean air can also be described by a zero mass flux boundary condition, for further details, see [5]. Initial conditions with respect to concentrations in each phase must be specified.

3. Nonequilibrium Mass Transfer Relationship

In the absence of pure phase, contaminant will be stored in the dissolved and the sorbed phases, in addition to the gas phase. We assume that a continuous film of water, being the wetting fluid, coats the soil grains so that gas-to-solid mass transfer is possibly only via the water phase. The relevant mass transfer process is therefore between the soil gas and the dissolved phase, since the sorbed phase is neglected. We assume that the process is nonequilibrium. A mass balance equation for

the stationary water phase, with mass transfer across the air-water interface, can be written as (see [4], [5], [19]):

$$\theta \frac{\partial C_w}{\partial t} = -\varepsilon_a k_s (K_H C_w - C_a), \tag{2}$$

where K_H is Henry's constant (dimensionless). The right-hand side expresses air-water transfer, with k_s being the corresponding first order mass transfer coefficient $[T^{-1}]$. C_w represents the average concentration in the water, while C_a / K_H represents the equilibrium concentration at the air-water interface, according to Henry's law $C_a = K_H C_w$.

4. Formulation of the Problem

We combine equations (1) and (2), and using the boundary conditions (continuity of the gas phase at the top of the column and chemical equilibrium at the bottom of the column) and the initial conditions, we obtain the following transport system:

$$\begin{cases} \frac{\partial C_a}{\partial t} - \operatorname{div}(D\nabla C_a - C_a \vec{v}) = k_s (K_H C_w - C_a) \text{ in } Q_T, \\ \frac{\partial C_w}{\partial t} = -\alpha (K_H C_w - C_a) \text{ in } Q_T, \\ C_a(x, t) = C^* \quad \text{on} \quad \Gamma_1 \times J, \\ D\nabla C_a . n = 0 \quad \text{on} \quad \Gamma_2 \times J, \\ C_a(x, 0) = C_a^0(x) \quad \text{in} \quad \Omega, \\ C_w(x, 0) = C_w^0(x) \quad \text{in} \quad \Omega, \end{cases} \tag{3}$$

where $\alpha = \frac{\varepsilon_a}{\theta} k_s$, Ω is a bounded polygonal domain in R^2 represents a vertical column of an unsaturated zone of a non deformable rigid porous media, with Lipschitz boundary Γ defined as $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$, and $J =]0, T[$ is a time interval with $T > 0$, $Q_T = \Omega \times J$ and C^* is the saturation concentration in the gaseous phase, C_a^0 and C_w^0 are the initial concentrations in the gaseous and liquid phases, respectively. From the equations $(3)_2$ and $(3)_6$, the function C_w writes under the form:

$$C_w(x,\,t) = C_w^0(x)e^{-\alpha K_H t} + \alpha\int_0^t e^{-\alpha K_H(t-s)}C_a(x,\,s)ds, \quad \forall (x,\,t) \in Q_T. \quad (4)$$

By inserting this formula in equation $(3)_1$, we get

$$\frac{\partial C_a}{\partial t} - \operatorname{div}(D\nabla C_a - C_a\vec{v})$$

$$= k_s K_H C_w^0(x)e^{-\alpha K_H t} + \alpha k_s K_H \int_0^t e^{-\alpha K_H(t-s)}C_a(x,\,s)ds - k_s C_a \text{ in } Q_T. \quad (5)$$

Whence the equation $(3)_2$ is eliminated. Therefore, the inverse problem which we shall consider is the following: For each $T > 0$, if the concentration C_a verifies the equation (5) and conditions $(3)_3$ - $(3)_5$, and in addition if the concentration $C_T(.,T)$ is given, does this allow to determine uniquely the function C_w^0 ? To answer this question, it is very important to have a time independent function, to be identified in the right-hand side of equation (5), in doing so, we use the following change of variables:

$$u(x,\,t) = \frac{1}{k_s K_H}\, e^{\alpha K_H t} C_a(x,\,t) \text{ in } Q_T,\; u_0(x) = \frac{1}{k_s K_H}\, C_a^0(x) \text{ in } \Omega,$$

$$f(x) = C_w^0(x) \text{ in } \Omega,\; \gamma(t) = \frac{C^*}{k_s K_H}\, e^{\alpha K_H t} \text{ in } J.$$

We denote by $r(x) = k_s - \alpha K_H$. Under these notations, the inverse problem (P) writes:

For $T > 0$ and γ, u_0 and h_T given functions, find the pair $(u,\,f)$ verifying:

$$(P)\begin{cases} (P_1)\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(D\nabla u - \vec{v}u) + r(x)u = f + \alpha k_s K_H \int_0^t u(s)ds \text{ in } Q_T, \\ u(x,\,t) = \gamma(t) \quad \text{on} \quad \Gamma_1 \times J, \\ D\nabla u.\vec{n} = 0 \quad \text{on} \quad \Gamma_2 \times J, \\ u(x,\,0) = u_0(x) \quad \text{in} \quad \Omega, \\ u(x,\,T) = h_T(x) \quad \text{in} \quad \Omega, \end{cases} \end{cases}$$

where D is a uniformly positive function in $\overline{\Omega} \times J$, and r is the given absorption/reaction coefficient.

5. Direct Problem

5.1. Variational formulation of problem (P_1)

We use the standard spaces $L^p(\Omega)$, $1 \leq p \leq +\infty$, together with the Sobolev spaces $H^S(\Omega)$, where S is a nonnegative real number. We denote by $H^1_{\Gamma_D}(\Omega) = \{\phi \in H^1(\Omega) : \phi / \Gamma_1 = 0\}$, and $H^{-1}(\Omega)$ stands for the dual space of the standard Sobolev space $H^1_0(\Omega)$. For simplicity, we denote by $(.,.)$ the scalar product on $L^2(\Omega)$ and by $\|.\|$ the associated norm. In all that follows, for any t, $0 < t \leq T$, and any Banach space X provided with the norm $\|.\|_X$, we denote by $L^2(0, t; X)$, respectively $L^\infty(0, t; X)$, the space of measurable functions ϕ from $]0, t[$ in X such that $\|\phi\|_{L^2(0, t; X)} = \left(\int_0^t \|\phi(., s)\|_X^2 ds\right)^{\frac{1}{2}} < +\infty$, respectively, $\|\phi\|_{L^\infty(0,t; X)}$ $= ess \sup_{0 \leq s \leq t} \|\phi(., s)\|_X < +\infty$.

Finally, we introduce the space $C^0(0, t; X)$ of continuous functions ϕ from $[0, t]$ in X. We need also in the sequel the following spaces

$$W = \left\{\phi \in L^2(0, T; H^1(\Omega)); \frac{\partial \phi}{\partial t} \in L^2(0, T; L^2(\Omega))\right\},$$

and,

$$W_0 = \left\{\phi \in L^2(0, T; H^1_{\Gamma_D}(\Omega)); \frac{\partial \phi}{\partial t} \in L^2(0, T; L^2(\Omega))\right\}.$$

We define the following bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to R$, $a(\phi, \psi) = (D\nabla\phi, \nabla\psi) + (\text{div}(\vec{v}\phi), \psi) + (r\phi, \psi)$. Let us state the following assumptions,

(A_1) D is a positive and continuously time-differentiable function such that

$$\forall x \in \overline{\Omega},\ \forall t \in]0,\ T[,\ 0 < D_{\min} \le D(x,\ t) \le D_{\max} < +\infty.$$

(A_2) $\vec{v} \in \mathcal{C}^1(\overline{\Omega})^d$; (A_3) $u_0 \in H^1(\Omega)$; (A_4) $f \in L^2(\Omega)$; (A_5). There exists a constant $\beta \ge 0$ s.th $\operatorname{div}(\vec{v}) + r \ge \frac{\beta}{2}$ a.e in Ω. The problem (P_1) admits the following variational formulation:

$$(PV)\begin{cases} \text{Find } u \in W \text{ such that } u - \gamma \in L^2(0,\ T;\ H^1_{\Gamma_D}(\Omega)) \text{ and} \\ (\frac{\partial u}{\partial t},\ \phi) + a(u,\ \phi) = (f,\ \phi) + \delta(\int_0^t u(s)ds,\ \phi).\ \forall \phi \in H^1_{\Gamma_D}(\Omega), \text{ for a.e. } t \in]0,\ T[, \\ u(.,\ 0) = u_0 \text{ a.e. in } \Omega. \end{cases}$$

5.2. Existence and uniqueness of problem (P_1)

5.2.1. Uniqueness

Lemma 5.1. *Under the assumptions* (A_1)-(A_5), *the problem* (P_1) *admits at most one solution.*

Proof. Supposing that u_1 and u_2 are two solutions of problem (P_1), we get

$$(\frac{\partial u_1}{\partial t},\ \phi) + a(u_1,\ \phi) = (f,\ \phi) + \delta(\int_0^t u_1(s)ds,\ \phi),\ \forall \phi \in H^1_{\Gamma_D}(\Omega), \qquad (a)$$

$$(\frac{\partial u_2}{\partial t},\ \phi) + a(u_2,\ \phi) = (f,\ \phi) + \delta(\int_0^t u_2(s)ds,\ \phi),\ \forall \phi \in H^1_{\Gamma_D}(\Omega). \qquad (b)$$

Denoting by $\omega = u_1 - u_2$ and subtracting (b) from (a), and by taking $\phi = \omega$ and integrating over $]0,\ t[$, we derive

$$\int_0^t (\frac{\partial w}{\partial t},\ w) + \int_0^t a(w(s),\ w(s))ds = \delta \int_0^t (w(s),\ \int_0^s w(\tau)d\tau)ds.$$

In view of the definition of $a(.,\ .)$, and the fact that $w(0) = 0$, we obtain:

$$\frac{1}{2}\|w(t)\|^2 + \int_0^t \|D^{\frac{1}{2}}\nabla w(s)\|^2 ds + \int_0^t \int_\Omega (r(x) + \operatorname{div}(\vec{v}))w^2(s)dsdx \le$$

$$\delta \int_0^t (w(s),\ \int_0^s w(\tau)d\tau)ds + \int_0^t \int_\Omega |\vec{v}.\nabla w||w|dsdx,$$

owing to assumptions (A_1)-(A_5) and using the Young's inequality for $\varepsilon > 0$, we get

$$\frac{1}{2}\|w(t)\|^2 + (D_{\min} - \frac{\|\vec{v}\|_\infty}{2}\varepsilon)\int_0^t \|\nabla w(s)\|^2 ds \le \frac{1}{2}(\delta T + \frac{\|\vec{v}\|_\infty}{\varepsilon})\int_0^t \|w(s)\|^2 ds.$$

By taking $\psi(t) = \|w(t)\|^2$, multiplying by 2 the previous inequation, and choosing $\varepsilon = D_{\min}\|\vec{v}\|_\infty^{-1}$, we have

$$\psi(t) \le (\delta T + D_{\min}^{-1}\|\vec{v}\|_\infty^2)\int_0^t \psi(s)ds, \ \forall t \in]0, T]; \ \psi(0) = 0.$$

Subsequently, by applying the Gronwall's lemma, we get $\psi = 0$. Therefore $u_1 = u_2$, which yields the result.

5.2.2. Existence

We shall prove the existence of a weak solution of (P_1) by using the Galerkin method. We first begin by constructing a dense subspace family $\{V_m\}_{m>0}$ in $H^1_{\Gamma_D}(\Omega)$, following this, we prove the existence of a solution sequence $\{u_m - \gamma\}_{m>0} \in V_m$ of problem (P_1) by replacing the space $H^1_{\Gamma_D}(\Omega)$ with V_m. Finally, we take the limits as $m \to +\infty$. In fact, from (PV), we get

$$(\frac{\partial(u-\gamma)}{\partial t}, \phi) + a(u-\gamma, \phi) = (f, \phi) + \delta(\int_0^t (u-\gamma)(s)ds, \phi) - (\frac{\partial \gamma}{\partial t}, \phi) -$$

$$a(\gamma, \phi) + \delta(\int_0^t \gamma(s)ds, \phi), \ \forall \phi \in V = H^1_{\Gamma_D}(\Omega).$$

Let $\{V_m\}_{m>0}$ be a sequence of the space V with finite dimension, such that $\bigcup_m V_m$ is dense in V. One defines the following problem,

$$(P_m)\begin{cases} \text{Find } u_m - \gamma \in V_m \text{ such that} \\ (\frac{\partial(u_m-\gamma)}{\partial t}, \phi) + a(u_m - \gamma, \phi) - \delta(\int_0^t (u_m-\gamma)(s)ds, \phi) = (F(t), \phi) \ \forall \phi \in V, \\ (u_m - \gamma)(0) = \pi_m(u_0 - \gamma(0)), \end{cases}$$

where $(F(t), \phi) = (f, \phi) - (\frac{\partial \gamma}{\partial t}, \phi) - a(\gamma, \phi) + \delta(\int_0^t \gamma(s)ds, \phi)$, and π_m is the projection operator of $L^2(\Omega)$ onto V_m, $((Id - \pi_m)\psi, \phi) = 0 \; \forall \psi \in L^2(\Omega)$, $\forall \phi \in V_m$.

Theorem 5.1. *Supposing that* $F \in \mathcal{C}^0([0, T]; L^2(\Omega))$, *then for all* $m > 0$ *there exists a unique solution* $u_m - \gamma \in \mathcal{C}^1([0, t]; V_m)$ *of problem* (P_m). *Furthermore, under the assumptions* (A_1)-(A_5), *there exist two nonnegative constants* c_T *and* c'_T *depending on* $\|\vec{v}\|_\infty$, $\|\mathrm{div}(\vec{v})\|_\infty$, $\|\gamma\|_{H^1(0,T)}$, $\|u_0\|_{H^1(0,T)}$ *and* $\|f\|$, *and not depending on* m, *such that*

$$\|u_m\|_W \leq c_T \text{ and } \|u_m\|_{L^\infty(0,T;H^1(\Omega))} \leq c'_T.$$

Proof. We split the proof in two steps.

Step 1: Existence and uniqueness of problem (P_m)

We have, $(u_m - \gamma)(t) = \sum_{k=1}^{m} gk(t)\omega_k$, with $\{\omega_k\}_{k=1,\ldots,m}$ be a basis of V_m. Let us introduce the following notations, $g(t) = (g_1(t), \ldots, g_m(t))^t$; $g_0 = (\pi_m(u_0 - \gamma(0)), \ldots, \pi_m(u_0 - \gamma(0))_m)^t$, $F_j(t) = (F(t), w_j)$ and $\mathcal{F}(t) = (F_1(t), \ldots, F_m(t))^t$. We denote by A a square matrix of order $m \times m$ whose elements are given by, $(A)_{ij} = (D\nabla\omega_j, \nabla\omega_i) + (\mathrm{div}(\vec{v}\omega_j), \omega_i)$. Therefore, we rewrite the problem (P_m) as :

Find $g : [0, T] \to \mathbf{R}^m$ verifying

$$\begin{cases} \dfrac{dg(t)}{dt} = -Ag(t) + \delta\displaystyle\int_0^t g(s)ds + \mathcal{F}(t), \; t > 0, \\ g(0) = g_0. \end{cases}$$

Then, there exists a unique solution g of class $\mathcal{C}^1([0, T]; \mathbf{R}^m)$ (see, for instance, [22]), whence the existence and uniqueness of $(u_m)_m$ is proved. It remains to prove that the sequence (u_m) is bounded.

Step 2: A priori estimates

By taking ϕ equal to $u_m - \gamma$ in (P_m), and owing to assumption (A_5), we get

$$\frac{1}{2}\frac{d}{dt}\|(u_m-\gamma)(t)\|^2 + \|D^{\frac{1}{2}}\nabla(u_m-\gamma)(t)\|^2 + \frac{\beta}{2}\|(u_m-\gamma)(t)\|^2 \le \|f\|\|u_m-\gamma\| + \frac{\delta}{2}\frac{d}{dt}\|\int_0^t (u_m-\gamma)(s)ds\|^2$$

$$-(\dot{\gamma},\, u_m-\gamma) + \delta(\int_0^t \gamma(s)ds,\, (u_m-\gamma)(t)) - ((\operatorname{div}(\vec{v})+r)\gamma,\, u_m-\gamma) - \int_\Omega \vec{v}.\nabla(u_m-\gamma)(u_m-\gamma)dx.$$

Setting $\Lambda_m = \frac{1}{2}\frac{d}{dt}\|(u_m-\gamma)(t)\|^2 + \|D^{\frac{1}{2}}\nabla(u_m-\gamma)(t)\|^2 + \frac{\beta}{2}\|(u_m-\gamma)(t)\|^2$.
Therefore, by applying the Young's inequality for $\varepsilon > 0$, we obtain

$$\Lambda_m \le \frac{1}{2}\left(2+\delta+\|\operatorname{div}(\vec{v})+r\|_\infty + \varepsilon\|\vec{v}\|_\infty\|D^{-\frac{1}{2}}\|_\infty\right)\|u_m-\gamma\|^2 + \frac{1}{2}\|f\|^2 + \frac{\delta}{2}\frac{d}{dt}\|\int_0^t (u_m-\gamma)(s)ds\|^2 + \frac{\delta}{2}\|\int_0^t \gamma(s)ds\|^2 + \frac{1}{2}\|\dot{\gamma}(t)\|^2 + \frac{\|\operatorname{div}(\vec{v})+r\|_\infty}{2}\|\gamma\|^2 + \frac{\|\vec{v}\|_\infty\|D^{-\frac{1}{2}}\|_\infty}{2\varepsilon}\|D^{\frac{1}{2}}\nabla(u_m-\gamma)\|^2.$$

By taking $\varepsilon = \|\vec{v}\|_\infty D^{-\frac{1}{2}}\|_\infty$, multiplying by 2 the previous inequation and integrating over $]0,\, t[$, we derive

$$2\int_0^t \Lambda_m(s)ds \le \left(2+\delta+\|\mathrm{div}(\vec{v})+r\|_\infty+\|\vec{v}\|_\infty^2\|D^{-\frac{1}{2}}\|_\infty^2\right)$$

$$\int_0^t\|(u_m-\gamma)(s)\|^2ds+T\|f\|^2+\delta\|\int_0^t(u_m-\gamma)(s)ds\|^2$$

$$+|\Omega|\left(\delta\int_0^t\left(\int_0^\tau\gamma(s)ds\right)^2d\tau+\int_0^t\dot{\gamma}^2(s)ds+\|\mathrm{div}(\vec{v})+r\|_\infty\int_0^t\gamma^2(s)ds\right)$$

$$+\int_0^t\|D^{\frac{1}{2}}\nabla(u_m-\gamma)(s)\|^2ds.$$

Let $t\in\,]0,T]$, we denote by $\widetilde{\Lambda}_m=\|\pi_m(u_0-\gamma(0))\|^2+2\int_0^t\Lambda_m(s)ds$

$-\int_0^t\|D^{\frac{1}{2}}\nabla(u_m-\gamma)(s)\|^2ds$. Then,

$$\widetilde{\Lambda}_m\le\left(2+\delta+\|\mathrm{div}(\vec{v})+r\|_\infty+\|\vec{v}\|_\infty^2\|D^{-\frac{1}{2}}\|_\infty^2\right)$$

$$\int_0^t\|(u_m-\gamma)(s)\|^2ds+\delta\|\int_0^t(u_m-\gamma)(s)ds\|^2+T\|f\|^2$$

$$+|\Omega|\left(\delta\int_0^t\left(\int_0^\tau\gamma(s)ds\right)^2d\tau+\int_0^t\dot{\gamma}^2(s)ds+\|\mathrm{div}(\vec{v})+r\|_\infty\int_0^t\gamma^2(s)ds\right)$$

$$+\|\pi_m(u_0-\gamma(0))\|^2,$$

using that $\|\pi_m(u_0-\gamma(0))\|\le\|u_0-\gamma(0)\|$ and $\|\int_0^t(u_m-\gamma)(s)ds\|^2\le T$

$\int_0^t\|(u_m-\gamma)(s)\|^2ds$, we get

$$\widetilde{\Lambda}_m\le\left(2+\delta+\|\mathrm{div}(\vec{v})+r\|_\infty+\|\vec{v}\|_\infty^2\|D^{-\frac{1}{2}}\|_\infty^2+\delta T\right)$$

$$\int_0^t \|(u_m - \gamma)(s)\|^2 ds + \|u_0 - \gamma(0)\|^2 + T\|f\|^2$$

$$+ |\Omega|(1 + \|\mathrm{div}(\vec{v}) + r\|_\infty + \delta T^2)\|\gamma\|^2_{H^1(0,\,T)}.$$

We have

$$\|(u_m - \gamma)(t)\|^2 + \int_0^t \|D^{\frac{1}{2}}\nabla(u_m - \gamma)(s)\|^2 ds$$

$$+ \beta\int_0^t \|(u_m - \gamma)(s)\|^2 ds \le \widetilde{\Lambda}_m,$$

then we get

$$\|(u_m - \gamma)(t)\|^2 \le \mu_1 + \mu_2\int_0^t \|(u_m - \gamma)(s)\|^2 ds, \tag{6}$$

where $\mu_1 = \|u_0 - \gamma(0)\|^2 + T\|f\|^2 + |\Omega|(1 + \|\mathrm{div}(\vec{v}) + r\|_\infty + \delta T^2)\|\gamma\|^2_{H^1(0,\,T)}$ and $\mu_2 = 2 + \delta + \|\mathrm{div}(\vec{v}) + r\|_\infty + \|\vec{v}\|^2_\infty \|D^{-\frac{1}{2}}\|^2_\infty + \delta T$. By applying the Gronwall's lemma, we obtain

$$\int_0^t \|(u_m - \gamma)(s)\|^2 ds \le T\mu_1 e^{T\mu_2}, \tag{7}$$

and

$$\|(u_m - \gamma)(t)\|^2 + \int_0^t \|D^{\frac{1}{2}}\nabla(u_m - \gamma)(s)\|^2 ds$$

$$+ \beta\int_0^t \|(u_m - \gamma)(s)\|^2 ds \le \mu_1 + T\mu_1\mu_2 e^{T\mu_2};\ \forall t \in]0,\, T]. \tag{8}$$

As we have,

$$\int_0^t \|D^{\frac{1}{2}}\nabla(u_m - \gamma)(s)\|^2 ds \le c_1, \tag{9}$$

and $\int_0^t \|(u_m - \gamma)(s)\|^2 \le c_2$, where $c_1 = \mu_1(1 + T\mu_2 e^{T\mu_2})$ and $c_2 = T\mu_1 e^{T\mu_2}$. Then,

$$\|u_m - \gamma\|^2_{L^2(0,T;H^1(\Omega))} \leq c_3, \tag{10}$$

where $c_3 = D_{\min}^{-1} c_1 + c_2$. This implies that $(u_m)_m$ is bounded in $L^2(0, T; H^1(\Omega))$. It remains to bound $\frac{\partial u_m}{\partial t}$ in $L^2(0, T; L^2(\Omega))$. By taking $\phi = \frac{\partial(u_m - \gamma)}{\partial t}$ in (P_m), we get

$$\begin{aligned} \| \frac{\partial(u_m - \gamma)}{\partial t} \|^2 &+ \int_\Omega D\nabla(u_m - \gamma).\nabla((u_m - \gamma)_t)dx \\ &+ \int_\Omega \mathrm{div}(\vec{v}(u_m - \gamma)).(u_m - \gamma)_t dx \\ &+ \int_\Omega r(x)(u_m - \gamma)(u_m - \gamma)_t dx \\ = \int_\Omega f(u_m - \gamma)_t dx &+ \delta(\int_0^t (u_m - \gamma)(s)ds, (u_m - \gamma)_t) \\ &- (\frac{d\gamma}{dt}, (u_m - \gamma)_t) - a(\gamma, (u_m - \gamma)_t) \\ &+ \delta(\int_0^t \gamma(s)ds, (u_m - \gamma)_t). \end{aligned}$$

Then, using the assumption (A_5) and setting $\sigma_m(t) = \| \frac{\partial(u_m - \gamma)}{\partial t} \|^2 + \frac{1}{2} \frac{d}{dt} \|D^{\frac{1}{2}} \nabla(u_m - \gamma)\|^2 + \frac{\beta}{2} \frac{d}{dt} \|u_m - \gamma\|^2$, and using the Young's inequality, this yields,

$$\begin{aligned} \sigma_m(t) \leq \frac{\varepsilon}{2}\|f\|^2 &+ \frac{3}{\varepsilon} \| \frac{\partial(u_m - \gamma)}{\partial t} \|^2 \\ &+ \frac{\varepsilon\delta^2}{2} \|\int_0^t (u_m - \gamma)ds\|^2 + \frac{\varepsilon}{2}\|\dot{\gamma}\|^2 + \frac{\varepsilon\delta^2}{2} \| \\ &\int_0^t \gamma(s)ds\|^2 + \frac{\varepsilon}{2}\|\mathrm{div}(\vec{v}) + r\|^2_\infty \|\gamma\|^2 + \end{aligned}$$

$$\frac{\varepsilon}{2}\|\vec{v}\|_\infty^2\|D^{-\frac{1}{2}}\|_\infty^2\|D^{\frac{1}{2}}\nabla(u_m-\gamma)\|^2,$$

which gives,

$$2(1-\frac{3}{\varepsilon})\|\frac{\partial(u_m-\gamma)}{\partial t}\|^2+\frac{d}{dt}\|D^{\frac{1}{2}}\nabla(u_m-\gamma)\|^2$$

$$+\beta\frac{d}{dt}\|u_m-\gamma\|^2\le\varepsilon\|f\|^2+\varepsilon\delta^2\|\int_0^t(u_m-\gamma)ds\|^2+\varepsilon\|\dot{\gamma}\|^2+$$

$$\varepsilon\delta^2\|\int_0^t\gamma(s)ds\|^2+\varepsilon\|\mathrm{div}(\vec{v})+r\|_\infty^2\|\gamma\|^2$$

$$+\varepsilon\|\vec{v}\|_\infty^2\|D^{-\frac{1}{2}}\|_\infty^2\|D^{\frac{1}{2}}\nabla(u_m-\gamma)\|^2.$$

Setting, $\Sigma_m(t)=2(1-\frac{3}{\varepsilon})\|\frac{\partial(u_m-\gamma)}{\partial t}\|^2+\frac{d}{dt}\|D^{\frac{1}{2}}\nabla(u_m-\gamma)\|^2+\beta\frac{d}{dt}\|u_m-\gamma\|^2$, and $\widetilde{\Sigma}_m(t)=\int_0^t\Sigma_m(s)ds+\|D^{\frac{1}{2}}\nabla(\pi_m(u_0-\gamma(0))\|^2+\beta\|\pi_m\ (u_0-\gamma(0))\|^2$. By using $\|\pi_m(u_0-\gamma(0))\|\le\|u_0-\gamma(0)\|$ and owing to assumption (A_1), this yields

$$\widetilde{\Sigma}_m\le(D_{\max}+\beta)\|u_0-\gamma(0)\|_{H^1(\Omega)}^2+\varepsilon T\|f\|^2+\varepsilon\delta^2T^2\int_0^T\|(u_m-\gamma)(s)^2ds$$

$$+\varepsilon|\Omega|(1+\delta^2T^2+\|\mathrm{div}(\vec{v})+r\|_\infty^2\|\gamma\|_{H^1(0,T)}^2$$

$$+\varepsilon\|\vec{v}\|_\infty^2\|D^{-\frac{1}{2}}\|_\infty^2\int_0^t\|D^{\frac{1}{2}}\nabla(u_m-\gamma)(s)\|^2ds.\ (11)$$

By choosing $\varepsilon=6$, we obtain

$$\int_0^t\|\frac{\partial(u_m-\gamma)}{\partial t}\|^2ds\le(D_{\max}+\beta)\|u_0-\gamma(0)\|_{H^1(\Omega)}^2$$

$$+6T\|f\|^2+6\delta^2T^2\int_0^T\|(u_m-\gamma)(s)\|^2ds$$

$$+6|\Omega|(1+\delta^2T^2+\|\mathrm{div}(\vec{v})+r\|_\infty^2)\|\gamma\|_{H^1(0,T)}^2$$

$$+ 6\|\vec{v}\|_\infty^2 \|D^{-\frac{1}{2}}\|_\infty^2 \int_0^t \|D^{\frac{1}{2}} \nabla(u_m - \gamma)(s)\|^2 ds.$$

So that

$$\| \frac{\partial(u_m - \gamma)}{\partial t} \|^2_{L^2(0,T;L^2(\Omega))} \leq c_4, \tag{12}$$

where from the expressions (7) and (9), we have $c_4 = (D_{\max} + \beta)\|u_0 - \gamma(0)\|^2_{H^1(\Omega)} + 6T\|f\|^2 + 6\delta^2 T^2 c_2 + 6|\Omega|(1 + \delta^2 T^2 + \|\mathrm{div}(\vec{v}) + r\|^2_\infty)\|\gamma\|^2_{H^1(0,T)} + 6\|\vec{v}\|^2_\infty D^{-1}_{\min} c_1$. Therefore, we deduce that $\frac{\partial u_m}{\partial t}$ is bounded in $L^2(0, T; L^2(\Omega))$. Then $(u_m)_m$ is bounded in W. Subsequently, there exists a subsequence denoted by $(u_m)_m$ converges weakly to $u \in W$. Since $(u_m)_m$ is bounded in $L^2(0, T; H^1(\Omega))$ and $\frac{\partial u_m}{\partial t}$ is bounded in $L^2(0, T; L^2(\Omega))$, then according to the Simon's theorem (see [23]), this implies that u_m converges strongly to u in $L^2(Q_T)(*)$. From the expressions (10) and (12), we derive $\|u_m - \gamma\|^2_W \leq c_5$, where $c_5 = c_3 + c_4$. Accordingly, we can get easily $\|u_m\|_W \leq c_T$, where $c_T = \left\{2\left(c_5 + |\Omega|\|\gamma\|^2_{H^1(0,T)}\right)\right\}^{\frac{1}{2}}$. In view of (11), we derive $\|\nabla(u_m - \gamma)\|^2 \leq D^{-1}_{\min} c_4$. So $\|u_m - \gamma\|^2_{H^1(\Omega)} \leq c_6$, where $c_6 = \mu_1 e^{T\mu_2} + D^{-1}_{\min} c_4$. Whence, we can get easily $\|u_m\|_{L^\infty(0,T;H^1(\Omega))} \leq c'_T$, where $c'_T = c_6^{\frac{1}{2}} + |\Omega|\|\gamma\|_{L^\infty(0,T)}$. This proves the estimates quoted in Theorem 5.1. Let $\phi \in V_k \subset V_m (k < m)$, one has

$$(\frac{\partial(u_m - \gamma)}{\partial t}, \phi) \underset{m \to +\infty}{\to} (\frac{\partial(u - \gamma)}{\partial t}, \phi).$$

Owing to the definition of a, we have

$$a(u_m - \gamma, \phi) = (D\nabla(u_m - \gamma), \nabla\phi) + (\mathrm{div}(\vec{v}(u_m - \gamma)), \phi) + (r(x)(u_m - \gamma), \phi).$$

According to $(*)$, we obtain

$$(r(x)(u_m - \gamma),\ \phi) \underset{m\to+\infty}{\to} (r(x)(u - \gamma),\ \phi).$$

Using that u_m is bounded in $L^2(0,\ T; H^1(\Omega))$, we get

$$(D\nabla(u_m - \gamma), \nabla\phi) \underset{m\to+\infty}{\to} (D\nabla(u - \gamma),\ \nabla\phi).$$

Since we have

$$(\mathrm{div}(\vec{v}(u_m - \gamma)),\ \phi) = (\mathrm{div}(\vec{v})(u_m - \gamma),\ \phi) + (\vec{v}.\nabla(u_m - \gamma),\ \phi),$$

whence, using $(*)$, we obtain $(\mathrm{div}(\vec{v})(u_m - \gamma),\ \phi) \underset{m\to+\infty}{\to} (\mathrm{div}(\vec{v})(u - \gamma),\ \phi)$, and using that u_m is bounded in $L^2(0,\ T; H^1(\Omega))$, we get $(\vec{v}.\nabla(u_m - \gamma),\ \phi) \underset{m\to+\infty}{\to} (\vec{v}.\nabla(u - \gamma),\ \phi)$. Finally, we have

$$a(u_m - \gamma,\ \phi) \underset{m\to+\infty}{\to} a(u - \gamma,\ \phi).$$

Using $(*)$, we get

$$\left(\int_0^t (u_m - \gamma)(s),\ \phi\right) \underset{m\to+\infty}{\to} \left(\int_0^t (u - \gamma)(s),\ \phi\right).$$

To sum up, combining all these limits, we obtain the following equation

$$\left(\frac{\partial(u - \gamma)}{\partial t},\ \phi\right) + a(u - \gamma,\ \phi) = (f,\ \phi) + \left(\int_0^t (u - \gamma)(s),\ \phi\right)$$

$$- \left(\frac{\partial \gamma}{\partial t},\ \phi\right) - a(\gamma,\ \phi) + \delta\left(\int_0^t \gamma(s)ds,\ \phi\right),\ \forall\phi \in V_k. \qquad \text{(E)}$$

It remains to prove that $u(x,\ 0) = u_0(x)\ \forall x \in \Omega$, for this reason, we prove that the sequence $(u_n)_n$ is a Cauchy's sequence in $\mathcal{C}^0([0,\ T];\ L^2(\Omega))$. In Fact, by taking ϕ equal to $u_m - u_n$ in (P_m), we derive

$$\left(\frac{\partial(u_m - \gamma)}{\partial t},\ u_m - u_n\right) + a(u_m - \gamma,\ u_m - u_n) = (f,\ u_m - u_n) +$$

$$(\int_0^t (u_m - \gamma)(s),\, u_m - u_n) - (\frac{\partial \gamma}{\partial t},\, u_m - u_n)$$

$$- a(\gamma,\, u_m - u_n) + \delta(\int_0^t \gamma(s)ds,\, u_m - u_n), \tag{13}$$

likewise, we get

$$(\frac{\partial (u_n - \gamma)}{\partial t},\, u_m - u_n) + a(u_n - \gamma,\, u_m - u_n)$$

$$= (f,\, u_m - u_n) + \delta\left(\int_0^t (u_n - \gamma)(s), u_m - u_n\right)$$

$$- (\frac{\partial \gamma}{\partial t},\, u_m - u_n) - a(\gamma,\, u_m - u_n) + \delta\left(\int_0^t \gamma(s)ds,\, u_m - u_n\right). \tag{14}$$

By subtracting (14) from (13), we have

$$(\frac{\partial (u_m - u_n)}{\partial t},\, u_m - u_n) + a(u_m - u_n,\, u_m - u_n)$$

$$= \delta\left(\int_0^t (u_m - u_n)(s)ds,\, u_m - u_n\right).$$

Then, owing to assumption (A_5) and by applying Young's inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|u_m - u_n\|^2 + \|D^{\frac{1}{2}}\nabla(u_m - u_n)\|^2 + \frac{\beta}{2}\|u_m - u_n\|^2 \leq$$

$$\frac{\delta}{2}\frac{d}{dt}\|\int_0^t (u_m - u_n)(s)ds\|^2 + \frac{1}{2}\|D^{\frac{1}{2}}\nabla(u_m - u_n)\|^2 + \frac{\|\vec{v}\|_\infty^2 \|D^{-\frac{1}{2}}\|_\infty^2}{2}\|u_m - u_n\|^2,$$

by integrating the previous inequation on the interval $]0,\, t[$, we derive

$$\frac{1}{2}\|(u_m - u_n)(t)\|^2 + \frac{1}{2}\int_0^t \|D^{\frac{1}{2}}\nabla(u_m - u_n)(s)\|^2 ds$$

$$\leq \frac{1}{2}\|(u_m - u_n)(0)\|^2 + \frac{1}{2}(\delta T + \|\vec{v}\|_\infty^2 \|D^{-\frac{1}{2}}\|_\infty^2)$$

$$\int_0^t \|(u_m - u_n)(s)\|^2 ds.$$

Whence, $\|(u_m - u_n)(t)\|^2 \leq \|(u_m - u_n)(0)\|^2 + (\delta T + \|\vec{v}\|_\infty^2 \|D^{-\frac{1}{2}}\|_\infty^2) \int_0^t \|(u_m - u_n)(s)\|^2 ds$. Owing to assumption (A_1), we obtain $\|(u_m - u_n)(t)\|^2(t) \leq \mu_1 + \mu_2 \int_0^t \|(u_m - u_n)(t)\|^2(s) ds$, where $\mu_1 = \|(u_m - u_n)(0)\|^2$, and $\mu_2 = \delta T + \|\vec{v}\|_\infty^2 D_{\min}^{-1}$. Therefore by applying Gronwall's lemma, we get $\|(u_m - u_n)(t)\|^2(t) \leq \mu_1 e^{\mu_2 T} \forall t \in [0, T]$. Whence,

$$\sup_{[0,T]} \|(u_n - u_m)(t)\|_{L^2} \leq \mu_1(n, m) c_7,$$

where $c_7 = e^{T\mu_2}$. Since $\mu_1(n, m) \to 0$ as $n, m \to +\infty$, then $(u_n)_n$ is a Cauchy's sequence in $\mathcal{C}^0([0, T]; L^2(\Omega))$. Whence, $u_n \underset{m \to +\infty}{\to} u$ in $\mathcal{C}^0([0, T]; L^2(\Omega))$ and $u_n(0) \underset{m \to +\infty}{\to} u(0)$ in $L^2(\Omega)$. Since, we have $(u_m - \gamma)(0) = \pi_m(u_0 - \gamma(0))$, then, for $m \to +\infty$, we deduce $u(0) = u_0$. Moreover, we have proved that $u \in \mathcal{C}^0([0, T], L^2(\Omega))$. By density we deduce that (E) is satisfied for all ϕ in $H^1_{\Gamma_D}(\Omega)$. To conclude, combining between all previous results, this leads to the following theorem.

Theorem 5.2. *Under the assumptions* (A_1) - (A_5), *problem* (P_1) *has a unique solution* $u \in W \cap \mathcal{C}^0([0, T]; L^2(\Omega))$ *such that* $u - \gamma \in L^2(0, T; H^1_{\Gamma_D}(\Omega))$, *in the following sense*:

$$\left(\frac{\partial u}{\partial t}, \phi\right) + a(u, \phi) = (f, \phi) + \delta\left(\int_0^t u(s) ds, \phi\right), \ \forall \phi \in H^1_{\Gamma_D}(\Omega), \text{ for a.e. } t \in]0, T[,$$

$$u(., 0) = u_0 \text{ a.e. in } \Omega.$$

Furthermore, we have the following estimates

$$\|u\|_W \leq c_T \text{ and } \|u\|_{L^\infty(0,T; H^1(\Omega))} \leq c'_T,$$

where the constants c_T *and* c'_T *are specified in Theorem* 5.1. *As the right-hand side of the first equation of problem* (P_1) *in* $\mathcal{C}^1(]0, T]; L^2(\Omega))$, *then we have* $u \in \mathcal{C}^1(]0, T]; L^2(\Omega))$.

Remark 1. The a priori estimates are directly deduced from ones quoted in Theorem 5.1, by passing to limits.

6. Inverse Problem

In [1] the authors proved the existence result by using FredHolm alternative in pondered spaces. In this work we prove the existence result by using FredHolm alternative in nonpondered spaces, moreover, we propose to reformulate the inverse problem like an optimization problem in least squares sense, and we prove the existence result in some admissible space. This method is very easy for the computation of the initial concentration *f*.

6.1. Existence of a solution of the inverse problem by FredHolm alternative

In order to prove the existence of a solution of the inverse problem (*P*), we decompose the solution of the problem (P_1) by the following manner. Let u^0 be the solution of the problem (P_1) with $f = 0$ and its initial and boundary conditions are different to zero (so that, $u_0 \neq 0$, $\gamma \neq 0$). A part of the Theorem 5.2 gives directly the existence and uniqueness of the solution u^0. Then we write the weak solution of the problem (P_1) with general data, f, γ, and u_0, as

$$u(., f) = u^0 + u^0(., f), \tag{15}$$

where $u^0(., f)$ is the solution of the problem (P_1) with $f \neq 0$ and its initial and boundary conditions are equal to zero (so that, $u_0 = 0$, $\gamma = 0$). Henceforth, we shall write $u(., f)$ instead of $u^0(., f)$ and h_T instead of $h_T - u^0(T)$. We define the following differential operator $\mathcal{A}_0$ on W_0 by $\mathcal{A}_0 u = -\text{div}(D\nabla u) + \vec{v}.\nabla u + k_s u$. Next,

we introduce the functional spaces IH and IH_0 defined as $IH = \{\phi \in H^2(\Omega) : D\nabla\phi.\vec{n} = 0 \text{ on } \Gamma_2,\ \phi = 0 \text{ on } \Gamma_1\}$; $IH_0 = L^2(\Omega)$. Let us define also the following operator $G : IH_0 \to IH_0$, as $f \to G(f) = \dfrac{\partial u(T, f)}{\partial t} + (\operatorname{div}(\vec{v}) - \alpha K_H)u(T, f) - \alpha k_s K_H \displaystyle\int_0^T u(s, f)ds$. By taking t equals to T in the first equation of problem (P_1), and taking into account the condition at the final time T, we obtain the equation that the function f has to satisfy

$$f - G(f) = \mathcal{A}_0 h_T \text{ on } IH_0. \tag{16}$$

Lemma 6.1. *Assuming that the hypotheses* (A_1)-(A_4) *are satisfied, and let* $h_T \in IH$, *then the problem* (P) *and the equation* (16) *are equivalent.*

Proof. We have from the Theorem 5.2 that $u^0 \in \mathcal{C}^0([0, T]; H^2(\Omega))$ then $h_T - u^0(T) \in IH$ if $h_T - \gamma(T) \in IH$. Let $f \in IH_0$ be a solution of the equation (16), then in view of the definition of G, we have

$$f - G(f) = f - \left[\frac{\partial u(T, f)}{\partial t} + (\operatorname{div}(\vec{v}) - \alpha K_H)u(T, f) - \alpha k_s K_H \int_0^T u(s, f)ds\right], \tag{17}$$

where $u(., f)$ is a solution of problem (P_1). Theorem 5.2 affirms that $u(., f) \in \mathcal{C}^0([0, T]; H^2(\Omega))$, hence $\mathcal{A}_0 u(T, f)$ is in $L^2(\Omega)$. We deduce from (16) that

$$f - G(f) = \mathcal{A}_0 u(T, f) \tag{18}$$

We combine (16), (17) and (18), we obtain

$$\mathcal{A}_0(h_T - u(T, f)) = 0 \text{ in } \Omega. \tag{19}$$

The boundary conditions of the function $h_T - u(T, f)$ are equal to zero, namely, $h_T - u(T, f) = 0$ on Γ_1; $D\nabla(h_T - u(T, f)).\vec{n} = 0$ on Γ_2. By multiplying (19) by $h_T - u(T, f)$ and integrating over Ω, we derive

$$\int_\Omega D\nabla(h_T - u(T, f)).\nabla(h_T - u(T, f))dx$$

$$+ k_s\int_\Omega (h_T - u(T, f))^2 dx =$$

$$- \int_\Omega \vec{v}.\nabla(h_T - u(T, f))(h_T - u(T, f))dx.$$

Then, owing to assumption (A_1) and applying the Young's inequality for $\varepsilon > 0$, we get

$$\left(D_{\min} - \frac{\|\vec{v}\|_\infty^2}{2}\varepsilon\right)\|\nabla(h_T - u(T, f))\|^2 + \left(k_s - \frac{1}{2\varepsilon}\right)\|h_T - u(T, f)\|^2 \leq 0.$$

Therefore, if the following condition $\|\vec{v}\|_\infty^2 \leq 4k_sD_{\min}$ is satisfied, we obtain $h_T = u(T, f)$. This proves that if $f \in IH_0$ satisfies the equation (16), then $(u(., f), f)$ is solution of problem (P). The reverse implication is obvious.

Lemma 6.2. *Assuming that the hypotheses* (A_1)-(A_5) *are satisfied, then the operator G is compact.*

Proof. Let $f \in IH_0$, then by virtue of Theorem 5.2, we obtain $u(., f) \in \mathcal{C}^1([0, T]; L^2(\Omega))$. We denote by $\omega = \frac{d}{dt}u(., f)$, a direct calculation gives that ω verifies

$$(P_2)\begin{cases}\frac{\partial\omega}{\partial t} + \mathcal{A}w = \alpha k_s K_H\int_0^t \omega(s)ds \text{ in } Q_T,\\ \omega(x, t) = 0 \quad \text{on } \Gamma_1 \times J,\\ D\nabla\omega.\vec{n} = 0 \quad \text{on } \Gamma_2 \times J,\\ \omega(x, 0) = f(x) \quad \text{in} \quad \Omega,\end{cases}$$

where the differential operator $\mathcal{A}$, is defined as $\mathcal{A}\omega = -\text{div}(D\nabla\omega - \vec{v}\omega) + r(x)\omega$. For a given f in IH_0, the existence of a weak solution of problem (P_2) is directly deduced from Theorem 5.2. By replacing $u_m - \gamma$ with $u(., f)$ in (6), we derive $\|u(t; f)\|^2 \leq \mu_1 + \mu_2\int_0^t \|u(s; f)\|^2 ds$. The Gronwall's lemma leads to

$$\int_0^t \|u(s;\, f)\|^2 ds \le T\mu_1 e^{T\mu_2}. \tag{20}$$

Owing to assumption (A_5) we derive also,

$$\|u(t;\, f)\|^2 + \int_0^t \|D^{\frac{1}{2}}\nabla u(s;\, f)\|^2 ds + \beta\int_0^t \|u(s;\, f)\|^2 ds \le \mu_1 + T\mu_1\mu_2 e^{T\mu_2}. \tag{21}$$

Then $\|u(t;\, f)\|^2 \le T\|f\|^2 + T^2\|f\|^2_{\mu_2} e^{T\mu_2}$. Whence,

$$\|u(t;\, f)\|^2 \le c_8\|f\|^2, \tag{22}$$

with $c_8 = T + T^2\mu_2 e^{T\mu_2}$. Owing to (20), we get

$$\int_0^t \|u(s;\, f)\|^2 ds \le c_9\|f\|^2, \tag{23}$$

where $c_9 = T^2 e^{T\mu_2}$. Therefore, we obtain,

$$\|\int_0^T u(s;\, f)ds\|^2 \le c_{10}\|f\|^2, \tag{24}$$

where $c_{10} = Tc_9$. By multiplying the first equation of problem (P_2) by ω, and integrating over Ω, we derive

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|^2 + \|D^{\frac{1}{2}}\nabla\omega(t)\|^2 + \int_\Omega (\mathrm{div}(\vec{v}) + r)\omega^2 dx$$

$$+ \int_\Omega \omega\vec{v}.\nabla\omega dx = \frac{\delta}{2}\frac{d}{dt}\|\int_0^t \omega(s)ds\|^2.$$

Owing to assumption (A_5), and integrating over $]0, t[$ and using Young's inequality, we obtain

$$\frac{1}{2}\|\omega(t)\|^2 + \frac{1}{2}\int_0^t \|D^{\frac{1}{2}}\nabla\omega(s)\|^2 ds + \frac{\beta}{2}$$

$$\int_0^t \|\omega(s)\|^2 ds \le \frac{1}{2}\|f\|^2 + \frac{1}{2}(\delta T$$

$$+ \|\vec{v}\|^2_\infty \|D^{-\frac{1}{2}}\|^2_\infty)\int^t \|\omega(s)\|^2 ds. \tag{25}$$

Therefore, we derive $\|\omega(t)\|^2 \leq \|f\|^2 + \left(\delta T + \|\vec{v}\|_\infty^2 \|D^{-\frac{1}{2}}\|_\infty^2\right)\int_0^t \|\omega(s)\|^2 ds$. By applying the Gronwall's lemma, we obtain

$$\|\omega(t)\|^2 \leq c_{11}\|f\|^2, \tag{26}$$

where $c_{11} = e^{T(\delta T + \|\vec{v}\|_\infty^2 D_{\min}^{-1})}$. Since, we have $\|G(f)\| \leq \|\omega(t)\| + (\|\mathrm{div}(\vec{v})\|_\infty + \alpha K_H)\|u(T, f)\| + \alpha k_s K_H \|\int_0^T u(s; f) ds\|$, combining (22), (24) and (26), we derive $\|G(f)\| \leq c_{12}\|f\|$, where $c_{12} = c_{11}^{\frac{1}{2}} + (\|\mathrm{div}\ (\vec{v})\|_\infty + \alpha K_H) c_8^{\frac{1}{2}} + c_{10}^{\frac{1}{2}}$. On the other hand, we have $\|\int_0^T \nabla u(s, f) ds\|^2 \leq T\int_0^T \|\nabla u(s, f)\|^2 ds$. In virtue of (21), we get

$$\int_0^t \|D^{\frac{1}{2}} \nabla u(s; f)\|^2 ds \leq \mu_1 + T\mu_1\mu_2 e^{T\mu_2}. \tag{27}$$

Then, owing to assumption (A_1), we obtain $\int_0^t \|\nabla u(t; f)\|^2 ds \leq c_{13}\|f\|^2$, with $c_{13} = D_{\min}^{-1} T(1 + T\mu_2 e^{T\mu_2})$. This yields,

$$\|\int_0^T \nabla u(t, f) ds\|^2 \leq c_{14}\|f\|^2, \tag{28}$$

where $c_{14} = Tc_{13}$. By multiplying the first equation of problem (P_1) by $\frac{\partial u(., f)}{\partial t}$ and using same techniques quoted in order to bound $\frac{\partial u_m}{\partial t}$, we deduce from (11) for $\varepsilon = 6$,

$$\|D^{\frac{1}{2}} \nabla u(s; f)\|^2 \leq 6T\|f\|^2 + 6\delta^2 T^2 \int_0^T \|u(s; f)\|^2 ds + 6\|\vec{v}\|_\infty^2 \|D^{-\frac{1}{2}}\|_\infty^2 \int_0^T \|D^{\frac{1}{2}} \nabla u(s; f)\|^2 ds.$$

Then, owing to inequalities (23), (27) and assumption (A_1) we obtain

$$\|\nabla u(s, f)\|^2 \leq c_{15}\|f\|^2, \tag{29}$$

where $c_{15} = TD_{\min}^{-1}(7 + 6\delta^2 Tc_9 + T\mu_2 e^{T\mu_2})$. From (25), we deduce $\int_0^T \|\nabla\omega(s)\|^2 ds \leq c_{17}\|f\|^2$, with $c_{17} = D_{\min}^{-1} + D_{\min}^{-1}(\delta T + \|\vec{v}\|^2 \quad D_{\min}^{-1}c_{11}T)$.

This implies that, $\|\nabla\omega(T)\| \leq c_{18}$ a.e. This yields that $\omega(T) \in H^1(\Omega)$ and we get $\|\omega(T)\|_{H^1(\Omega)} \leq c_{19}$. Finally, we obtain

$$\|G(f)\|_{H^1(\Omega)} \leq c_{20}. \tag{30}$$

Since the injection of $H^1(\Omega)$ in $L^2(\Omega)$ is compact, then by virtue of (30), there exists a subsequence still denoted by $G(f_n)$ which converges in $L^2(\Omega)$, therefore this yields the compacity of G in IH_0. Subsequently, from the Lemma 6.1, we have the equivalence between problem (P) and equation (16) in IH_0 for each $h_T \in IH$. Lemma 6.2 affirms that G is a compact operator from IH_0 onto IH_0 and we clearly have $\mathcal{A}_0 h_T \in IH_0$. The fredHolm alternative (see [8]) is valid and the uniqueness of the solution implies the existence of (u, f) solution of problem (P).

6.2. Solution for the inverse problem by optimization method

In order to compute the initial concentration f solution of the inverse problem, we consider the minimization problem

$$(\mathcal{P}) : \min_{f \in \mathcal{U}_{ad}} \mathcal{J}(f) = J(f, u(f)), \tag{31}$$

where $J(f, u(f)) = \frac{1}{2}\int_\Omega |u(x, T; f) - h_T(x)|^2 dx$, and $u(f)$ is the solution of the problem (P_1) and $\mathcal{U}_{ad}$ is given by $\mathcal{U}_{ad} = \{f \in L^2(\Omega) : \|f\| \leq c;\ c$ is a nonnegative constant$\}$.

Theorem 6.1. *Under the Assumptions* (A_1)-(A_3), *the minimization problem* $(\mathcal{P})$ *has at least one solution* $f \in \mathcal{U}_{ad}$.

Proof. Let $(f_n)_n$ be a minimizing sequence of the energy $\mathcal{J}(f_n) = \frac{1}{2}$ $\int_\Omega |u_n(x, T; f_n) - h_T(x)|^2 dx$, where $(u_n)_n$ is a solution of problem (P_1) and $f_n \in \mathcal{U}_{ad}$. Since $(u_n)_n$ is a solution of problem (P_1), then we have

$$\frac{\partial u_n}{\partial t} - \operatorname{div}(D(x, \nabla u_n)\nabla u_n - \vec{v}u_n) + r(x)u_n = f_n + \alpha K_H \int_0^t u_n(s)ds.$$

As $f_n \in \mathcal{U}_{ad}$, then $(f_n)_n$ is bounded in $L^2(\Omega)$, so there exists a subsequence still denoted $(f_n)_n$ converges weakly to f^*. From Theorem 5.2, we have the following estimate $\|u_n(x, T; f_n)\|_W \le c_T$. Then

$$u_n(x, T; f_n) \underset{n\to+\infty}{\to} u(x, T; f^*) \text{ in } L^2(\Omega).$$

According to the Lebesgue's theorem, we deduce

$$\frac{1}{2}\int_\Omega |u_n(x, T; f_n) - h_T(x)|^2 dx \underset{n\to+\infty}{\to} \frac{1}{2}\int_\Omega |u(x, T; f^*) - h_T(x)|^2 dx,$$

then f^* minimizes $\mathcal{J}(f)$ on the set $\mathcal{U}_{ad}$. Thus, we can use the gradient method to compute the initial concentration f^*.

7. Conclusion

In this paper we have studied an inverse problem permitting to compute an initial concentration of some pollution problem in porous media. We have proved the existence and uniqueness of the direct problem by using the Galerkin method in nonpondered spaces. Furthermore, we have proved the existence result for the inverse problem (*P*) by using FredHom alternative in nonponderd spaces and we have also reformulated the inverse problem (*P*) like an optimization problem which permits to compute the initial concentration *f*.

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