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COINCIDENCE AND FIXED POINTS FOR FAMILIES OF MULTIMAPS

YEW-DANN CHEN[1], TIAN-YUAN KUO[2], JYH-CHUNG JENG[3] and YOUNG-YE HUANG[1]

[1]Center for General Education
Southern Taiwan University
1 Nan-Tai St. Yung-Kang City
Tainan Hsien 710, Taiwan

[2]Fooyin University
151 Chin-Hsueh Rd.
Ta-Liao Hsiang
Kaohsiung Hsien 831, Taiwan

[3]Nan-Jeon Institute of Technology
Yanshoei, Tainan Hsien 737, Taiwan
e-mail: jhychung@mail.njtc.edu.tw

Abstract

In this paper we at first establish some new coincidence and fixed point results of Lan-Webb type for a family of multimaps on a product G-convex space, and then apply them to the study of nonempty intersection, minimax, equilibrium problems and system of inequalities.

1. Introduction and Preliminaries

Ansari et al. [1] established some coincidence point and fixed point theorems for families of multimaps on noncompact convex subsets of

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product topological vector spaces, which generalized the fixed point results of Lan and Webb [6]. In this paper we shall extend this type of coincidence and fixed point results from topological vector spaces to generalized convex spaces under slight different conditions. And then we apply them to study nonempty intersection, minimax and equilibrium problems, and system of inequalities.

We begin with recalling some notations and terminology concerned with multimaps which will be used throughout the paper.

For a nonempty set Y, 2^Y and $\langle Y \rangle$ denote the class of all subsets of Y and the class of all nonempty finite subsets of Y, respectively. For $A \in \langle Y \rangle$, $|A|$ denotes the cardinality of A. A multimap $T : X \to 2^Y$ is a function from a set X into the power set 2^Y of Y. Its graph is denoted by $\mathrm{Gr}(T)$. The notation $T : X \multimap Y$ stands for a multimap $T : X \to 2^Y$ having nonempty values. If $B \subseteq Y$, then the inverse image of B under T is $T^-(B) = \{x \in X : T(x) \cap B \neq \varnothing\}$.

All topological spaces are supposed to be Hausdorff. The closure of a subset A of a topological space is denoted by $\mathrm{cl}(A)$. Let X and Y be two topological spaces. A multimap $T : X \to 2^Y$ is said to be

(a) upper semicontinuous (u.s.c.) if $T^-(B)$ is closed in X for each closed subset B of Y;

(b) lower semicontinuous (l.s.c.) if $T^-(B)$ is open in X for each open subset B of Y.

(c) compact if $\mathrm{cl}(T(X))$ is a compact subset of Y.

It is well-known that $\mathrm{Gr}(T)$ is closed if Y is requlars and T is u.s.c. with closed values.

Lemma 1.1 [2]. *Suppose X and Y are two topological spaces, $f : X \times Y \to \mathbb{R}$ is u.s.c., and $S : X \multimap Y$ has compact values. Define $\hat{f} : X \to \mathbb{R}$ by*

$$\hat{f}(x) = \max_{y \in S(x)} f(x, y)$$

for all $x \in X$. *Then* $\hat{f}$ *is u.s.c. provided S is u.s.c.*

A subset A of a topological space X is compactly open (resp., compactly

closed) if for every nonempty compact subset K of X, $A \cap K$ is open (resp., closed) in K. The compactly interior of A is defined by

$$\operatorname{cint}(A) = \bigcup \{B \subseteq X : B \subseteq A \text{ and } B \text{ is compactly open in } X\},$$

cf. [3]. It is easy to see that cint(A) is compactly open, and A is compactly open if and only if $X \setminus A$ is compactly closed.

Definition 1.2 [3]**.** *Let X be a nonempty set and Y be a topological space. A family* $\{A_\lambda\}_{\lambda \in \Lambda}$ *of subsets of Y is said to be transfer compactly open if for any* $\lambda \in \Lambda$ *and any* $y \in A_\lambda$ *there is* $\alpha \in \Lambda$ *such that* $y \in \operatorname{cint}(A_\alpha)$. *A multimap* $T : X \to 2^Y$ *is said to be transfer compactly open valued if* $\{T(x) : x \in X\}$ *is a transfer compactly open family in Y. A function* $f : X \times Y \to \mathbb{R}$ *is said to be transfer compactly l.s.c.* (*resp., u.s.c.*) *on X if for each* $x \in X$ *and each* $\lambda \in \mathbb{R}$ *with* $x \in \{u \in X : f(u, y) > \lambda\}$ (resp., $x \in \{u \in X : f(u, y) < \lambda\}$) *there exists an* $y' \in Y$ *such that*

$$x \in \operatorname{cint}(\{u \in X : f(u, y') > \lambda\})$$

$$(\text{resp., } x \in \operatorname{cint}(\{u \in X : f(u, y') < \lambda\})).$$

Throughout this paper, for a family of topological spaces $\{E_i\}_{i \in I}$, the product spaces $\Pi_{i \in I} E_i$ and $\Pi_{i \in I \setminus \{i\}} E_i$ are denoted by E and E^i respectively. And any point $x \in E$ is expressed as $x = [x^i, x_i]$, where $x^i \in E^i$ and $x_i \in E_i$. For a subset A of E, $x^i \in E^i$ and $x_i \in E_i$, the sections $A[x^i]$ and $A[x_i]$ are defined as

$$A[x^i] = \{v_i \in E_i : [x^i, v_i] \in A\},$$

$$A[x_i] = \{u^i \in E^i : [u^i, x_i] \in A\}.$$

The following two lemmas are easy to derive.

Lemma 1.3. *Let* $\{A_i\}_{i \in I}$ *be a family of subsets of* $E = \Pi_{i \in I} E_i$. *Then the following two statements are equivalent.*

(1.3.1) *For any* $i \in I$, $A_i[x^i]$ *is nonempty and* $\{A_i[x_i] : x_i \in E_i\}$ *is a transfer compactly open family.*

(1.3.2) *For any* $i \in I$, $E^i = \bigcup_{x_i \in E_i} cint(A_i[x_i])$.

If X and Y are topological spaces and $T : X \to 2^Y$, then, putting $A_i = \mathrm{Gr}(T)$ in the above lemma, we see that $X = \bigcup_{y \in Y} \mathrm{cint}(T^-(y))$ is equivalent to that T has nonempty values and $T^- : Y \to 2^X$ is transfer compactly open valued.

Lemma 1.4. *Let* X *and* Y *be two topological spaces and* $f : X \times Y \to \mathbb{R}$. *Then* f *is transfer compactly l.s.c.* (*resp.*, *u.s.c.*) *on* X *if and only if for any* $\lambda \in \mathbb{R}$ *the multimap* $F : Y \to 2^X$ *defined by*

$$F(y) = \{x \in X : f(x, y) > \lambda\}$$

$$(resp., F(y) = \{x \in X : f(x, y) < \lambda\})$$

is transfer compactly open valued.

For $n \geq 0$, Δ_n denotes the standard n-simplex of $\mathbb{R}^{n+1}$, that is,

$$\Delta_n = \left\{\alpha = (\alpha_0, \cdots, \alpha_n) \in \mathbb{R}^{n+1} : \alpha_i \geq 0 \text{ for all } i \text{ and } \sum_{i=0}^{n} \alpha_i = 1\right\};$$

and $\{\mathbf{e}_0, \cdots, \mathbf{e}_n\}$, the standard basis of $\mathbb{R}^{n+1}$, is the set of the vertices of Δ_n.

The notion of a generalized convex space was first introduced by Park and Kim [9]:

Definition 1.5. *A generalized convex space or a G-convex space* $(E;\Gamma)$ *consists of a topological space E and a map* $\Gamma : \langle E\rangle \multimap E$ *such that*

(1.4.1) *for any* $A, B \in \langle E\rangle$, $A \subseteq B$ *implies* $\Gamma(A) \subseteq \Gamma(B)$; *and*

(1.4.2) *for each* $A = \{a_0, \cdots, a_n\} \in \langle E\rangle$ *with* $|A| = n+1$, *there exists a continuous function* $\varphi_A : \Delta_n \to \Gamma(A)$ *such that* $\varphi_A(\Delta_J) \subseteq \Gamma(B)$ *for any* $B \in \langle A\rangle$ *with* $|B| = |J|+1$, *where* $J \subseteq \{0, \cdots, n\}$ *and* Δ_J *denotes the face of* Δ_n *corresponding to B.*

In this paper, we assume that a G-convex space $(E;\Gamma)$ always satisfies the extra condition: $x \in \Gamma(\{x\})$ for any $x \in E$.

A subset K of a G-convex space $(E;\Gamma)$ is said to be Γ-*convex* if for any $A \in \langle K\rangle$, $\Gamma(A) \subseteq K$. For a nonempty subset Q of E, the Γ-*convex* hull of Q, denoted by Γ-*co*(Q), is defined by

$$\Gamma\text{-co}(Q) = \bigcap\{C : Q \subseteq C \subseteq E,\ C \text{ is } \Gamma\text{-}convex\}.$$

It is easy to see that Γ-co(Q) is the smallest Γ-*convex* subset of E containing Q.

Tan and Zhang [10] showed that the product of an arbitrary family of G-convex spaces is a G-convex space: Suppose $\{(E_i;\Gamma_i)\}_{i\in I}$ is any family of *G-convex spaces*. Let $E = \Pi_{i\in I}E_i$ be equipped with product topology. For each $i \in I$, let $\pi_i : E \to E_i$ be the projection. Define $\Gamma = \Pi_{i\in I}\Gamma_i : \langle E\rangle \multimap E$ by

$$\Gamma(A) = \Pi_{i\in I}\Gamma_i(\pi_i(A))$$

for each $A \in \langle E\rangle$. Then $(E;\Gamma)$ is a G-convex space.

The following Lemma 1.6 is well-known, cf. [5].

Lemma 1.6. *Let* $(E;\Gamma) = (\Pi_{i\in I}E_i; \Pi_{i\in I}\Gamma_i)$ *be the product G-convex space of a family of G-convex spaces* $(E_i;\Gamma_i)$, $i \in I$. *Then* $K = \Pi_{i\in I}K_i$ *is* Γ-*convex in E provided for each* $i \in I$, K_i *is a* Γ_i-*convex subset of* E_i.

2. Coincidence and Fixed Point Theorems

Following Park [8], we make the following definition.

Definition 2.1. *Let X be a topological space and $(E;\Gamma)$ a G-convex space. A multimap $T : X \to 2^E$ is called a generalized Φ-mapping if there exists a multimap $S : X \to 2^E$ satisfying that*

(2.1.1) *for any $x \in X$ and any $A \in \langle S(x)\rangle$, $\Gamma\text{-co}(A) \subseteq T(x)$;*

(2.1.2) $X = \bigcup_{y\in E} \operatorname{cint}(S^-(y))$.

The mapping S is called a companion mapping of T.

If T has nonempty and Γ-*convex* values and T^- is transfer compactly open valued, *then* T is a generalized Φ-mapping with itself as a companion mapping.

From now on, $(E;\Gamma) = (\Pi_{i\in I}E_i;\Pi_{i\in I}\Gamma_i)$ will always denote the product G-convex space of a family of G-convex spaces $(E_i;\Gamma_i)$, $i \in I$. And for each $i \in I$, the convex structure Γ_i is assumed to have the property that $\Gamma_i\text{-co}(C)$ or $\operatorname{cl}(\Gamma_i\text{-co}(C))$ is compact Γ_i-convex whenever C is a compact subset of E_i, and $E^i = \Pi_{j\in I\setminus\{i\}}E_j$ will be equipped with the product convex structure $\Gamma^i = \Pi_{j\in I\setminus\{i\}}\Gamma_j$.

Any Fréchet space has the property that $\operatorname{cl}(\operatorname{co}(C))$ is compact whenever C is a compact subset, cf. [11, Theorem 3.20].

We are now able to establish the following coincidence result on G-convex spaces.

Theorem 2.2. *Suppose the two multimaps $F_i : E_i \to 2^{E_i}$ and $T_i : E^i \to 2^{E_i}$ satisfy the following conditions*

(2.2.1) *$F_i^-\circ T_i$ is a generalized Φ-mapping with the companion mapping S_i.*

(2.2.2) *There exist a nonempty compact subset* K^i *of* E^i *and a nonempty compact subset* M_i *of* E_i *such that*

$$E^i \setminus K^i \subseteq \bigcup_{x_i \in M_i} \operatorname{cint}(S_i^-(x_i)).$$

Then there exists $\bar{x} \in E$ *such that* $T_i(\bar{x}^i) \cap F_i(\bar{x}_i) \neq \varnothing$ *for all* $i \in I$.

Proof. For each $i \in I$, define the multimap $H_i : E_i \to 2^{E^i}$ by

$$H_i(x_i) = E^i \setminus \operatorname{cint}(S_i^-(x_i))$$

for $x_i \in E_i$. Obviously, each $H_i(x_i)$ is compactly closed in E^i. Moreover, we claim that

(I) $\bigcap_{x_i \in M_i} H_i(x_i)$ is compact, and

(ii) $\bigcap_{x_i \in E_i} H_i(x_i) = \varnothing$.

It follows from (2.2.2) that

$$\begin{aligned}\bigcap_{x_i \in M_i} H_i(x_i) &= \bigcap_{x_i \in M_i} \{x^i \in E^i : x^i \notin \operatorname{cint}(S_i^-(x_i))\} \\ &\subseteq K^i,\end{aligned}$$

so

$$\begin{aligned}\bigcap_{x_i \in M_i} H_i(x_i) &= \left(\bigcap_{x_i \in M_i} H_i(x_i)\right) \cap K^i \\ &= \bigcap_{x_i \in M_i} (H_i(x_i) \cap K^i)\end{aligned}$$

is compact in K^i by noting that K^i is compact and $H_i(x_i) \cap K^i$ is closed in K^i for any $x_i \in M_i$. As for (ii), since $E^i = \bigcup_{x_i \in E_i} \operatorname{cint}(S_i^-(x_i))$ by (2.2.1), we have

$$\begin{aligned}\bigcap_{x_i \in E_i} H_i(x_i) &= \bigcap_{x_i \in E_i} (E^i \setminus \operatorname{cint}(S_i^-(x_i))) \\ &= E^i \setminus \left(\bigcup_{x_i \in E_i} \operatorname{cint}(S_i^-(x_i))\right) \\ &= \varnothing\end{aligned}$$

for each $i \in I$.

Now we show that there exists $A = \{a_{i,0}, \cdots, a_{i,l_i}\} \in \langle E_i \rangle$ such that

$$(\bigcap_{x_i \in M_i} H_i(x_i)) \bigcap (\bigcap_{j=0}^{l_i} H_i(a_{i,j})) = \varnothing. \tag{1}$$

If not, then for any $\{y_0, \cdots, y_n\} \in \langle E_i \rangle$ one has

$$(\bigcap_{x_i \in M_i} H_i(x_i)) \bigcap (\bigcap_{j=0}^{n} H_i(y_j)) \neq \varnothing,$$

so the family $\{R_i(y) : y \in E_i\}$ has the finite intersection property, where $R_i(y) = (\bigcap_{x_i \in M_i} H_i(x_i)) \bigcap H_i(y)$ for any $y \in E_i$. Noting that each $R_i(y)$ is compact in K^i, we conclude that $\bigcap_{y \in E_i} R_i(y) \neq \varnothing$, a contradiction to (ii). Hence (1) holds.

By (1) we have

$$E^i = (\bigcup_{x_i \in M_i} \operatorname{cint}(S_i^-(x_i))) \bigcup (\bigcup_{j=0}^{l_i} \operatorname{cint}(S_i^-(a_{i,j}))). \tag{2}$$

Let $Q_i = \Gamma_i\text{-co}(M_i \cup \{a_{i,0}, \cdots, a_{i,l_i}\})$ or $Q_i = \operatorname{cl}(\Gamma_i\text{-co}(M_i \cup (a_{i,0}, \cdots, a_{i,l_i}\}))$ and $Q^i = \Pi_{j \in I \setminus \{i\}} Q_j$. By our assumption on the convex structure Γ_i, each Q_i is a compact Γ_i-convex subset of E_i and Q^i is a compact Γ_i-convex subset of E^i. So it follows from (2) that there exists $\{b_{i,0}, \cdots, b_{i,m_i}\} \in \langle M_i \rangle$ such that

$$Q^i \subseteq \bigcup_{j=0}^{m_i} \operatorname{cint}(S_i^-(b_{i,j})) \bigcup (\bigcup_{j=0}^{l_i} \operatorname{cint}(S_i^-(a_{i,j}))). \tag{3}$$

Letting $\{c_{i,0} \cdots, c_{i,n_i}\} = \{a_{i,0}, \cdots, a_{i,l_i}, b_{i,0}, \cdots, b_{i,m_i}\}$, we rewrite (3) as

$$Q^i \subseteq \bigcup_{j=0}^{n_i} \operatorname{cint}(S_i^-(c_{i,j})).$$

Put $X_i = \Gamma_i\text{-co}(\{c_{i,0} \cdots, c_{i\,n_i}\})$ and $X^i = \Pi_{j \in I \setminus \{i\}} X_j$. By the definition of a G-convex space, there exists a continuous mapping $\varphi_i : \Delta_{n_i} \to X_i$ such that

$$\varphi_i(\Delta_J) \subseteq \Gamma_i(C) \subseteq X_i$$

for any $C \in \langle\{c_{i,0} \cdots, c_{i,n_i}\}\rangle$ with $|C| = |J| + 1$, where $J \in \{0, \cdots, n_i\}$. Meanwhile, since $X^i = \bigcup_{j=0}^{n_i}(\text{cint}\,(S_i^-(c_{i,j})) \cap X^i)$ is compact, there is a partition of unity $\{\lambda_{i,j}\}_{j=0}^{n_i}$ subordinated to $\{\text{cint}\,(S_i^-(c_{i,j})) \cap X^i\}_{j=0}^{n_i}$. For any $j \in \{0, \cdots, n_i\}$ and any $x^i \in X^i$, we have

$$\lambda_{i,j}(x^i) \neq 0 \Leftrightarrow x^i \in \text{cint}\,(S_i^-(c_{i,j})) \cap X^i$$

$$\Rightarrow c_{i,j} \in S_i(x^i). \tag{4}$$

Define a mapping $\psi_i : X^i \to \Delta_{n_i}$ by

$$\psi_i(x^i) = \sum_{j=0}^{n_i} \lambda_{i,j}(x^i)\mathbf{e}_j.$$

Clearly, ψ_i is continuous and $\psi_i(x^i) = \sum_{j \in J(x^i)} \lambda_{i,j}(x^i)\mathbf{e}_j$, where $J(x^i) = \{j \in \{0, \cdots, n_i\} : \lambda_{i,j}(x^i) \neq 0\}$. We see from (4) that $\{c_{i,j} : j \in J(x^i)\} \in \langle S_i(x^i)\rangle$, so in view of $F_i^- \text{o} T_i$ being a generalized Φ-mapping with the companion mapping S_i, we conclude that

$$\Gamma_i(\{c_{i,j} : j \in J(x^i)\}) \subseteq \Gamma_i\text{-co}(\{c_{i,j} : j \in J(x^i)\})$$

$$\subseteq F_i^- \text{o} T_i(x^i). \tag{5}$$

Define $f_i : X^i \to X_i$ by $f_i(x^i) = (\varphi_i \text{o} \psi_i)(x^i)$. For each $x^i \in X^i$, it follows from (5) that

$$f_i(x^i) = \varphi_i(\psi_i(x^i))$$

$$\in \varphi_i(\Delta_{J(x^i)})$$

$$\subseteq F_i^-(T_i(x^i)).$$

Let $\Delta = \Pi_{i\in I}\Delta_{n_i}$, which is a compact convex subset of the locally convex topological space $\Pi_{i\in I}\mathbb{R}^{n_i+1}$. Define two continuous mappings $\Omega : \Delta \to X = \Pi_{i\in I}X_i$ and $\Psi : X \to \Delta$ by

$$\Omega(t) = \Pi_{i\in I}\varphi_i(\pi_i(t)) \text{ for } t \in \Delta, \text{ and}$$

$$\Psi(x) = \Pi_{k\in I}\psi_k(\pi^k(x)) \text{ for } x \in X,$$

where π_i is the projection of Δ onto Δ_{n_i}, and π^k is the projection of X onto X^k. Since $\Psi \circ \Omega : \Delta \to \Delta$ is continuous, it has a fixed point $\bar{t} \in \Delta$ by Tychonoff's fixed point theorem. Let $\bar{x} = \Omega(\bar{t})$. Noticing that

$$\begin{aligned}
\bar{x} &= \Omega(\bar{t}) \\
&= \Omega(\Psi \circ \Omega(\bar{t})) \\
&= (\Omega \circ \Psi)(\bar{x}) \\
&= \Omega(\Pi_{k\in I}\psi_k(\bar{x}^k)) \\
&= \Pi_{i\in I}\varphi_i(\psi_i(\bar{x}^i)) \\
&= \Pi_{i\in I}f_i(\bar{x}^i),
\end{aligned}$$

we see that

$$\bar{x}^i = f_i(\bar{x}^i) \in F_i^-(T_i(\bar{x}^i))$$

for any $i \in I$. This says that $F_i(\bar{x}_i) \cap T_i(\bar{x}^i) \neq \varnothing$.

When $F_i(x_i) = \{x_i\}$ for any $x_i \in E_i$, Theorem 2.2 becomes a fixed point theorem of Lan-Webb type.

Theorem 2.3. *Suppose the multimap $T_i : E^i \to 2^{E_i}$ satisfies the following conditions*:

(2.2.1) *T_i is a generalized Φ-mapping with the companion mapping S_i.*

(2.2.2) *There exist a nonempty compact subset* K^i *of* E^i *and a nonempty compact subset* M_i *of* E_i *such that*

$$E^i \setminus K^i \subseteq \bigcup_{x_i \in M_i} \mathrm{cint}\,(S_i^-(x_i)).$$

Then there exists $\bar{x} \in E$ *such that* $\bar{x}_i \in T_i(\bar{x}^i)$ *for all* $i \in I$.

3. Applications

The fixed point theorem of Lan-Webb type in Section 2 can be used to derive generalization of the von Neumann intersection theorem for sets with convex sections.

Theorem 3.1. *Let* $\{A_i\}_{i \in I}$ *be a family of subsets of* $E = \Pi_{i \in I} E_i$. *Suppose the following conditions hold*:

(3.1.1) *For each* $i \in I$ *and each* $x^i \in E^i$, $A_i[x^i]$ *is nonempty and* Γ_i*-convex.*

(3.1.2) *For each* $i \in I$, $\{A_i[x_i] : x_i \in E_i\}$ *is a transfer compactly open family.*

(3.1.3) *There exists a nonempty compact subset* K^i *of* E^i *and a nonempty compact subset* M_i *of* E_i *such that*

$$E^i \setminus K^i \subseteq \bigcup_{x_i \in M_i} \mathrm{cint}\,(A_i[x_i]).$$

Then there is $\bar{x} \in E$ *such that* $\bar{x} \in \bigcap_{i \in I} A_i$.

Proof. For each $i \in I$, define $T_i : E^i \to 2^{E_i}$ by $T_i(x^i) = A_i[x^i]$ for all $x^i \in E^i$. It is easy to check that T_i is a generalized Φ-mapping with itself as a companion mapping. Then the conclusion follows from Theorem 2.3.

This theorem is a generalization to Theorem 2.3 of [6], which in turn extends Theorem 2 of [7].

An immediate consequence of the above theorem is the result about existence of solution for a system of imequalities. To formulate it we at first recall that a real-valued function f on a G-convex space $(E; \Gamma)$ is said to be quasiconcave (resp., quasiconvex) if for any $\alpha \in \mathbb{R}$, the set $\{x \in E : f(x) \geq \alpha\}$ (resp., $\{x \in E : f(x) \leq \alpha\}$) is Γ-convex. If f is quasiconcave (resp., quasiconvex), then for any $\alpha \in R$, the set $X = \{x \in E : f(x) > \alpha\}$ (resp., $X = \{x \in E : f(x) < \alpha\}$) is Γ-convex. To see this, observe that for any $A = \{x_1, \cdots, x_n\} \in \langle X \rangle$, the set $Y = \{x \in E : f(x) \geq \min\{f(x_1), \cdots, f(x_n)\}\}$ (resp., $Y = \{x \in E : f(x) \leq \max\{f(x_i), \cdots, f(x_n)\}\}$) is Γ-convex. and $Y \subseteq X$. So $\Gamma(A) \subseteq Y \subseteq X$, which shows that X is Γ-convex.

Theorem 3.2. *Let $\{t_i\}_{i \in I}$ be a family of real numbers, and for each $i \in I$ suppose $f_i, g_i : E^i \times E_i \to \mathbb{R}$ are functions satisfying the following conditions*:

(3.2.1) *$f_i(x) \leq g_i(x)$ for all $x \in E$.*

(3.2.2) *For each $x^i \in E^i$, $x_i \mapsto g_i(x^i, x_i)$ is quasiconcave on E_i.*

(3.2.3) *f_i is transfer compactly l.s.c. on E^i.*

(3.2.4) *For each $x^i \in E^i$, there is an $x_i \in E_i$ such that $f_i(x^i, x_i) > t_i$.*

(3.2.5) *There exist a nonempty compact subset K^i of E^i and a nonempty compact subset M_i of E_i such that for any $x^i \in E^i \setminus K^i$ there is a $y_i \in M_i$ with*

$$x^i \in \operatorname{cint}(\{u^i \in E^i : f_i(u^i, y_i) > t_i\}).$$

Then there is an $\bar{x} \in E$ such that $g_i(\bar{x}) > t_i$ for all $i \in I$.

Proof. For each $i \in I$, let $A_i = \{x \in E : f_i(x) > t_i\}$ and $B_i = \{x \in E : g_i(x) > t_i\}$. Then for any $x^i \in E^i$, $A_i[x^i] \neq \varnothing$ by (3.2.4), and $A_i[x^i] \subseteq B_i[x^i]$ by (3.2.1). Moreover, it follows from (3.2.2) that

each $B_i[x^i]$ is Γ_i-convex, so $\Gamma_i\text{-co}(A_i[x^i]) \subseteq B_i[x^i]$. Also, we have

$$E^i = \bigcup_{x_i \in E_i} \text{cint}\,(A_i[x_i]) \subseteq \bigcup_{x_i \in E_i} \text{cint}\,(B_i[x_i])$$

by (3.2.3), and

$$E^i \setminus K^i \subseteq \bigcup_{y_i \in M_i} \text{cint}\,(A_i[y_i]) \subseteq \bigcup_{y_i \in M_i} \text{cint}\,(B_i[y_i])$$

by (3.2.5). Thus all the requirements of Theorem 3.1 are satisfied, and so $\bigcap_{i \in I} B_i \neq \varnothing$. Any $\bar{x} \in \bigcap_{i \in I} B_i$ satisfies that $g_i(\bar{x}) > t_i$ for all $i \in I$.

We now apply Theorem 2.3 directly to deduce another result about existence of solution for a system of inequalities.

Theorem 3.3. *For any $i \in I$, suppose $f_i : E^i \to \mathbb{R}$ and $g_i : E \to \mathbb{R}$ satisfy the following conditions*:

(3.3.1) *For any $x_i \in E_i$, $x^i \mapsto g_i(x^i, x_i)$ is l.s.c. on E^i.*

(3.3.2) *Each f_i is u.s.c. on E^i.*

(3.3.3) *For any $x = [x^i, x_i] \in E$, the set $\{y_i \in E_i : g_i(x^i, y_i) > f_i(x^i)\}$ is nonempty and* Γ_i-convex.

(3.3.4) *There exist a nonempty compact subset K^i of E^i and a nonempty compact subset M_i of E_i such that for any $x^i \in E^i \setminus K^i$ there is a $y_i \in M_i$ with*

$$x^i \in \text{cint}\,(\{u^i \in E^i : g_i(u^i, y_i) > f_i(x^i)\}).$$

Then there is an $\bar{x} \in E$ such that $g_i(\bar{x}) > f_i(\bar{x}^i)$ for all $i \in I$,

Proof. For any $i \in I$, Define $T_i : E^i \to 2^{E_i}$ by

$$T_i(x^i) = \{y_i \in E_i : g_i(x^i, y_i) > f_i(x^i)\}$$

for $x^i \in E^i$. We at first claim that each T_i is a Φ-mapping with itself as a companion mapping. For this it suffices to show that

(i) each $T_i(x^i)$ is nonempty and Γ_i-convex, and

(ii) For any $y_i \in E_i$, $T_i^-(y_i)$ is open in E^i.

Obvioulsy, (i) follows from (3.3.3) immediately and (ii) holds by noting that

$$T_i^-(y_i) = \{x^i \in E^i : g_i(x^i, y_i) > f_i(x^i)\}$$

$$= \bigcup_{t\in\mathbb{R}} (\{x^i \in E^i : g_i(x^i, y_i) > t\} \cap \{x^i \in E^i : t > f_i(x^i)\})$$

and for any $t \in \mathbb{R}$ both of $\{x^i \in E^i : g_i(x^i, y_i) > t\}$ and $\{x^i \in E^i : t > f_i(x^i)\}$ are open by (3.3.1) and (3.3.2) respectively.

Next, (3.3.4) says that $E^i \setminus K^i \subseteq \bigcup_{y_i\in M_i} \operatorname{cint}(T_i^-(y_i))$. So, by Theorem 2.3, there is an $\bar{x} \in E$ such that $\bar{x}_i \in T_i(\bar{x}^i)$ for all $i \in I$. This says that $g_i(\bar{x}) > f_i(\bar{x}^i)$ for all $i \in I$.

We like to mention that a similar result to the above theorem was obtained by Idzik [3, Theorem 5]. Another application of Theorem 2.3 is the following equilibrium existence theorem.

Theorem 3.4. *Let I be a nonempty finite index set. Suppose for any $i \in I$, the following conditions hold*:

(3.4.1) *$S_i : E^i \multimap E_i$ is a u.s.c. compact map with closed Γ_i-convex values and $S_i^-(y_i)$ is compactly open for any $y_i \in E_i$.*

(3.4.2) *$f_i, g_i : E \to \mathbb{R}$ are u.s.c. on E_i and E, respectively, with $f_i(x) \le g_i(x)$ for any $x \in E$.*

(3.4.3) *For each $y_i \in E_i$, $x^i \mapsto g_i(x^i, y_i)$ is l.s.c.*

(3.4.4) *The map $x^i \mapsto h_i(x^i) = \max_{y_i\in S_i(x^i)} f_i(x^i, y_i)$ is l.s.c.*

(3.4.5) *For each $x^i \in E^i$, $y_i \mapsto g_i(x^i, y_i)$ is quasiconcave.*

Then there exists an $\bar{x} \in E$ *such that* $\bar{x}_i \in S_i(\bar{x}^i)$ *and* $g_i(\bar{x}) \geq h_i(\bar{x}^i)$ *for all* $i \in I$.

Proof. For any $i \in I$, let $K_i = \mathrm{cl}(S_i(E^i))$. Then K_i is a nonempty compact subset of E_i. Also, for any $i \in I$ and any $k \in \mathbb{N}$, define $T_{(i,k)} : E^i \to 2^{K_i}$ by

$$T_{(i,k)}(x^i) = \left\{y_i \in S_i(x^i) : g_i(x^i, y_i) > h_i(x^i) - \frac{1}{k}\right\}.$$

Since f_i is u.s.c. on E_i and $S_i(x^i)$ is compact, there is $\hat{y}_i \in S_i(x^i)$ such that $h_i(x^i) = f_i(x^i, \hat{y}_i)$. This shows that $\hat{y}_i \in T_{(i,k)}(x^i)$, and hence each $T_{(i,k)}(x^i)$ is nonempty. By (3.4.5), each $T_{(i,k)}(x^i)$ is Γ_i-convex. Moreover, for each $y_i \in E_i \setminus K_i$, $T^-_{(i,k)}(y_i) = \varnothing$, and for each $y_i \in K_i$, the set $\{x^i \in E^i : g_i(x^i, y_i) > h_i(x^i) - \frac{1}{k}\}$ is open by noting that h_i is u.s.c. (cf. Lemma 1.1) and $x^i \mapsto g_i(x^i, y_i)$ is l.s.c., which together with $S_i^-(y_i)$ compactly open shows that

$$\begin{aligned} & T^-_{(i,k)}(y_i) \\ &= \left\{x^i \in E^i : y_i \in S_i(x^i) \text{ and } g_i(x^i, y_i) > h_i(x^i) - \frac{1}{k}\right\} \\ &= S_i^-(y_i) \cap \left\{x^i \in E^i : g_i(x^i, y_i) > h_i(x^i) - \frac{1}{k}\right\} \end{aligned}$$

is compactly open. Consequently,

$$E^i = \bigcup_{y_i \in E_i} T^-_{(i,k)}(y_i) = \bigcup_{y_i \in E_i} \mathrm{cint}\,(T^-_{(i,k)}(y_i)).$$

Furthermore, it is easy to see that $E^i \setminus K^i \subseteq \bigcup_{y_i \in K_i} \mathrm{cint}\,(T^-_{(i,k)}(y_i))$, so by Theorem 2.3, there exists an $\bar{x}_{(k)} = [\bar{x}^{(i,k)}, \bar{x}_{(i,k)}] \in E$ such that $\bar{x}_{(i,k)} \in T_{(i,k)}(\bar{x}^{(i,k)})$ for any $i \in I$ and any $k \in \mathbb{N}$, that is,

$$\bar{x}_{(i,k)} \in S_i(\bar{x}^{(i,k)})$$

and

$$g_i(\bar{x}^{(i,k)}, \bar{x}_{(i,k)}) > \max_{v \in S_i(\bar{x}^{(i,k)})} f_i(\bar{x}^{(i,k)}, v) - \frac{1}{k}.$$

Since I is finite and each K_i is compact, we may assume that $\bar{x}_{(k)} \to \bar{x} \in K = \Pi_{i \in I} K_i$. Also, since $\mathrm{Gr}(S_i)$ is closed, we see that $\bar{x}_i \in S_i(\bar{x}^i)$ for any $i \in I$. By (3.4.2) and (3.4.4), we have

$$\begin{aligned} g_i(\bar{x}^i, x_i) &\geq \limsup_{k \to \infty} g_i(\bar{x}^{(i,k)}, \bar{x}_{(i,k)}) \\ &\geq \limsup_{k \to \infty} \left[\max_{v \in S_i(\bar{x}^{(i,k)})} f_i(\bar{x}^{(i,k)}, v) - \frac{1}{k} \right] \\ &\geq \liminf_{k \to \infty} \left[\max_{v \in S_i(\bar{x}^{(i,k)})} f_i(\bar{x}^{(i,k)}, v) - \frac{1}{k} \right] \\ &\geq \max_{v \in S_i(\bar{x}^i)} f_i(\bar{x}^i, v) = h_i(\bar{x}^i) \end{aligned}$$

for all $i \in I$, where the last inequality follows from (3.4.4).

Corollary 3.5. *Suppose all assumptions are as in Theorem 3.4. If $f_i = g_i$ for all $i \in I$, then there exists an $\bar{x} \in E$ such that $\bar{x}_i \in S_i(\bar{x}^i)$ and $g_i(\bar{x}^i, \bar{x}_i) = \max_{v \in S_i(\bar{x}^i)} g_i(\bar{x}^i, v)$ for all $i \in I$.*

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