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CONSTANT CURVED HOLOMORPHIC 2-SPHERES IN $G(2, 4)$

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Abstract

A complete classification for holomorphic 2-spheres with constant Gaussian curvature immersed in the complex Grassmann manifold $G(2, 4)$ is given.

Preliminary

It is well-known that, up to a $U(n+1)$-equivalence, constant curved holomorphic 2-spheres in the complex projective space CP^n can be expressed by C^n-valued holomorphic polynomial as follows:

$$V_n|_C = \pi_1 \circ f = \pi_1 \circ (1, \sqrt{n}z, \ldots, \sqrt{\binom{n}{j}}z^j, \ldots, z^n), \qquad (0.1)$$

where $V_n : S^2 = C \cup \{\infty\} \to CP^n$ is the Veronese embedding,

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$\pi_1 : C^{n+1} \setminus \{0\} \to CP^n$ is the natural projective ([1]). It is clear that $|f|^2 = (1 + z\bar{z})^n$ and the induced metric is

$$ds^2 = \frac{n}{(1 + z\bar{z})^2} dzd\bar{z} = \partial\bar{\partial} \ln|f|^2 dzd\bar{z} \tag{0.2}$$

on C, where $\partial = \partial / \partial z$, $\bar{\partial} = \partial / \partial\bar{z}$. As a generalization of $CP^n = G(1, n + 1)$, we may intend to give an analytic expression for holomorphic 2-spheres with constant Gaussian curvature in the complex Grassmann manifold $G(k, n)$. In the simplest situation, we proved the following theorem.

Theorem A. *Let* $\varphi : S^2 \to G(2, 4)$ *be a holomorphic immersion with constant (induced) Gaussian curvature K. Set* $S^2 = C \cup (\infty)$. *Then* φ *is* $U(4)$*-equivalent to one of the following curves, at most up to an isometry of* S^2.

(1) $K = 4$, $\varphi|_C = \pi \circ \begin{pmatrix} 1 & 0 & z & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$;

(2) $K = 2$, $\varphi|_C = \pi \circ \begin{pmatrix} 1 & 0 & \cos 2t\ z^2 & \sqrt{2} \sin t\ z \\ 0 & 1 & \sqrt{2} \cos t\ z & 0 \end{pmatrix}$, *where* $t \in R$;

(3) $K = 4 / 3$, $\varphi|_C = \pi \circ \begin{pmatrix} 1 & 0 & \sqrt{3}z^2 & 0 \\ 0 & 1 & \sqrt{8/3}z & \sqrt{1/3}z \end{pmatrix}$;

(4) $K = 1$, $\varphi|_C = \pi \circ \begin{pmatrix} 1 & 0 & 2z^3 & \sqrt{3}z^2 \\ 0 & 1 & \sqrt{3}z^2 & 2z \end{pmatrix}$.

This theorem is provided by the first author in [8] without proof. In this note we will give a complete proof. We think that it may be helpful for solving the above problem in general $G(k, n)$. All conceptions and notations inherit those in [7] and [8].

1. General Results

Let $M(k, n)$ be the complex vector space consists of all $k \times n$ matrices. We consider $G(k, n)$ as the orbit space $\mathfrak{M}(k, n)/GL(k, C)$, where $\mathfrak{M}(k, n)$ is he set of all complex matrices with rank k and the complex general linear group $GL(k, C)$ acts on $\mathfrak{M}(k, n)$ by left multiplication. Denote by $\pi : \mathfrak{M}(k, n) \to G(k, n)$ the natural projection.

If $\varphi : S^2 \to G(k, n)$ is a holomorphic isometric immersion with constant Gaussian curvature K, then we have a canonical coordinate z on $S^2 \setminus \{\infty\}$ such that the induced metric is given by

$$ds^2 = \frac{4}{K(1 + z\bar{z})^2} dzd\bar{z}. \tag{1.1}$$

Suppose the associated harmonic (bundle) sequence of φ is

$$\underline{\varphi} = \underline{\varphi}_0 \xrightarrow{\partial} \underline{\varphi}_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \underline{\varphi}_m \xrightarrow{\partial} \underline{0}. \tag{1.2}$$

According to [7] we have a set of C^n– valued holomorphic polynomial $\{f_1, \ldots, f_k\}$ such that $f_1 \wedge \cdots \wedge f_k$ has no zero and

$$\varphi|_C = \pi \circ A(f_1, \ldots, f_k),$$

where $A(f_1, \ldots, f_k)$ is the $M(k, n)$- valued holomorphic polynomial with f_j the j-th row. Since the Plücker embedding $\mathrm{Pl} : G(k, n) \to CP^{N-1}$ $: \pi(A(\overline{u}_1, \ldots, u_k)) \mapsto \pi_1(u_1 \wedge \ldots \wedge u_k)$ $(N = \binom{n}{k})$ is a holomorphic isometry, up to a multiple we may set

$$|f_1 \wedge \ldots \wedge f_k|^2 = (1 + z\bar{z})^d, \tag{1.3}$$

where $d = \deg(\varphi)$ is the degree of φ, which is equal to the degree of $\wedge^k C^n$– valued holomorphic polynomial $f_1 \wedge \ldots \wedge f_k$, and $d = 4/K$ by (0.2) and (1.1).

Suppose

$$f_j(z)=\sum_{l=0}^{d_j}u_{jl}z^l \quad (d_j=\deg(f_j)), \tag{1.4}$$

where $u_{jl}\in C^n$. If we set

$$f_1\wedge\ldots\wedge f_k=\sum_{j=0}^{d}v_jz^j, \tag{1.5}$$

then

$$\langle v_j,\bar{v}_l\rangle=\binom{d}{j}\delta_{jl} \tag{1.6}$$

by (0.1), where

$$v_j=\sum_{l_1+\cdots+l_k=j}u_{1l_1}\wedge\cdots\wedge u_{kl_k}\in\wedge^kC^n.$$

Thus to solve the problem proposed in section 1, is equivalent to solving the algebraic equations (1.6).

When rank $(\underline{\varphi}_1)=1$ we have

Proposition 1. *Suppose* φ *is full and* $\underline{\varphi}$ *does not contain any proper trivial subbundle. If* rank $(\underline{\varphi}_1)=1$, *then* $\underline{\varphi}=\underline{\gamma}\oplus\underline{\gamma}_1\oplus\cdots\oplus\underline{\gamma}_{k-1}$, *where* $\gamma=V_{n-1}$ *is he Veronese embedding in (0.1), and* $\underline{\gamma}_j$ *is the j-th member in the harmonic sequence of* V_{n-1}.

Proof. Consider the $\bar{\partial}$-transformation of $\underline{\varphi}_1$ [3]:

$$\underline{0}\xleftarrow{\bar{\partial}}\underline{\gamma}=\underline{\psi}_l\xleftarrow{\bar{\partial}}\cdots\xleftarrow{\bar{\partial}}\underline{\psi}_1\xleftarrow{\bar{\partial}}\underline{\varphi}_1.$$

Then we have an orthogonal decomposition $\underline{\varphi}=\underline{\alpha}\oplus\underline{\psi}_1\oplus\cdots\oplus\underline{\psi}_l$. Since the holomorphic subbundle $\underline{\alpha}$ of $\underline{\varphi}$ is holomorphic in $\underline{C}^n$, and is anti-holomorphic in $\underline{C}^n$ by $\underline{\alpha}\subset\underline{\mathrm{Ker}}(\partial_\varphi)$, $\underline{\alpha}$ is a trivial subbundle of $\underline{\varphi}$. By assumption rank $(\underline{\alpha})=0$. Thus $\underline{\varphi}=\underline{\gamma}\oplus\underline{\gamma}_1\oplus\cdots\oplus\underline{\gamma}_{k-1}$, where

$\gamma = \psi_l : S^2 \to CP^{n-1}$ is holomorphic, and $\underline{\gamma}_j$ is the j-th member in the harmonic sequence of ∂-transformation of γ.

Take C^n-valued holomorphic polynomial g such that $\gamma = \pi o g$. Then by using Lemma 3 of [6] we get a set of C^n-valued holomorphic polynomial $\{g = g_0, g_1, \ldots, g_{k-1}\}$ such that $\underline{\gamma} \oplus \cdots \oplus \underline{\gamma}_j$ is spanned by $\{g_0, \ldots, g_j\}$ for $j = 0, 1, \ldots, k-1$ and $g_0 \wedge \cdots \wedge g_{k-1}$ has no zero. By definition of ∂-transformation we have

$$g_0 \wedge \cdots \wedge g_{j-1} \wedge \partial g_{j-1} = p_j g_0 \wedge \cdots \wedge g_j \quad (j = 1, \ldots, k-1)$$

for some holomorphic polynomial p_j. Hence

$$\partial\bar{\partial} \ln|g_0 \wedge \cdots \wedge g_{k-1}|^2 = \frac{4}{K(1 + z\bar{z})^2} = \frac{d}{(1 + z\bar{z})^2} = \partial\bar{\partial} \ln(1 + z\bar{z})^d \quad (1.7)$$

by (1.1) and (1.3), where $d = \deg(\varphi)$. Denote by d_j the degree of $g_0 \wedge \cdots \wedge g_j$. (2.7) shows that $\ln \frac{|g_0 \wedge \cdots \wedge g_{k-1}|^2}{(1+z\bar{z})^d}$ is constant on C since $\frac{|g_0 \wedge \cdots \wedge g_{k-1}|^2}{(1+z\bar{z})^d}$ is bounded. Up to a suitable multiple of g_{k-1}, we may set

$$|g_0 \wedge \cdots \wedge g_{k-1}|^2 = (1 + z\bar{z})^d.$$

Applying Lemma 1 of [6], we get

$$\frac{d}{(1+z\bar{z})^2} = \partial\bar{\partial} \ln|g_0 \wedge \cdots \wedge g_{k-1}|^2$$

$$= \frac{|g_0 \wedge \cdots \wedge g_{k-2}|^2 |g_0 \wedge \cdots \wedge g_{k-1} \wedge \partial g_{k-1}|^2}{|g_0 \wedge \cdots \wedge g_{k-1}|^4}.$$

This implies

$$|g_0 \wedge \cdots \wedge g_{k-2}|^2 = c_{k-2}(1 + z\bar{z})^{d_{k-2}}$$

by irreducibility of the polynomial $1+z\bar{z}$. One gets $|g_0|^2 = c_0(1+z\bar{z})^{d_0}$ by using induction. Hence γ has constant Gaussian curvature.

To simplify equation (1.6), the following proposition is useful.

Proposition 2. *Up to a choice of unitary basis of C^n (or equivalently, a $U(n)$-equivalence), for any chosen pole ∞ there exist C^n-valued holomorphic polynomials $f_1, \ldots, f_k$ with $|f_1 \wedge \cdots \wedge f_k|^2 = (1+z\bar{z})^d$ such that $\varphi|_C = \pi \circ A(f_1, \ldots, f_k)$, where $A = A(f_1, \ldots, f_k)$ is in the form*

$$A(f_1, \ldots, f_k) = \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix} = \begin{pmatrix} & zg_1 \\ I_k & \vdots \\ & zg_k \end{pmatrix}, \tag{1.8}$$

where I_k is the unit matrix of order k and $g_1, \ldots, g_k$ are holomorphic polynomials valued in C^{n-k}.

Proof. Take $\{f_1, \ldots, f_k\}$ such that (1.3) is satisfied. From (1.4) we get

$$f_1 \wedge \cdots \wedge f_k = u_{10} \wedge \cdots \wedge u_{k0} + z\sum_{l=1}^{d} v_l z^{l-1}.$$

Then by (1.6),

$$|u_{10} \wedge \cdots \wedge u_{k0}|^2 = 1, \quad \langle u_{10} \wedge \cdots \wedge u_{k0}, \bar{v}_l \rangle = 0$$

for $l = 1, \ldots, d$. Thus $\langle f_1 \wedge \cdots \wedge f_k, \bar{u}_{10} \wedge \cdots \wedge \bar{u}_{k0} \rangle = 1$. By a suitable choice of basis of C^n we can assume $u_{10} \wedge \cdots \wedge u_{k0} = \varepsilon_1 \wedge \cdots \wedge \varepsilon_k$, here $\{\varepsilon_1, \cdots, \varepsilon_k\}$ is a part of the natural basis of C^n. Set $f_j = \sum_{l=1}^{n} f_{jl}\varepsilon_l = (f_{j1}, \cdots, f_{jn})$, i.e.,

$$A = \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix} = \begin{pmatrix} f_{11} & \cdots & f_{1k} & f_{1\,k+1} & \cdots & f_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{k1} & \cdots & f_{kk} & f_{k\,k+1} & \cdots & f_{kn} \end{pmatrix} := (A_1, A_2).$$

Then $1 = \langle f_1 \wedge \cdots \wedge f_k, \varepsilon_1 \wedge \cdots \wedge \varepsilon_k \rangle = \det A_1$. This means A_1^{-1} is still

a $M(k, k)$-valued holomorphic polynomial and therefore $\widetilde{A} = A_1^{-1}A = (I_k, \widetilde{A}_2)$ is a holomorphic polynomial valued in $M(k, n)$. If $\widetilde{A} = \widetilde{A}(\widetilde{f}_1, \cdots, \widetilde{f}_k)$, then $\widetilde{f}_1 \wedge \cdots \wedge \widetilde{f}_k = \det(A_1^{-1})\ f_1 \wedge \cdots \wedge f_k = f_1 \wedge \cdots \wedge f_k = \varepsilon_1 \wedge \cdots \wedge \varepsilon_k + z\sum_{l=0}^{d-1} v_l z^l$. Thus $|\widetilde{f}_1 \wedge \cdots \wedge \widetilde{f}_k|^2 = (1 + z\bar{z})^d$. For $j > k$, the j-th component of $\widetilde{f}_k$ is

$$\widetilde{f}_{kj} = \langle \widetilde{f}_1 \wedge \cdots \wedge \widetilde{f}_k, \varepsilon_1 \wedge \cdots \wedge \varepsilon_{k-1} \wedge \varepsilon_j \rangle = z\sum_{l=1}^{d} \langle v_l, \varepsilon_1 \wedge \cdots \wedge \varepsilon_{k-1} \wedge \varepsilon_j \rangle z^{l-1}.$$

So $\widetilde{f}_k = \varepsilon_k + zg_k$ for some C^{n-k}-valued holomorphic polynomial g_k. Similarly $\widetilde{f}_j = \varepsilon_j + zg_j$ for $j = 1, \cdots, k-1$.

To prove Theorem A, we need the following well-known lemma.

Lemma 3. *Let $A \in M(k, k)$. Then there are $U_1, U_2 \in U(k)$ such that U_1AU_2 is diagonal and the elements on the diagonal are nonnegative real.*

2. Proof of Theorem A

Now we focus our attention on holomorphic immersion $\varphi : S^2 \to G(2, 4)$.

A. Some special cases.

If $\underline{\varphi}$ contains a trivial subbundle $\underline{C}^1$, then we have orthogonal decomposition $\underline{\varphi} = \underline{C}^1 \oplus \underline{\gamma}$ with $\gamma : S^2 \to CP^2$. Just as in proof of Proposition 1 we see $\gamma = V_1$ or V_2 according to deg $(\varphi) = 1$ or 2. Then φ is $U(4)$-equivalent to (1) in Theorem A, or to (2) in Theorem A when $t = \pi/2$.

If φ is not full, then φ is into $G(2, 3) \cong G(1, 3) = CP^2$. Thus $\deg(\varphi) = 1$ or 2. This is in the case of (1), or (2) in Theorem A when $t = 0$.

Excluding the above cases, if the hypothesis in Proposition 1 holds for φ, then

$$\varphi|_C = \pi \circ \begin{pmatrix} 1 & \sqrt{3}z & \sqrt{3}z^2 & z^3 \\ 0 & \sqrt{3} & 2\sqrt{3}z & 3z^2 \end{pmatrix}.$$

This is equivalent to (4) of Theorem A by being left multiplied with $\begin{pmatrix} -1 & z \\ 0 & 1/\sqrt{3} \end{pmatrix}$ and changing $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ into $\{-\varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_3\}$.

B. In case rank $\underline{\varphi}_1 = 2$.

From now on we set φ is full and rank $(\underline{\varphi}_1) = 2$. According to Proposition 2, we may set $\varphi|_C = \pi \circ A(f_1, f_2)$, where

$$A(f_1, f_2) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & zg_1 & zg_2 \\ 0 & 1 & zg_3 & zg_4 \end{pmatrix} \tag{2.1}$$

and $|f_1 \wedge f_2|^2 = (1 + z\bar{z})^d$. Note that $d \le 5$ since $\mathrm{Pl} \circ \varphi$ is into $\pi_1(\wedge^2 C^4) = CP^5$.

We take a basis $\{\tilde{\varepsilon}_1 = \varepsilon_1 \wedge \varepsilon_2, \tilde{\varepsilon}_2 = -\varepsilon_2 \wedge \varepsilon_3, \tilde{\varepsilon}_3 = -\varepsilon_2 \wedge \varepsilon_4, \tilde{\varepsilon}_4 = \varepsilon_1 \wedge \varepsilon_3, \tilde{\varepsilon}_5 = \varepsilon_1 \wedge \varepsilon_4, \tilde{\varepsilon}_6 = \varepsilon_3 \wedge \varepsilon_4\}$ for $\wedge^2 C^4 \cong C^6$, where $\{\varepsilon_1, \ldots, \varepsilon_4\}$ is the natural basis of C^4. Then $f_1 \wedge f_2$ can be expressed by its components as

$$f_1 \wedge f_2 = (1, zg_1, zg_2, zg_3, zg_4, z^2 h), \tag{2.2}$$

where

$$h = g_1 g_4 - g_2 g_3. \tag{2.3}$$

Thus $\deg(g_j) \le d - 1$, $\deg(h) \le d - 2$ if non-vanishing. (2.1) can also be written as

$$A(f_1, f_2) = (I_2, C_1 z + \cdots + C_d z^d), \tag{2.4}$$

where $C_j \in M(2, 2)$ for $j = 1, \ldots, 5$. According to Lemma 3 we may assume one of C_j's is diagonal with nonnegative real diagonal elements.

We have C^4-valued holomorphic polynomials f_3, f_4 such that $f_1 \wedge \cdots \wedge f_4$ has no zero and

$$\begin{cases} \partial f_1 \equiv p_1 f_3 & (\mathrm{mod}\, f_1, f_2), \\ \partial f_2 \equiv p_2 f_4 & (\mathrm{mod}\, f_1, f_2, f_3), \end{cases} \tag{2.5}$$

where p_1, p_2 are holomorphic polynomials. Then by the orthogonalizing process we have a unitary frame $\{e_1, \ldots, e_4\}$ with

$$e_1 \wedge \cdots \wedge e_j = \frac{f_1 \wedge \cdots \wedge f_j}{|f_1 \wedge \cdots \wedge f_j|}, \quad j = 1, 2, 3, 4. \tag{2.6}$$

(see Lemma 3 and Lemma 4 of [6]).

Write down the generalized Frenet formula for φ [7]:

$$\begin{cases} de_1 = i\,\rho_1 e_1 + \lambda_1 \omega e_2 + a_1 \omega e_3, \\ de_2 = -\bar{\lambda}_1 \bar{\omega} e_1 + i\,\rho_2 e_2 + a_2 \omega e_3 + a_3 \omega e_4, \\ de_3 = -a_1 \bar{\omega} e_1 - \bar{a}_2 \bar{\omega} e_2 + i\,\rho_3 e_3 + \lambda_2 \omega e_4, \\ de_4 = -a_3 \bar{\omega} e_2 - \bar{\lambda}_2 \bar{\omega} e_3 + i\,\rho_4 e_4, \end{cases} \tag{2.7}$$

where $\omega = \dfrac{\sqrt{d}}{1 + z\bar{z}} dz$. By definition of induced metric we have

$$|a_1|^2 + |a_2|^2 + |a_3|^2 = 1. \tag{2.8}$$

Since $f_1 \wedge \cdots \wedge f_4 = (\det B(f_1, \ldots, f_4))\varepsilon_1 \wedge \cdots \wedge \varepsilon_4$ has no zero, $\det B$ is constant. We may set $f_1 \wedge \cdots \wedge f_4 = \varepsilon_1 \wedge \cdots \wedge \varepsilon_4$ by a suitable multiple of f_4. From (2.5)-(2.7) we get

$$a_1 a_3 = \frac{p_1 p_2}{d \cdot (1 + z\bar{z})^{d-2}}. \tag{2.9}$$

From (2.5) we have also

$$f_1 \wedge f_2 \wedge \partial f_1 \wedge \partial f_2 = p_1 p_2 f_1 \wedge \cdots \wedge f_4 = p_1 p_2 \varepsilon_1 \wedge \cdots \wedge \varepsilon_4. \tag{2.10}$$

It is clear that $F = \left|\det\begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix}\right|^2 = |a_1|^2|a_3|^2$ is dependent neither on the frame $\{e_1, \ldots, e_4\}$ nor on ω, so is a global function on S^2. The number of zeros of F is known as *ramification index* of φ, which are isolated by (2.9), and denoted by $r(\varphi)$ (cf. [7]). From (2.9) we see the zeros of F coincide with that of $p_1 p_2$ if ∞ is not a ramification point. In contrast if $\deg(p_1 p_2) < r(\varphi)$, then ∞ is a ramification. The order of a zero z_0 of $p_1 p_2$ is called the *order of ramification point* z_0. From (2.9) we know $r(\varphi) = 2(d-2)$ in this case, since $|a_1, a_3|^2$ is bounded by (2.8). Thus $d \geq 2$.

In the following we proceed separately according to $\deg(\varphi)$.

C. When $d = 2$.

If $C_2 \neq 0$ in (2.4), we may set $C_2 = \operatorname{diag}(a, 0)$ with $a > 0$ a real number. Because h is constant in (2.2) and $\deg(g_1) = 1$, we have $g_4 = 0$ by (2.3). (2.2) becomes

$$f_1 \wedge f_2 = (1, a_0 z + a z^2, b_0 z, c_0 z, 0, b_0 c_0 z^2)$$

$$= (1, 0, 0, 0, 0, 0) + (0, a_0, b_0, c_0, 0, 0) z + (0, a, 0, 0, 0, b_0 c_0) z^2.$$

From (1.6) we know $a_0 = 0$ by $\langle v_1, \bar{v}_2 \rangle = 0$. By assumption we see $b_0 c_0 \neq 0$ otherwise φ is not full or $\operatorname{rank}(\underline{\varphi}_1) < 2$. By a suitable multiple of f_2 and appropriate rotations of ε_2 and ε_4, we may set $b_0, c_0 \in R$. Then $b_0^2 + c_0^2 = 2$ by (1.6). Thus $b_0 = \sqrt{2}\sin t$, $c_0 = \sqrt{2}\cos t$ for some $t \in R$. Again by (1.6) we get $a = \pm\cos 2t$. This is essentially $U(4)$-equivalent to (2) of Theorem A by replacing f_1 with $-f_1$ if necessary.

If $C_2 = 0$ in (2.4), we may set $C_1 = \operatorname{diag}(a, b)$. Then $a^2 + b^2 = 2$ and $ab = 1$ by (1.6). Therefore $a = b = 1$ and φ is equivalent to (2) of Theorem A when $t = \pi/4$.

D. When $d \geq 3$.

Since $r(\varphi) = 2(d-2)$, up to an isometry of S^2, we may take $z = 0$ to be a ramification point in this case. By Lemma 3 and (2.10) we may set $C_1 = \mathrm{diag}(a, 0)$. Then $f_1 \wedge f_2 = \varepsilon_1 \wedge \varepsilon_2 + az\varepsilon_3 \wedge \varepsilon_2 + z^2 g(z)$ for some $\wedge^2 C^4$-valued holomorphic polynomial g by (2.1). In the same way as in proof of Proposition 1 we get $a = \sqrt{d}$ and $zg_1 = \langle f_1 \wedge f_2, \varepsilon_3 \wedge \varepsilon_2 \rangle = \sqrt{d}z$. Thus $g_1 = \sqrt{d}$.

(2.1) and (2.2) can be rewritten as

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \sqrt{d}z & z^2 g_2 \\ 0 & 1 & z^2 g_3 & z^2 g_4 \end{pmatrix}, \tag{2.11}$$

$$f_1 \wedge f_2 = (1, \sqrt{d}z, z^2 g_2, z^2 g_3, z^2 g_4, z^3 h), \tag{2.12}$$

respectively, where

$$h = \sqrt{d} g_4 - z g_2 g_3. \tag{2.13}$$

Thus $\deg(g_j) \leq d-2$ for $j = 2, 3, 4$ whenever $g_j \neq 0$ and $\deg(h) \leq d-3$ if $h \neq 0$.

(i) When $d = 3$ we get $h = \alpha_0$, a constant, and $zg_2 g_3 = \sqrt{3} g_4 - \alpha_0$ by (2.13).

If $g_2 g_3 \neq 0$, then $\sqrt{3} g_4 = \alpha_0 + z g_2 g_3 \neq 0$. Thus $\deg(g_4) \leq 1$ and $g_2 = a_0, g_3 = b_0$ by (2.13). From (1.6) and (2.12) we get $\alpha_0 = 0$ since $a_0 b_0 \neq 0$. Therefore $|a_0 b_0|^2 = 3$ and $|a_0|^2 + |b_0|^2 = 3$ by (1.6). This is impossible. So $g_2 g_3 = 0$.

If $g_2 = 0$, then $g_4 = \dfrac{\alpha_0}{\sqrt{3}}$ by (2.13) and $\alpha_0 \neq 0$ otherwise φ is not full. We can assume $\alpha_0 > 0$ by a rotation of ε_4. The right hand side of (2.12) becomes $(1, \sqrt{3}z, 0, b_0 z^2 + b_1 z^3, \dfrac{\alpha_0}{\sqrt{3}} z^2, \alpha_0 z^3)$. Then by (1.6) $b_0 b_1 = 0$. b_1 has to be 0, otherwise $b_0 = 0$, which forces $\alpha_0 = 3, \alpha_0^2 + |b_1|^2 = 1$ by (1.6),

a contradiction. Thus $\alpha_0 = 1, |b_0| = \sqrt{8/3}$ by (1.6). Up to a multiple of f_2 and a $U(4)$-equivalence, we may set $b_0 = \sqrt{8/3}$. We then get (3) of Theorem A by $U(4)$-equivalence

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{8}}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & -\frac{\sqrt{8}}{3} \end{pmatrix} \in U(4).$$

Similarly, if $g_3 = 0$, we get

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \sqrt{3}z & \sqrt{8/3}z^2 \\ 0 & 1 & 0 & \sqrt{1/3}z^2 \end{pmatrix}.$$

By a holomorphic isometry $z \mapsto w = z^{-1}$ of S^2, φ can be expressed as

$$\varphi = \pi \begin{pmatrix} 1 & 0 & \sqrt{3}z & \sqrt{8/3}z^2 \\ 0 & 1 & 0 & \sqrt{1/3}z^2 \end{pmatrix} = \pi \begin{pmatrix} 1 & -\sqrt{8} & \sqrt{3}z & 0 \\ 0 & 1 & 0 & \sqrt{1/3}z^2 \end{pmatrix}$$

$$= \pi \begin{pmatrix} w & -\sqrt{8}w & \sqrt{3} & 0 \\ 0 & w^2 & 0 & \sqrt{1/3} \end{pmatrix} = \pi \begin{pmatrix} -\sqrt{1/3}w & \sqrt{8/3}w & -1 & 0 \\ 0 & \sqrt{3}w^2 & 0 & 1 \end{pmatrix}.$$

This is in the case (3) of Theorem A by rechoosing basis of C^4.

Evidently $g_2 = g_3 = 0$ is impossible.

(ii) When $d = 4$ we may set

$$h = \alpha_0 + \alpha_1 z, \quad g_4 = c_0 + c_1 z + c_2 z^2 \tag{2.14}$$

by (2.12). If $c_2 = 0$, then $g_2 = a_0, g_3 = b_0$ by (2.13) and consequently $\alpha_0 = 2c_0, a_0 b_0 = 2c_1 - \alpha_1$. Using (2.6) we get $\alpha_0 = 2c_0 = 0, |\alpha_1| = 1, |c_1| = 2$ and $|a_0|^2 + |b_0|^2 = 6$. From

$$3 = \frac{1}{2}(|a_0|^2 + |b_0|^2) \geq |a_0||b_0| = |2c_1 - \alpha_1| \geq 2|c_1| - |\alpha_1| = 3$$

we know $|a_0| = |b_0| = \sqrt{3}$. By rotating f_2 and $\varepsilon_2, \varepsilon_4$ we may set

$b_0 = \sqrt{3}, c_1 = 2$. The above inequality implies $\alpha_1 = 1$ and therefore $a_0 = (2c_1 - \alpha_1)/b_0 = \sqrt{3}$. This is just (4) of Theorem A, although it contradicts to rank $\underset{\sim}{\varphi}_1 = 2$.

Suppose $c_2 \neq 0$ in (2.14). Then $g_2 g_3 \neq 0$ by (2.13) and therefore $\deg(g_2 g_3) = 1$. Without loss of generality we may set

$$g_2 = a_0, \quad g_3 = b_0 + b_1 z, \quad a_0 b_1 \neq 0.$$

Also we have

$$\alpha_0 = 2c_0, \quad \alpha_1 = 2c_1 - a_0 b_0, \quad 2c_2 = a_0 b_1$$

by (2.13). Similarly, from (1.6) one gets

$$c_0 = \alpha_0 = c_1 = \alpha_1 = b_0 = 0, \quad |c_2| = 1, \ |b_1| = 2, \ |a_0| = \sqrt{6},$$

contradicts to $2c_2 = a_0 b_1$.

(iii) When $d = 5$ the mapping $\mathrm{Pl} \circ \varphi : S^2 \to CP^5$ is full, and hence neither g_j nor h vanishes by (2.12). Since $\deg(h) \leq 2$ and $\deg(g_4) \leq 3$, we have $\deg(g_2 g_3) \leq 2$ by (2.13). Keep it in mind.

Now

$$p_1 p_2 = \sqrt{5}\partial(z^2 g_4) - \partial(z^2 g_2)\partial(z^2 g_3) \tag{2.15}$$

by (2.10). Thus $\deg(p_1 p_2) \leq 4$. Because $r(\varphi) = 6$ when $d = 5$, (2.15) means that for any chosen ramification point $x \in S^2$, the antipodal point of x is a ramification point with order ≥ 2. Consequently there is only a pair of ramification point with order ≥ 2. We may assume the ramification point $z = 0$ is of minimum order. Thus

$$p_1 p_2 = cz^2, \quad \text{or} \quad p_1 p_2 = cz^3 \tag{2.16}$$

for some constant c. Combine it with (2.13) gives

$$h = \alpha_1 z + \alpha_2 z^2, \quad g_4 = c_1 z + c_2 z^2 + c_3 z^3.$$

We will prove the case $d = 5$ does not occur.

If $\deg(g_2) = 2$, we may set $g_2 = a_0 + a_1 z + a_2 z^2$, $g_3 = b_0$, $a_2 b_0 \neq 0$. Then $\sqrt{5}c_3 = a_2 b_0$ by (2.13), and $5\sqrt{5}c_3 = 8a_2 b_0$ by (2.15), a contradiction. Similarly $\deg(g_3) = 2$ is impossible. If $\deg(g_2) = \deg(g_3) = 1$, there is still a contradiction between (2.13) and (2.15) by setting $g_2 = a_0 + a_1 z$, $g_3 = b_0 + b_1 z$ with $a_1 b_1 \neq 0$.

Suppose $\deg(g_2 g_3) = 1$. Without loss of generality we set $g_2 = a_0$, $g_3 = b_0 + b_1 z$. Then $c_3 = 0$, $\alpha_1 = \sqrt{5}c_1 - a_0 b_0$, $\alpha_2 = \sqrt{5}c_2 - a_0 b_1$ by (2.13). (2.12) is now

$$f_1 \wedge f_2 = (1, \sqrt{5}z, a_0 z^2, b_0 z^2 + b_1 z^3, c_1 z^3 + c_2 z^4, \alpha_1 z^4 + \alpha_2 z^5).$$

Using (1.6) we get

$$|\alpha_2| = 1,\ \alpha_1 = 0,\ |c_2| = \sqrt{5},\ c_1 = 0,\ |b_1| = \sqrt{10},\ b_0 = 0,\ |a_0| = \sqrt{10}.$$

Thus we have a contradiction:

$$10 = |a_0 b_1| = |\sqrt{5}c_2 - \alpha_2| \leq |\sqrt{5}c_2| + |\alpha_2| = 6.$$

It is easy to see that the case $\deg(g_2 g_3) = 0$ can not occur by (2.13) and (1.6).

E. Notes on Theorem A.

(1) For $K = 2$ there is a classification in [4]. The reader should compare (2) of Theorem A with (4.13) and (4.15) in [4]. Note that the two expressions in [4] are equivalent to each other by parameter transformation $\psi \to -\psi$ and coordinate transformation $w \to -w$.

(2) For constant curved holomorphic curves M^2 in $G(2, 4)$ there are three local expressions in [5, p. 89, p. 95 and p. 100] which are correspondent to (2) when $t = 0$, (4) and (2) when $t = \frac{\pi}{4}$ of Theorem A, respectively.

(3) In contrast to E. Calabi's result [2], (2) of Theorem A shows that, in general, the rigidity theorem does not hold in $G(k, n)$ when $k > 1$. To see this one can compute the global function $4F = 4|a_1|^2|a_3|^2 = \sin^2(2t)$ by means of (2.9) and (2.10). Obviously F is invariant under holomorphic isometric of $G(2, 4)$. Hence for $t \neq t'$ the corresponding φ_t and $\varphi_{t'}$ are not congruent to each other.

(4) The embedding φ in (3) of Theorem A has two ramification points. This is a distinguished phenomenon different from holomorphic S^2 in CP^n.

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