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ANTI-SYMMETRIC ORTHOGONAL ANTI-SYMMETRIC SOLUTIONS OF MATRIX EQUATION AND THE OPTIMAL APPROXIMATION

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Abstract

An $n \times n$ real P is said to be an anti-symmetric orthogonal matrix if $P^T = P$ and $P^2 = I$. An $n \times n$ real X is said to be an anti-symmetric orthogonal anti-symmetric matrix with respect to the symmetric orthogonal-matrix P if $PXP = X$ and $X^T = -X$. By applying the generalized singular value decomposition (GSVD) of matrices, this paper provides the necessary and sufficient conditions for the existence and the expression for the anti-symmetric orthogonal anti-symmetric with a symmetric orthogonal matrix P solutions of the matrix equation $A^T XA = B(A \in R^{n\times m}, B \in R^{m\times m})$. In addition, in solution set of the equation, the expression of the optimal approximation solution to the given matrix and of least-norm solution are derived.

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1. Introduction

We first introduce some symbols to be used. Let $R^{n\times m}$, $R_n^{n\times n}$ be the set of all $n\times m$ real matrices and all $n\times n$ nonsingular matrices, respectively. Let $OR^{n\times n}$, $SOR^{n\times n}$ be the set of all $n\times n$ orthogonal matrices and symmetric orthogonal matrices, respectively. Denoted by the superscripts T and I_n be the transpose and identity matrix with order n, respectively. In space $R^{n\times m}$, we define inner product by $\langle A, B\rangle = tr(B^TA)$ for all $A, B\in R^{n\times m}$. For a mtarix $A\in R^{n\times m}$, $\|A\|$ represents its Frobenius norm, $rank(A)$ represents its rank. $A*B$ represents Hadamard product of matrices A and B.

Definition 1.1. Assume $X\in R^{n\times n}$, for given generalized matrix P, $P^T=P$, $P^TP=I_n$. Matrix X is said to be *orthogonal-antisymmetric* with respect to P, if $(PX)^T=-PX$. Matrix X is said to be *antisymmetric orthogonal-antisymmetric* with respect to P, if $(PX)^T=-PX$, $X^T=-X$. The set of all antisymmetric orthogonal antisymmetric matrices is denoted by $ASR_r^{n\times n}(P)$. X is also reflexive matrix with respect to P, $X=PXP$. The set of all $n\times n$ reflexive matrices is denoted by $R_r^{n\times n}(P)$.

Remark. In the following part, we always consider generalized matrix P as in Definition 1.1.

In this paper, we considr the following problems.

Problem I. Given $A\in R^{n\times m}$, $B\in R^{m\times m}$, find $X\in ASR_r^{n\times n}(P)$, such that

$$A^TXA=B. \tag{1}$$

Problem II. For given $X^*\in R^{n\times n}$, find $\hat{X}\in S_E$, satisfying

$$\|\hat{X}-X^*\|=\min_{X\in S_E}\|X-X^*\|, \tag{2}$$

where S_E is the set of solutions of problem I.

The linear matrix equation $A^T XA = B$ has been widely discussed. For instance, Chu [1], Dai [3] and He [5] have studied the solvability conditions of the common solution, symmetric solution and the symmetric positive definite solution. Peng [7] has discussed the necessary and sufficient conditions for the expression of the centrosymmetry solution, and he has studied the optimal approximation of given matrix equation. And Peng et al. [9] have obtained orthogonal-symmetric solution and optimal approximation. Howerver, there is no result of the orthogonal-antisymmetric solution, optimal approximation, and least-norm solution of the matrix equation. In this paper, by applying the generalized singular value decomposition of matrices, this paper provides the necessary and sufficient conditions for the existence and the expression for the anti-symmetric orthogonal anti-symmetric with a symmetric orthogonal matrix P solutions of the matrix equation $A^T XA = B(A \in R^{n\times m}, B \times R^{m\times m})$. In addition, in solution set of the equation, the expression of the optimal approximation solution to the given matrix and of least-norm solution are derived.

2. The Solution of Problem I

Lemma 1 [8]**.** *The necessary and sufficient conditions of* $X \in ASR_r^{n\times n}(P)$ *is*

$$X = U\begin{pmatrix} X_1 & O \\ O & X_2 \end{pmatrix}U^T \tag{3}$$

and

$$X_1 \in ASR_r^{r\times r},\ X_2 \in ASR_{n-r}^{(n-r)\times(n-r)},\ X_1^T = -X_1,$$

$$X_2^T = -X_2,\ r = rank\left[\frac{1}{2}(I + P)\right], \tag{4}$$

U is exclusively confirmed by P,

$$P = U\begin{pmatrix} I_r & o \\ o & -I_{n-r} \end{pmatrix}U^T. \tag{5}$$

Set

$$U^T A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},\ A \in R^{r\times n},\ A_2 \in R^{(n-r)\times n}. \tag{6}$$

Applying the generalized singular value decomposition (GSVD) on $[A_1^T,\ A_2^T]$, we obtain

$$A_1^T = W\Sigma_1 U_1^T,\ A_2^T = W\Sigma_2 V_1^T, \tag{7}$$

here $W \in R_m^{m\times m}$, $U_1 \in OR^{r\times r}$, $V_1 \in OR^{(n-r)\times(n-r)}$ and

$$\Sigma_1 = \begin{pmatrix} I_k & & \\ & D_1 & \\ & & O_1 \\ & O & \end{pmatrix} \begin{matrix} k \\ s \\ r-k-s \\ m-r \end{matrix},$$
$$\begin{matrix} \ \ k & \ \ s & \ \ r-k-s \end{matrix}$$

$$\Sigma_2 = \begin{pmatrix} O_2 & & \\ & D_2 & \\ & & I_{t-k-s} \\ & O & \end{pmatrix} \begin{matrix} k \\ s \\ t-k-s \\ m-t \end{matrix}, \tag{8}$$
$$\begin{matrix} n-r+k-t & \ \ s & \ \ t-k-s \end{matrix}$$

$t = rank(A_1^T,\ A_2^T)$, $k = t - rank(A_2^T)$, $s = rank(A_1^T) + rank(A_2^T) - t$, $D_1 = diag(a_1,\ a_2,\ ...,\ a_s) > 0$, $D_2 = diag(b_1,\ b_2,\ ...,\ b_s) > 0$, O_1, O_2, O are all null matrices.

Set partition matrix of $W^{-1}BW^{-T}$ as

$$W^{-1}BW^{-T} = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix} \begin{matrix} k \\ s \\ t-k-s \\ m-t \end{matrix}, \tag{9}$$
$$\begin{matrix} \ \ k & \ \ s & \ \ t-k-s & \ \ m-t \end{matrix}$$

then we can get the following theorem.

Theorem 1. *Given* $P \in SOR^{n\times n}$, $A \in R^{n\times m}$, $B \in R^{m\times m}$, *if* $U^T A$

and $W^{-1}BW^{-T}$ *could be as* (6) *and* (9), *respectively, and the generalized singular value decomposition of matrix* $[A_1^T, A_2^T]$ *be* (7). *The Eq.* (1) *has anti-symmetric orthogonal anti-symmetric solutions if and only if*

$$B_{ij}^T = -B_{ji} \quad (i, j = 1, 2, 3, 4),\ B_{14} = O,\ B_{24} = O,$$

$$B_{34} = O,\ B_{13} = O,\ B_{44} = O. \tag{10}$$

In this case, it has common expression of anti-symmetric orthogonal anti-symmetric solution

$$X = U\begin{pmatrix} U_1\begin{pmatrix} B_{11} & B_{12}D_1^{-1} & X_{13} \\ -D_1^{-1}B_{12}^T & X_{22} & X_{23} \\ -X_{13}^T & -X_{23}^T & X_{33} \end{pmatrix}U_1^T & O \\ O & V_1\begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ -Y_{12}^T & D_2^{-1}(B_{22} - D_1X_{22}D_1)D_2^{-1} & D_2^{-1}B_{23} \\ -Y_{13}^T & -B_{32}^TD_2^{-1} & B_{33} \end{pmatrix}V_1^T \end{pmatrix}U^T, \tag{11}$$

and $X_{13}, X_{23}, Y_{12}, Y_{13}$ *are arbitrary matrix,* X_{22}, X_{33}, Y_{11} *are arbitrary corresponding anti-symmetric matrix.*

Proof. We first prove the necessity of (10). If matrix equation $A^TXA = B$ is consistent, where $X \in ASR_r^{n\times n}(P)$, by Lemma 1, expression of X is (3). Because X is solution of $A^TXA = B$, and $X^T = -X$, we have that $B^T = -B$, and

$$B_{ij}^T = -B_{ji} \quad (i, j = 1, 2, 3, 4). \tag{12}$$

Put (3) in $A^TXA = B$, we can get

$$(U^TA)^T\begin{pmatrix} X_1 & O \\ O & X_2 \end{pmatrix}U^TA = B, \tag{13}$$

then

$$A_1^TX_1A_2 + A_2^TX_2A_1 = B \tag{14}$$

from (7) and (14), we have

$$W\Sigma_1(U_1^T X_1 V_1)\Sigma_2^T W^T + W\Sigma_2(V_1^T X_2 U_1)\Sigma_1^T W^T = B.$$

Because W is nonsingular, so we get

$$\Sigma_1(U_1^T X_1 U_1)\Sigma_1^T + \Sigma_2(V_1^T X_2 V_1)\Sigma_2^T = W^{-1}BW^{-T}. \tag{15}$$

Let

$$U_1^T X_1 U_1 = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ -X_{12}^T & X_{22} & X_{23} \\ -X_{13}^T & -X_{23}^T & X_{33} \end{pmatrix} \begin{matrix} k \\ s \\ r-k-s \end{matrix} \tag{16}$$

$$V_1^T X_2 V_1 = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ -Y_{12}^T & Y_{22} & Y_{23} \\ -Y_{13}^T & -Y_{23}^T & Y_{33} \end{pmatrix} \begin{matrix} n-r+k-t \\ s \\ t-k-s \end{matrix}, \tag{17}$$

here $X_{ii}^T = -X_{ii}$, $Y_{ii}^T = -Y_{ii}$, $(i = 1, 2, 3)$,

then we obtain

$$\Sigma_1(U_1^T X_1 U_1)\Sigma_1^T = \begin{pmatrix} X_{11} & X_{12}D_1 & O & O \\ -D_1X_{12}^T & D_1X_{22}D_1 & O & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix} \begin{matrix} k \\ s \\ r-k-s \\ m-r \end{matrix}$$

$$= \begin{pmatrix} X_{11} & X_{12}D_1 & O & O \\ -D_1X_{12}^T & D_1X_{22}D_1 & O & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix} \begin{matrix} k \\ s \\ r-k-s \\ m-r \end{matrix}, \tag{18}$$

and

$$\Sigma_2(V_1^T X_2 V_1)\Sigma_2^T = \begin{pmatrix} O & O & O & O \\ O & D_2Y_{22}D_2 & D_2Y_{23} & O \\ O & -Y_{23}^T D_2 & Y_{33} & O \\ O & O & O & O \end{pmatrix} \begin{matrix} k \\ s \\ t-k-s \\ m-t \end{matrix}. \tag{19}$$

By (18) and (19) to (15),

$$\begin{pmatrix} X_{11} & X_{12}D_1 & O & O \\ -D_1X_{12}^T & D_1X_{22}D_1 + D_2Y_{22}D_2 & D_2Y_{23} & O \\ O & -Y_{23}^TD_2 & Y_{33} & O \\ O & O & O & O \end{pmatrix} \begin{matrix} k \\ s \\ t-k-s \\ m-t \end{matrix}$$

$$\begin{matrix} \\ k \end{matrix} \quad \begin{matrix} \\ s \end{matrix} \quad \begin{matrix} \\ t-k-s \end{matrix} \quad \begin{matrix} \\ m-t \end{matrix}$$

$$= \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix} \tag{20}$$

comparing both sides of (20) and by (12), we get (10) and

$$X_{12} = B_{12}D_1^{-1},\ Y_{23} = D_2^{-1}B_{23},\ X_{11} = B_{11},\ Y_{33} = B_{33}, \tag{21}$$

$$Y_{22} = D_2^{-1}(B_{22} - D_1X_{22}D_1)D_2^{-1}. \tag{22}$$

From (21), (22), (16), (17) and (3), we get (11).

Now we prove the sufficient (11). Let

$$X_{10} = U_1 \begin{pmatrix} B_{11} & B_{12}D_1^{-1} & O \\ -D_1^{-1}B_{12}^T & O & O \\ O & O & O \end{pmatrix} U_1^T,$$

$$X_{20} = V_1 \begin{pmatrix} O & O & O \\ O & D_2^{-1}B_{22}D_2^{-1} & D_2^{-1}B_{23} \\ O & -B_{23}^TD_2^{-1} & O \end{pmatrix} V_1^T$$

and

$$X_0 = U \begin{pmatrix} X_{10} & 0 \\ 0 & X_{20} \end{pmatrix} U^T.$$

By Lemma 1, we know that $X_0 \in ASR_r^{n\times n}(P)$, and

$$A^TX_0A = A^TU \begin{pmatrix} & X_{10} \\ X_{20} & \end{pmatrix} U^TA = A_1^TX_{10}A_2 + A_2^TX_{20}A_1$$

$$= W\begin{pmatrix} B_{11} & B_{12} & O & O \\ -B_{21} & B_{22} & B_{23} & O \\ O & -B_{32} & B_{33} & O \\ O & O & O & O \end{pmatrix} W^T = B.$$

Hence X_0 is solution of matrix equation $A^T XA = B$.

3. The Expression of Solution of Problem II

Lemma 2. *If we supposed* $E = (e_{ij}) \in R^{n\times n}$, *then we could get a unique solution* G *that satisfies* $h(G) = \| G - E \|^2 + \|G^T + F\|^2 = \min$ *and* $G = \frac{1}{2}(E - F^T)$.

Lemma 3. *If we supposed* $E = (e_{ij}) \in R^{n\times n}$, *then we could get a exclusively anti-symmetric matrix* $G(g_{ij}) \in ASR^{n\times n}$ *that satisfies* $g(G) = \| G - E \|^2 = \min$ *and* $G = \frac{1}{2}(E - E^T)$.

Lemma 4. *If we supposed* $E = (e_{ij}) \in R^{n\times n}$, $F(f_{ij}) \in R^{n\times n}$, $D = diag(\sigma_1, \sigma_2, ..., \sigma_n) > 0$, *and* $G(g_{ij}) \in ASR^{n\times n}(G^T = -G)$, *then we could get an exclusively anti-symmetric matrix* $G(g_{ij}) \in ASR^{n\times n}$ *that satisfies* $h(G) = \| G - E \|^2 + \| DGD - F \|^2 = \min$, *and* $G = \Phi * (E - E^T + D(F - F^T)D)$, *where* $\Phi = \Phi(\phi_{ij}) \in R^{n\times n}$, $\phi_{ij} = \frac{1}{2(1 + \sigma_i^2\sigma_j^2)}$.

Proof. For $G(g_{ij}) \in ASR_n^{n\times n}$, $g_{ii} = 0$, $g_{ij} = -g_{ji}$, we have that

$$\begin{aligned} h(G) &= \| G - E \|^2 + \| DGD - F \|^2 \\ &= Trace((G^T - E^T)(G - E) + (DG^T D - F^T)(DGD - F)) \\ &= -Trace((G + E^T)(G - E) + (DGD + F^T)(DGD - F)) \end{aligned}$$

$$= -\sum_{i=1}^{n}((g_{ii}^2 - e_{ii}^2) + (\sigma_i^2 g_{ii}^2 - f_{ii}^2))$$

$$+ \sum_{1\le i\le j\le n}((g_{ij} - e_{ij})^2 + (g_{ij} + e_{ji})^2 + (\sigma_i g_{ij}\sigma_j - f_{ij})^2 + (\sigma_i g_{ij}\sigma_j + f_{ji})^2).$$

When calculate the derivative of above expression with g_{ij} and set $\frac{\partial h(G)}{\partial g_{ij}} = 0$, we can get

$$g_{ij} = \frac{1}{2(1+\sigma_i^2\sigma_j^2)}(e_{ij} - e_{ji} + \sigma_i(f_{ij} - f_{ji})\sigma_j),\ (1 \le i < j \le n).$$

$\Phi = \Phi(\phi_{ij}) \in R^{n\times n}$, $\phi_{ij} = \frac{1}{2(1+\sigma_i^2\sigma_j^2)}$, it is clear that $G = \Phi * (E - E^T + D(F - F^T)D)$.

Theorem 2. *Suppose* $X^* \in R^{n\times n}$, *matrix equation* $A^T XA = B(A \in R^{n\times m}, B \in R^{m\times m})$ *is consistent in* $ASR_r^{n\times n}(P)$, *it is solution in* S_E, *then* $\|X - X^*\| = \min$ *has exclusively solution* $\hat{X}$ *in* S_E, $\hat{X}$ *the same symbols as Theorem* 1,

Let

$$U^T X^* U = \begin{pmatrix} K_{11}^* & K_{12}^* \\ K_{21}^* & K_{22}^* \end{pmatrix} \begin{matrix} r \\ n-r \end{matrix} \tag{23}$$

$$U_1^T K_{11}^* U_1 = \begin{pmatrix} X_{11}^* & X_{12}^* & X_{13}^* \\ X_{21}^* & X_{22}^* & X_{23}^* \\ X_{31}^* & X_{32}^* & X_{33}^* \end{pmatrix} \begin{matrix} k \\ s \\ r-k-s \end{matrix},$$

$$\begin{matrix} k & s & r-k-s \end{matrix}$$

$$V_1^T K_{22}^* V_1 = \begin{pmatrix} Y_{11}^* & Y_{12}^* & Y_{13}^* \\ Y_{21}^* & Y_{22}^* & Y_{23}^* \\ Y_{31}^* & Y_{32}^* & Y_{33}^* \end{pmatrix} \begin{matrix} n-r+k-t \\ s \\ t-k-s \end{matrix}. \tag{24}$$

$$\begin{matrix} n-r+k-t & s & t-k-s \end{matrix}$$

We can get

$$\hat{X}=U\begin{pmatrix} U_1\begin{pmatrix} B_{11} & B_{12}D_1^{-1} & \hat{X}_{13} \\ -D_1^{-1}B_{12}^T & \hat{X}_{22} & \hat{X}_{23} \\ -\hat{X}_{13}^T & -\hat{X}_{23}^T & \hat{X}_{33} \end{pmatrix} U_1^T & O \\ O & V_1\begin{pmatrix} \hat{Y}_{11} & \hat{Y}_{12} & \hat{Y}_{13} \\ -\hat{Y}_{12}^T & D_2^{-1}(B_{22}-D_1\hat{X}_{22}D_1)D_2^{-1} & D_2^{-1}B_{23} \\ -Y_{13}^T & -B_{23}^TD_2^{-1} & B_{33} \end{pmatrix} V_1^T \end{pmatrix} U^T, \tag{25}$$

here

$$\hat{X}_{13}=\frac{1}{2}(X_{13}^*-X_{31}^{*T}),\ \hat{X}_{23}=(X_{23}^*-X_{32}^{*T}),\ \hat{Y}_{13}=\frac{1}{2}(Y_{13}^*-Y_{31}^{*T}),$$

$$\hat{Y}_{12}=\frac{1}{2}(Y_{12}^*-Y_{21}^{*T}),\ \hat{X}_{33}=\frac{1}{2}(X_{33}^*-X_{33}^{*T}),\ \hat{Y}_{11}=\frac{1}{2}(Y_{11}^*-Y_{11}^{*T}),$$

$$\hat{X}_{22}=\Phi(X_{22}^*-X_{22}^{*T}+D_2^{-1}D_1(Y_{22}^*-Y_{22}^{*T})D_2^{-1}D_1)-2\Phi*D_2^{-2}D_1B_{22}D_2^{-2}D_1$$

and $\Phi=\Phi(\phi_{ij})\in R^{s\times s}$, $\phi_{ij}=\dfrac{b_i^2b_j^2}{2(b_i^2b_j^2+a_i^2a_j^2)}$, $(i,\ j=1,\ 2,\ ...,\ s)$.

Proof. For (9), (10), (24), (25) and invariance of Frobenius norm of orghogonal transposition, we have that problem $\|X-X^*\|=\min$ and $X\in S_E$ is equivalent to

$$\|\hat{X}_{13}-X_{13}^*\|^2+\|\hat{X}_{13}^T+X_{31}^*\|^2=\min,\quad \|\hat{Y}_{12}-Y_{12}^*\|^2+\|\hat{Y}_{12}^T+Y_{21}^*\|^2=\min,$$

$$\|\hat{X}_{23}-X_{23}^*\|^2+\|\hat{X}_{23}^T+X_{32}^*\|^2=\min,\quad \|\hat{Y}_{13}-Y_{13}^*\|^2+\|\hat{Y}_{13}^T+Y_{31}^*\|^2=\min, \tag{26}$$

$$\|\hat{X}_{33}-X_{33}^*\|^2=\min,\qquad \|\hat{Y}_{11}-Y_{11}^*\|^2=\min, \tag{27}$$

$$\|\hat{X}_{22}-X_{22}^*\|^2+\|D_2^{-1}(B_{22}-D_1\hat{X}_{22}D_1)D_2^{-1}-Y_{22}^*\|^2=\min, \tag{28}$$

by (26) and Lemma 2

$$\hat{X}_{13}=\frac{1}{2}(X_{13}^*-X_{31}^{*T}),\quad \hat{X}_{23}=\frac{1}{2}(X_{23}^*-X_{32}^{*T}),$$

$$\hat{Y}_{12} = \frac{1}{2}(Y_{12}^* - Y_{21}^{*T}), \quad \hat{Y}_{13} = \frac{1}{2}(Y_{13}^* - Y_{31}^{*T}),$$

by (27) and Lemma 3

$$\hat{X}_{33} = \frac{1}{2}(X_{33}^* - X_{33}^{*T}), \quad \hat{Y}_{11} = \frac{1}{2}(Y_{11}^* - Y_{11}^{*T}),$$

by (28) nd Lemma 4

$$\hat{X}_{22} = \Phi * (X_{22}^* - X_{22}^{*T} + D_2^{-1}D_1(Y_{22}^{*T} - Y_{22}^* + 2D_2^{-1}B_{22}D_2^{-1})D_2^{-1}D_1),$$

here $\Phi = \Phi(\phi_{ij}) \in R^{s\times s}$, $\phi_{ij} = \dfrac{b_i^2 b_j^2}{2(b_i^2 b_j^2 + a_i^2 a_j^2)}$.

We can get the following corollary if we supposed the X^* to be null matrix in Theorem 1:

Corollary 1. *If matrix equation* $A^T XA = B\,(A \in R^{n\times m},\ B \in R^{m\times m})$ *is consistent in* $ASR_r^{n\times n}(P)$, *then it's least-norm solution* $\hat{X}$ *exists and exclusively,* $\hat{X}$ *can be denoted as*

$$\hat{X} = U\begin{pmatrix} U_1\begin{pmatrix} B_{11} & B_{12}D_1^{-1} & O \\ -D_1^{-1}B_{12}^T & \hat{X}_{22} & O \\ O & O & O \end{pmatrix}U_1^T & O \\ O & V_1\begin{pmatrix} O & O & O \\ O & D_2^{-1}(B_{22} - D_1\hat{X}_{22}D_1)D_2^{-1} & D_2^{-1}B_{23} \\ O & -B_{23}^T D_2^{-1} & B_{23} \end{pmatrix}V_1^T \end{pmatrix}U^T,$$

here

$$\hat{X}_{22} = -\Psi * B_{22},\ \Psi = \Psi(\varphi_{ij}) \in R^{s\times s},\ \phi_{ij} = \frac{a_i a_j}{b_i^2 b_j^2 + a_i^2 a_j^2},(i, j = 1,\ 2, ..., s).$$

According to Theorem 1 and Theorem 2, we give the optimal approximation algorithm for solving Problem II as following:

Step 1: Given $A \in R^{n\times m}$, $B \in R^{m\times m}$, $P \in SOR^{n\times n}$, arbitrary matrix $X^* \in R^{n\times n}$;

Step 2: Compute orthogonal matrix U by (5);

Step 3: Compute A_1, A_2, W, Σ_1, Σ_2, U_1 and V_1 by (6) and GSVD;

Step 4: Compute unblocking matrix B_{ij} $(i, j = 1, 2, 3, 4)$ by (9), if it satisfied condition of (10), otherwise stop.

Step5: Compute (25) by (23) and (24), then compute $\hat{X}_{ij}$, $\hat{Y}_{ij}$ by (25), compute optimal approximation $\hat{X}$ at last.

Example : Given symmetric orthogonal symmetric matrix P and coefficient matrix of A and B as following,

$$P=\begin{pmatrix} -0.0035 & 0.1995 & 0.3172 & 0.6947 & -0.0354 & -0.4997 & -0.2209 & 0.2533 & 0.1144 \\ 0.1995 & -0.0470 & 0.7622 & -0.1889 & 0.4493 & 0.0242 & 0.0999 & -0.2929 & 0.2076 \\ 0.3172 & 0.7622 & 0.2050 & -0.2095 & -0.3718 & 0.1159 & 0.1741 & 0.1322 & -0.1817 \\ 0.6947 & -0.1889 & -0.2095 & 0.4631 & 0.0118 & 0.1662 & 0.4105 & -0.1645 & 0.0056 \\ -0.0354 & 0.4493 & -0.3718 & 0.0118 & 0.7816 & -0.1000 & 0.1347 & 0.1305 & -0.0491 \\ 0.4997 & 0.0242 & 0.1159 & 0.1662 & -0.1000 & 0.1661 & 0.7334 & 0.1840 & 0.3154 \\ -0.2209 & 0.0999 & 0.1741 & 0.4105 & 0.1347 & 0.7334 & -0.2669 & -0.0408 & -0.3369 \\ 0.2533 & -0.2929 & 0.1322 & -0.1645 & 0.1305 & 0.1840 & -0.0408 & 0.8677 & -0.0050 \\ 0.1144 & 0.2076 & -0.1817 & 0.0056 & -0.0491 & 0.3154 & -0.3369 & -0.0050 & 0.8339 \end{pmatrix}$$

$$A=\begin{pmatrix} 0.77 & 0.80 & 0.25 & 0.40 & 1.10 & 0.50 & 0.45 & 0.89 \\ 0.32 & 0.60 & 0.23 & 0.43 & 0.96 & 0.66 & 0.09 & 0.36 \\ 0.46 & -0.48 & -0.02 & 0.32 & -0.27 & -0.08 & 0.04 & 0.22 \\ -0.04 & -0.08 & -0.08 & 0.01 & -0.29 & -0.15 & 0.01 & -0.29 \\ 0.14 & 0.55 & 0.19 & 0.30 & 0.82 & 0.57 & 0.04 & 0.21 \\ 0.56 & 0.46 & 0.20 & 0.44 & 0.80 & 0.48 & 0.22 & 0.54 \\ 0.36 & 0.54 & 0.08 & 0.04 & 0.48 & 0.08 & 0.35 & 0.42 \\ 0.13 & 0.61 & 0.12 & 0 & 0.65 & 0.26 & 0.20 & 0.34 \\ 0.05 & -0.13 & -0.01 & -0.11 & -0.06 & -0.11 & 0.02 & 0.22 \end{pmatrix}$$

$$B=\begin{pmatrix} 0 & 0.28 & -0.03 & -0.27 & 0.26 & -0.01 & -0.01 & 0.30 \\ -0.28 & 0 & -0.29 & -0.68 & -0.24 & -0.33 & -0.36 & 0.05 \\ 0.03 & 0.29 & 0 & -0.13 & 0.20 & -0.01 & 0.10 & 0.14 \\ 0.27 & 0.68 & 0.13 & 0 & 0.70 & 0.22 & 0.31 & 0.49 \\ -0.26 & 0.24 & -0.12 & -0.70 & 0 & -0.28 & -0.17 & 0.20 \\ 0.01 & 0.33 & 0.01 & -0.22 & 0.28 & 0 & 0.08 & 0.26 \\ 0.01 & 0.36 & -0.10 & -0.31 & 0.17 & -0.08 & 0 & 0.16 \\ -0.30 & -0.05 & -0.14 & -0.49 & -0.20 & -0.26 & -0.16 & 0 \end{pmatrix}$$

if

$$X^* = \begin{pmatrix} 0.58 & 0.31 & 0.56 & 0.95 & 0.37 & 0.55 & 0.72 & 0.19 & 0.51 \\ 0.26 & 0.86 & 0.86 & 0.62 & 0.84 & 0.13 & 0.99 & 0.67 & 0.89 \\ 0.78 & 0.43 & 0.21 & 0.45 & 0.54 & 0.42 & 0.10 & 0.38 & 0.64 \\ 0.14 & 0.05 & 0.30 & 0.64 & 0.81 & 0.05 & 0.45 & 0.07 & 0.20 \\ 0.77 & 0.88 & 0.58 & 0.97 & 0.48 & 0.79 & 0.13 & 0.68 & 0.06 \\ 0.51 & 0.25 & 0.90 & 0.42 & 0.11 & 0.55 & 0.70 & 0.85 & 0.80 \\ 0.58 & 0.28 & 0.52 & 0.97 & 0.14 & 0.77 & 0.30 & 0.09 & 0.42 \\ 0.51 & 0.22 & 0.66 & 0.51 & 0.78 & 0.99 & 0.06 & 0.39 & 0.89 \\ 0.14 & 0.59 & 0.67 & 0.54 & 0.26 & 0.08 & 0.17 & 0.37 & 0.99 \end{pmatrix}$$

then we can get the anti-symmetric orthogonal anti-symmetric optimal approximation solution of X^* as :

$$\hat{X} = \begin{pmatrix} -0.0000 & -0.4103 & 0.3426 & 0.1140 & 0.2606 & 0.0695 & -0.2180 & -0.4187 & -0.0362 \\ 0.4103 & -0.0000 & -0.2709 & -0.3987 & 0.2000 & 0.5209 & -0.7158 & -0.0182 & 0.0627 \\ -0.3426 & 0.2709 & -0.0000 & -0.0114 & 0.0020 & -0.7536 & 0.3221 & 0.1379 & 0.4758 \\ -0.1140 & 0.3987 & 0.0114 & 0.0000 & -0.1031 & -0.1176 & 0.2407 & -0.4859 & -0.1928 \\ -0.2606 & -0.2000 & -0.0020 & 0.1031 & 0.0000 & -0.1446 & 0.2277 & 0.3948 & 0.3669 \\ -0.0695 & -0.5209 & 0.7536 & 0.1176 & 0.1446 & 0.0000 & -0.1978 & -0.2816 & 0.2610 \\ 0.2180 & 0.7158 & -0.3221 & -0.2407 & -0.2277 & 0.1978 & 0.0000 & -0.0632 & -0.0433 \\ 0.4187 & 0.0182 & -0.1379 & 0.4859 & -0.3948 & 0.2816 & 0.0632 & 0.0000 & 0.0469 \\ 0.0362 & -0.0627 & -0.4758 & 0.1928 & -0.3669 & -0.2610 & 0.0433 & -0.0469 & -0.0000 \end{pmatrix}$$

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