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PROJECTION PROBLEM FOR REGULAR POLYGONS

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Abstract

In this paper a question posed about an affine invariant functional defined by Lutwak et al. [5] is studied and the answer for regular polygons is given.

1. Introduction

If K is a convex body (a compact, convex subset with nonempty interior) in n-dimensional Euclidean space R^n, then its support function $h(K, \cdot) : S^{n-1} \to R$ is defined by

$$h(K, u) = \max\{u \cdot x : x \in K\},$$

where $u \cdot x$ denotes the standard inner product of u and x. The projection body $\prod K$ of K can be defined as the convex body whose support function, for $u \in S^{n-1}$, is given by

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$$h(\prod K, u) = vol_{n-1}\left(K\middle|u^{\perp}\right),$$

where vol_{n-1} denotes $(n-1)$-dimensional volume and $K|u^{\perp}$ denotes the image of the orthogonal projection of K onto the codimension 1 subspace orthogonal to u.

For more information about convex bodies, see the books of Gardner [3] and Schneider [7].

An important unsolved problem is Schneider's projection problem [6, 7, 8]: What is the least upper bound, as K ranges over the class of origin-symmetric convex bodies, of the affine-invariant ratio

$$\left[V(\prod K)/\ V(K)^{n-1}\right]^{\frac{1}{n}},$$

where V is used to abbreviate vol_n. Just as mentioned in [5] in 1982, Schneider [6] conjectured that this ratio is maximized by parallelotopes. In [6], Schneider also presented applications of such results in stochastic geometry. However, in 1996, a counterexample was produced in [2] to show this is not the case.

In 2001, Lutwak et al. [5] introduced the following new centro-affine functional for a convex polytope and presented a modified version of Schneider's projection problem.

Definition. If P is a convex polytope in R^n which contains the origin in its interior, and $u_1, \cdots, u_N$ are the outer normal unit vectors to the faces of P, with $h_1, \cdots, h_N$ the corresponding distances of the faces from the origin and $a_1, \ldots, a_N$ the corresponding areas of the faces, then define $U(P)$ by

$$U(P)^n = \frac{1}{n^n} \sum_{u_{i_1} \wedge \ldots \wedge u_{i_n} \neq 0} h_{i_1} \cdots h_{i_n} a_{i_1} \cdots a_{i_n}.$$

Based on the above definition, Lutwak et al. proved the following theorem.

Theorem 1. *If K is an origin-symmetric convex polytope in R^n, then*

$$\frac{V(\prod K)}{U(K)^{\frac{n}{2}} V(K)^{\frac{n}{2}-1}} \leq 2^n \left(\frac{n^n}{n!}\right)^{\frac{1}{2}},$$

with equality if and only if K is a parallelotope.

Further, they posed two questions.

Question A. If P is an origin-symmetric convex polytope in R^n, then is it the case that

$$U(P) \geq n^{-1}(n!)^{\frac{1}{n}} V(P),$$

with equality if and only if P is a parallelotope?

Recently, He et al. [4] resolved Question A in the affirmative.

The John ellipsoid of a convex body is the largest (in volume) ellipsoid that is contained in the body. The John point of a convex body is the center of the John ellipsoid of the body. A convex body in R^n is said to be in *John position* if its John ellipsoid is the standard unit ball in R^n.

The other question posed by Lutwak et al. in [5] is:

Question B. Suppose P is a convex polytope in R^n with its John point at the origin. Is it the case that

$$U(P) \geq \frac{[(n+1)!]^{\frac{1}{n}}}{n+1} V(P),$$

with equality if and only if P is a simplex?

In this paper, we give a partial answer to Question B in R^2, obtaining the following theorem.

Theorem 2. *If P is a regular n-polygon in R^2 with its John point at the origin, then for $n \geq 6$*

$$U(P) \geq \frac{\sqrt{6}}{3} V(P),$$

with equality if and only if P is a regular hexagon.

For two convex bodies K, L in R^n, the Hausdorff metric $\delta(K, L)$ is defined by

$$\delta(K, L) = \sup_{u \in S^{n-1}} |h(K, u) - h(L, u)|.$$

Note that the functional U is centro-affine invariant in that,

$$U(\phi P) = U(P),$$

for all $\phi \in SL(n)$ and all regular polygons P with John point at the origin. Denote the set of regular polygon in R^2 by $\mathcal{P}$. Define the set $\mathcal{C}$ of convex polygons in R^2 by

$$\mathcal{C} = \{K | \forall \varepsilon > 0, \exists P \in \mathcal{P} \text{ and } \phi \in SL(n) \text{ such that } \delta(K, \phi P) < \varepsilon \}.$$

Then we can get the following corollary from Theorem 2.

Corollary. *If* $K \in \mathcal{C}$, *then for* $n \geq 6$

$$U(K) \geq \frac{\sqrt{6}}{3} V(K).$$

2. The Proof of the Main Result

The well-known theorem of John characterizes the ellipsoid of maximal volume contained in a convex body and is stated as a lemma. See [1].

Lemma 1. *Let C be a convex body in* R^n. *The ellipsoid of maximal volume in C is* B_2^n, *if and only if C contains* B_2^n *and there are contact points* $(u_i)_1^m$ *and positive number* $(c_i)_1^m$ *so that*

(a) $\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n$, *and*

(b) $\sum_{i=1}^{m} c_i u_i = 0.$

Condition (a) shows that the $(u_i)_1^m$ behave rather like an orthonormal basis in that we can resolve the Euclidean norm as a (weighted) sum of squares of inner products. This condition is equivalent to the statement that, for all x in R^n,

$$|x|^2 = \sum_{i=1}^{m} c_i \langle x, u_i \rangle^2,$$

where $\langle , \rangle$ is the usual Euclidean inner product and $|\cdot|$ is the induced norm by this inner product.

At first, we will show that the inscribed circle of regular polygon is its John circle by the method different from the usual affine transformation. For this, one need only prove that the tangent points of the regular polygon with its inscribed circle satisfy the condition (a) and (b) of Lemma 1.

Without loss of generalization, we suppose that a regular polygon P contains the unit circle B as its inscribed circle with $u_1, u_2, \dots, u_n$ arrayed anticlockwise as the unit outer normal vectors of the sides of P. Suppose that $z_1, z_2, \dots, z_n$ are the complex numbers corresponding to $u_1, u_2, \dots, u_n$. Then a equivalent condition to condition (b) is that

$$\sum_{i=1}^{n} c_i z_i = 0,$$

for some c_i.

Suppose that u_1 agrees with the unit vector on the x-axis and the angle between u_i and u_{i+1} is θ. Then

$$z_1 = \cos 0 + i \sin 0,$$

$$z_i = \cos(i-1)\theta + i \sin(i-1)\theta, \quad i = 2, 3, \dots, n.$$

Observe that $z_1, z_2, \ldots, z_n$ are the n roots of the complex equation

$$z^n = 0.$$

Thus by Viete's theorem we obtain

$$z_1 + z_2 + \ldots + z_n = 0.$$

Let $c_i = \frac{2}{n}$, $i = 1, 2, \ldots, n$. Then

$$\sum_{i=1}^{n} c_i z_i = 0.$$

Thus condition (b) is satisfied.

Next, we will prove that the u_i satisfy condition (a) of Lemma 1, that is, for any $x \in R^n$, the following equality holds:

$$|x|^2 = \sum_{i=1}^{m} c_i \langle x, u_i \rangle^2, \tag{1}$$

where $c_i = \frac{2}{n}$, $i = 1, \ldots, n$.

Equality (1) is equivalent to

$$\frac{n}{2} = \sum_{i=1}^{n} \langle u, u_i \rangle^2, \tag{2}$$

for all $u \in S^1$.

Let that $u = (\cos\varphi, \sin\varphi)$ and note that $u_i = (\cos(i-1)\theta, \sin(i-1)\theta)$, because $u_i (i = 1, 2, \ldots, n)$ correspond to the roots of $z^n = 1$.

We then have

$$\sum_{i=1}^{n} \langle u, u_i \rangle^2 = \sum_{i=1}^{n} [\cos(i-1)\theta \cos\varphi + \sin(i-1)\theta \sin\varphi]^2$$

$$= \sum_{i=1}^{n} \cos^2[(i-1)\theta + \varphi]$$

$$= \sum_{i=1}^{n} \frac{\cos[2(i-1)\theta + 2\varphi] + 1}{2}$$

$$= \frac{n}{2} + \frac{1}{2} \sum_{i=1}^{n} \cos[2(i-1)\theta + 2\varphi].$$

Note that $G_1 = \{z_i = \cos(i-1)\theta + i\sin(i-1)\theta, (i = 1, \dots n), \cdot\}$ is a cyclic group. If we denote $z_i' = \cos[2(i-1)\theta] + i\sin[2(i-1)\theta]\,(i = 1, \dots, n)$, then $G_2 = \{z_i', (i = 1, \dots, n), \cdot\}$ is a cyclic subgroup of G_1. Thus there must exist a complex equation $(z')^n = 1$ such that z_i' are the n roots. By Viete's theorem, we have

$$\sum_{i=1}^{n} \cos[2(i-1)\theta + 2\varphi] = \sum_{i=1}^{n} \cos[2(i-1)\theta] = 0.$$

Thus condition (a) is satisfied and we have the conclusion that the inscribed circle of regular polygon is its John circle.

The following lemma is also needed in the proof of Theorem 2.

Lemma 2. *If P is a polygon with n sides, then the pair number of mutually parallel sides is no more than* $\left[\frac{n}{2}\right]$.

Proof.

Case 1. For $n = 3, 4, 5,$ the result is obvious.

Case 2. For $n \geq 6$, suppose that a_i is a side of P with outer unit normal vector u_i. Then the number of sides parallel to a_i is no more than one. In fact, if not, suppose that $a_j, a_k, j \neq k$ with outer unit normal vectors u_j, u_k, respectively, are both parallel to a_i. Then the angle θ_{ij} between u_i and u_j is π and the angle θ_{ik} between u_i and u_k is also π. This contradicts the fact that $\theta_{ij} + \theta_{ik} < 2\pi$. Thus we get that if P is a polygon with n sides, then the pair number of mutually parallel sides is no more than $\left[\frac{n}{2}\right]$.

Proof of Theorem 2. *By definition of* $U(P)$ *and Lemma 2, we have*

$$\begin{aligned} U(P)^2 &= \frac{1}{4}\sum_{u_{i_1}\wedge u_{i_2}\neq 0} h_{i_1}h_{i_2}a_{i_1}a_{i_2} \\ &= \frac{1}{4}\sum_{i\neq j} a^2 - \frac{1}{4}\sum_{u_{i_1}\wedge u_{i_2}=0} a^2 \\ &\geq \frac{1}{4}n(n-1)a^2 - \frac{1}{4}2\left[\frac{n}{2}\right]a^2 \\ &\geq \frac{1}{4}n(n-1)a^2 - \frac{1}{4}na^2 \\ &= \frac{1}{4}a^2\left(n^2-2n\right). \end{aligned}$$

By directly computing, we have

$$V(P)^2 = \frac{1}{4}n^2a^2.$$

Thus

$$U(P) \geq \frac{\sqrt{6}}{3}V(P),$$

is equivalent to

$$\frac{1}{4}a^2\left(n^2-2n\right) \geq \frac{2}{3}\frac{1}{4}n^2a^2,$$

or

$$n^2 - 6n \geq 0.$$

So for $n \geq 6$ we have

$$U(P) \geq \frac{\sqrt{6}}{3}V(P).$$

For $n = 6$ when P is a regular hexagon, we have

$$U(P)^2 = \frac{1}{4}(\underbrace{a^2 + \ldots + a^2}_{24}) = 6a^2$$

and

$$V(P)^2 = 9a^2,$$

so

$$U(P) = \frac{\sqrt{6}}{3} V(P).$$

Remark

(i) For $n = 3$ when P is a regular triangle, we have

$$U(P)^2 = \frac{1}{4}\left(a^2 + a^2 + a^2 + a^2 + a^2 + a^2\right) = \frac{3}{2} a^2$$

and

$$V(P)^2 = \frac{9}{4} a^2,$$

so

$$U(P) = \frac{\sqrt{6}}{3} V(P).$$

(ii) For $n = 4$, by directly computing, we have

$$U(P)^2 = \frac{1}{4}\left(a^2 + a^2 + a^2 + a^2 + a^2 + a^2 + a^2 + a^2\right) = 2a^2$$

and

$$V(P)^2 = 4a^2,$$

so

$$U(P) < \frac{\sqrt{6}}{3} V(P).$$

This is a counterexample to Question B in R^2.

(iii) For $n = 5$, by directly computing, we have

$$U(P)^2 = \frac{1}{4}(\underbrace{a^2 + \ldots + a^2}_{20}) = 5a^2$$

and

$$V(P)^2 = \frac{25}{4}a^2,$$

so

$$U(P) > \frac{\sqrt{6}}{3}V(P).$$

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