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VISCOSITY APPROXIMATION TO COMMON FIXED POINTS OF k_n-LIPSCHITZIAN NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract

Let E be a real Banach space with uniformly Gâteaux differentiable norm possessing uniform normal structure. K is a nonempty bounded closed convex subset of E, and $\{T_n\}$ $(n = 1, 2, \ldots)$ is a sequence of k_n- Lipschitzian nonexpansive mappings from K into itself such that $\lim_{n\to\infty} k_n = 1$ and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and f be a contraction on K. Under sutiable conditions on sequence $\{t_n\}$, we show the sequence $\{x_n\}$ defined as $x_n = (1 - \frac{t_n}{k_n})f(x_n) + \frac{t_n}{k_n} T_n x_n$ exists and converges strongly to a fixed point of a mapping T. And we apply it to prove the iterative process defined by $x_1 \in K$ and $x_{n+1} = (1 - \frac{t_n}{k_n})f(x_n) + \frac{t_n}{k_n} T_n x_n$ converges strongly to the same point.

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1. Introduction

Let E be a real Banach space with dual E^* and K be a nonempty closed convex subset of E. Let $J : E \to 2^{E^*}$ denote the normalized duality mapping defined by $J(x) = \{f \in E^*, \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}$, $\forall x \in E$. $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the sequel, we shall denote the single-valued duality mapping by j.

A mapping $T : K \to K$ is called *Lipschitzian* if there exists a positive constant L such that $\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in K$.

T is also called L-*Lipschitzian*, if T is L_1-*Lipschitzian* and $L_1 < L_2$, then T is L_2-*Lipschitzian*. Throughout the paper, we assume that every *Lipschitzian* mapping is L-*Lipschitzian* with $L \geq 1$. If $L = 1$, T is known as a nonexpansive mapping and is asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\| \quad \forall x, y \in K$ for all integers $n \geq 0$. We denote $F(T) = \{x \in E;\ Tx = x\}$.

In [3], Moudafi proposed a viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping in Hilbert spaces. If H is a Hilbert space, $T : K \to K$ a nonexpansive self-mapping of a nonempty closed convex subset K of h, and $f : K \to K$ is a contraction, he proved the following theorems.

Theorem M1 [3, Theorem 2.1]**.** *The sequence* $\{x_n\}$ *generated by the scheme,*

$$x_n = \frac{1}{1 + \varepsilon_n} Tx_n + \frac{\varepsilon_n}{1 + \varepsilon_n} f(x_n),\ n \geq 0,$$

converges strongly to the unique solution of the variational inequality:

$$\tilde{x} \in F(T)$$

such that

$$\langle (I - f)p, \tilde{x} - x \rangle \leq 0 \quad \textit{for all } x \in F(T),$$

where $\{\varepsilon_n\}$ *is a sequence of positive numbers tending to zero.*

Theorem M2 [3, Theorem 2.2]**.** *With an initial* $z_0 \in K$ *defined the sequence* $\{z_n\}$ *by*

$$z_{n+1} = \frac{1}{1 + \varepsilon_n} Tz_n + \frac{\varepsilon_n}{1 + \varepsilon_n} f(z_n).$$

Supposed that $\lim_{n \to \infty} \varepsilon_n = 0$, $\sum_{n=0}^{\infty} \varepsilon_n = \infty$; *and* $\lim_{n \to \infty} \left| \frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n} \right| = 0$. *Then* $\{z_n\}$ *converges strongly to the unique solution of the variational inequality:*

$$\tilde{x} \in F(T)$$

such that

$$\langle (I - f)p, \tilde{x} - x \rangle \leq 0 \quad \textit{for all } x \in F(T).$$

Xu [7] studied the viscosity approximation methods proposed by Moudafi for a nonexpansive mapping in a uniformly smooth Banach space. If $\prod_K$ denotes the set of all contractions on K, he proved the following theorems:

Theorem HKX1 [7, Theorem 4.1]**.** *Let* E *be a uniformly smooth Banach space,* K *a closed convex subset of* E, *and* $T : K \to K$ *a nonexpansive mapping with* $F(T) \neq \emptyset$, *and* $f \in \prod_K$. *Then the path* $\{x_t\}$ *defined by*

$$x_t = tf(x_t) + (1 - t)Tx_t, \; t \in (0, 1),$$

converges strongly to a point in $F(T)$. *If we define* $Q : \prod_K \to F(T)$ *by* $Q(f) = \lim_{t \to 0} x_t$, *then* $Q(f)$ *solves the variational inequality*:

$$\langle (I - f)Q(f), j(Q(f) - x) \rangle \leq 0 \quad x \in F(T), \quad f \in \prod_K.$$

Theorem HKX2 [7, Theorem 4.2]**.** *Let E be a uniformly smooth Banach space, K a closed convex subset of E, and $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f \in \prod_K$. Assume that $\alpha_n \subset (0, 1)$ satisfies the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) *either* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$ *or* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$.

Then the sequence $\{x_n\}$ generated by

$$x_0 \in K,\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n,$$

converges strongly to a point in $F(T)$.

Recently, Shazad and Udomene [5] obtained fixed point solutions of variational inequalities for an asymptotically nonexpansive mapping defined on a real Banach space with uniformly Gâteaux differentiable norm possessing uniform normal structure. Nilsrakoo and Saejung [4] established weak and strong convergence theorem for a countable family of certain Lipschitzian mappings and in a real Hilbert space. They proved the following theorem.

Theorem W. N., S. S. [4, Theorem 5]**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}(n = 1, 2, \ldots)$ is a sequence of L_n– Lipschitzian nonexpansive mappings from C into itself such that $\sum_{n=0}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_n x_n,$$

for all $n \in N$, *where* $\{\alpha_n\}$ *is a sequence in* $[0, 1)$ *with* $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$.

Let $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ *for any bounded subset B of C and T be a mapping of C into itself defined by* $Tz = \lim_{n\to\infty} T_n z$ *for all* $z \in C$ *and suppose that* $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. *Then* $\{x_n\}$ *converges weakly to* $\omega \in F(T)$. *Moreover*, $\lim_{n\to\infty} P_{F(T)} x_n = \omega$.

In this paper, we studied the countable family of certain Lipschitzian mappings in the real Banach space, and proved the sequence converges strongly to the unique solution of some variational inequality in our more general space. Our theorem also extends Theorems 3.1 and 3.2 of [5] to more general class of nonexpansive mappings.

2. Preliminaries

Let $S := \{x \in E : \|x\| = 1\}$ denote the unit sphere of the Banach space E, The space E is said to have a Gâteaux differentiable norm if the limit

$$\lim_{n\to\infty} \frac{\|x + ty\| - \|x\|}{t} = 0, \qquad (*)$$

exists for each $x, y \in S$, and we call E smooth; and E is said to have a uniformly Gâteaux differentiable norm if for each $y \in S$ the limit $(*)$ is attained uniformly for $x \in S$. Further, E is said to be uniformly smooth if the limit $(*)$ exists uniformly for $(x, y) \in S \times S$. If E is smooth, then any duality mapping on E is single-valued, and if E has a uniformly Gâteaux differentiable norm, then the duality mapping is norm-to-weak$*$ uniformly continuous on bounded sets.

A bounded convex subset K of a Banach space E is said to have normal structure if every convex subset H of K that contains more than one point contains a point $x_0 \in H$ such that $sup\{\|x_0 - y\|, y \in H\} < d(H)$, where $d(H) = sup\{\|x - y\|, x, \ y \in H\}$ denotes the diameter of H. A Banach space E is said to have normal structure if every bounded, convex

subset of E has normal structure. E is said to have uniform normal structure if there exists $0 < c < 1$, such that for any subset K as above, there exists $x_0 \in K$ such that $sup\{\|x_0 - y\|, y \in K\} < c(d(K))$. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure.

Suppose a linear continuous functional μ on l^∞ such that $\|\mu\| = \mu(1) = 1$. Then μ is called a *mean* on N if and only if the inequalities $inf\{a_n; n \in N\} \leq \mu(a) \leq sup\{a_n; n \in N\}$ hold for each $a = (a_1, a_2, \dots) \in l^\infty$ we denote by $\mu_n(a_n)$ instead of $\mu(a)$. A mean is called a *Banach limit* if $\mu_n(a_n) = \mu_n(a_{n+1})$.

Lemma 2.1 [6]**.** *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm, let S be an index set, let $\{x_t : t \in S\}$ be a bounded set of E, and let μ be a mean on S. Let $u \in C$, then $\mu_t\|x_t - u\|^2 = \min_{y \in C} \mu_t\|x_t - y\|^2$ if and only if $\mu_t\langle z - u, J(x_t - u)\rangle \leq 0$ for all $z \in C$, where J is the duality mapping of E.*

Lemma 2.2. *Let E be an arbitrary real Banach space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle \tag{2}$$

$\forall x, y \in E$ and $\forall j(x + y) \in J(x + y)$.

Lemma 2.3 [2]**.** *Let $\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty, \{c_n\}_{n=0}^\infty$ be sequences of nonnegative numbers satisfying*

$$a_{n+1} \leq (1 - \omega_n)a_n + b_n + c_n, \forall n \geq 0,$$

where $\{\omega_n\}_{n=0}^\infty \subset [0, 1]$. If

$$\sum_{n=0}^\infty \omega_n = \infty, b_n = o(\omega_n) \text{ and } \sum_{n=0}^\infty c_n < \infty,$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 [1]. *Let K be a nonempty bounded closed convex subset of a reflexive Banach space E. And suppose that K has normal structure. If ϕ is a mapping of K into itself which does not increase distances, then ϕ has a fixed point.*

Lemma 2.5 [7]. *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\beta_n,\ n \geq 0,$$

where $\{\gamma_n\} \subset (0, 1)$ and $\{\beta_n\}$ is real number sequence such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\lim_{n\to\infty} \beta_n = 0$.

Then $\{a_n\}$ converges to zero, as $n \to \infty$.

3. Main Results

Theorem 3.1. *Let E be a real Banach space with uniformly Gâteaux differentiable norm possessing uniform normal structure. Suppose K is a nonempty bounded closed convex subset of E, and $\{T_n\}$ $(n = 1, 2, \ldots)$ is a sequence of k_n–Lipschitzian nonexpansive mappings from K into itself such that $\lim_{n\to\infty} k_n = 1$ and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let T be a mapping of K into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in K$ with $\lim_{n\to\infty} \sup_{z\in K} \|T_n z - Tz\| = 0$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. $f : K \to K$ a contraction with constant $\alpha \in [0, 1)$. Let $\{t_n\}_n \subset (0, \frac{(1-\alpha)k_n}{k_n - \alpha})$ be such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} \frac{k_n - 1}{k_n - t_n} = 0$. Then*

(i) *For each integer $n \geq 1$, there is a unique $x_n \in K$ such that*

$$x_n = (1 - \frac{t_n}{k_n})f(x_n) + \frac{t_n}{k_n} T_n x_n; \tag{3.1}$$

(ii) *Then as* $n \to \infty$, x_n *converges strongly to some fixed point* p *of* T *such that* p *is the unique solution in* F *to the following variational inequality*:

$$\langle (I - f)p,\ j(p - x^*)\rangle \le 0 \quad \textit{for all } x^* \in F. \tag{3.2}$$

Proof. At first, we show that there exists unique solution to the equality (3.1). In fact, for each integer $n \ge 1$, by the conditions on t_n, the mapping $\phi_n x = (1 - \frac{t_n}{k_n})f(x) + \frac{t_n}{k_n} T_n x$ is a contraction. It follows there exists unique $x_n \in K$ such that $\phi_n x_n = x_n$.

Since K is bounded and $x_n \in K$, $\{x_n\}$, $\{f(x_n)\}$, $\{T_n x_n\}$ are bounded.

$$\|x_n - T_n x_n\| = (1 - \frac{t_n}{k_n})\|f(x_n) - T_n x_n\| \to 0.$$

By the condition $\lim_{n\to\infty} \sup_{z\in K} \|T_n z - Tz\| = 0$,

$$\|x_n - Tx_n\| \le \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\| \to 0.$$

Define the mapping $\psi : K \to R$ by

$$\psi(x) := \mu_n \|x_n - x\|^2, \quad \forall x \in K.$$

Since E is reflexive, $\psi(x) \to \infty$ as $\|x\| \to \infty$, and ψ is continuous and convex, we have that the set $C := \{y \in K : \psi(y) = \inf_{x\in K} \psi(x)\} \ne \emptyset$. C is closed and bounded. From $\|x_n - Tx_n\| \to 0$, $\psi(Tx) = \mu_n \|x_n - Tx\|^2 = \mu_n \|Tx_n - Tx\|^2 \le \mu_n \|x_n - x\|^2 = \psi(x)$, therefore, $T(C) \subset C$. By Lemma 2.4, T has a fixed point in C. Let $p \in C \cap F(T)$ and using Lemma 2.1, we have

$$\mu_n \langle x - p,\ j(x_n - p)\rangle \le 0, \quad \text{for all } x \in K.$$

It follows that

$$\mu_n \langle f(p) - p,\ j(x_n - p)\rangle \leq 0. \tag{a}$$

By the conditions on $\{T_n\}$, we have

$$\langle x_n - T_n x_n,\ j(x_n - x^*)\rangle \geq -(k_n - 1)\|x_n - x^*\|^2, \quad \forall x^* \in F(T).$$

By the definition of the sequence $\{x_n\}$, we have

$$\langle x_n - f(x_n),\ j(x_n - x^*)\rangle \leq \frac{k_n - 1}{k_n - t_n}\|x_n - x^*\|^2, \quad \forall x^* \in F(T).$$

Since K is bounded, it follows that

$$\limsup_{n\to\infty} \langle x_n - f(x_n),\ j(x_n - x^*)\rangle \leq 0, \quad \forall x^* \in F(T). \tag{b}$$

From $(1-\alpha)\|x_n - p\|^2 \leq \langle x_n - f(x_n),\ j(x_n - p)\rangle + \langle f(p) - p,\ j(x_n - p)\rangle$. And (a), (b), $\mu_n\|x_n - p\| = 0$. Thus, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to p$ as $k \to \infty$. Assume that there is another subsequence $\{x_{n_l}\} \subset \{x_n\}$ such that $x_{n_l} \to q$ as $l \to \infty$.

For $x_{n_k} \to p$ set $x^* = q$, by (b), $\langle p - f(p), j(p-q)\rangle \leq 0$. For $x_{n_l} \to q$ set $x^* = p$, by (b), $\langle q - f(q),\ j(q - p)\rangle \leq 0$.

$$\|p - q\|^2 \leq \langle f(p) - f(q),\ j(p - q)\rangle \leq \alpha\|p - q\|^2.$$

We must have $p = q$ and the uniqueness is proved. Thus, $x_n \to p$ as $n \to \infty$ and $p \in F$ is unique. Again, using (b), we have

$$\langle (I - f)p,\ j(p - x^*)\rangle \leq 0, \quad \text{for all } x^* \in F.$$

This concludes the proof.

Remark 1. For an asymptotically nonexpansive mapping T, set $T_n = T^n$, then we can get Theorem 3.1 of [5] as a corollary: *Let E be a real Banach space with uniformly Gâteaux differentiable norm possessing uniform normal structure, K a nonempty closed convex and bounded subset of E, $T : K \to K$ an asymptotically nonexpansive mapping with*

sequence $\{k_n\}_n \subset [1, \infty)$ *and* $f : K \to K$ *a contraction with constant* $\alpha \in [0, 1)$. *Let* $\{t_n\}_n \subset (0, \frac{(1-\alpha)k_n}{k_n - \alpha})$ *be such that* $\lim_{n\to\infty} t_n = 1$ *and* $\lim_{n\to\infty} \frac{k_n - 1}{k_n - t_n} = 0$. *Then*

(i) *For each integer* $n \geq 1$, *there is a unique* $x_n \in K$ *such that*

$$x_n = (1 - \frac{t_n}{k_n})f(x_n) + \frac{t_n}{k_n} T_n x_n;$$

(ii) *Then* as $n \to \infty$, x_n *converges strongly to some fixed point* p *of* T *such that* p *is the unique solution in* F *to the following variational inequality*:

$$\langle (I - f)p, j(p - x^*)\rangle \leq 0 \quad \textit{for all } x^* \in F.$$

Theorem 3.2. *Let* E *be a real Banach space with uniformly Gâteaux differentiable norm possessing uniform normal structure. Suppose* K *is a nonempty closed convex subset of* E, *and* $\{T_n\}$ $(n = 1, 2, \ldots)$ *is a sequence of* k_n*-Lipschitzian nonexpansive mappings from* K *into itself such that* $\lim_{n\to\infty} k_n = 1$ *and* $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. *Let* $\{t_n\}_n \subset (0, \min\{\frac{(1-2\alpha)k_n}{k_n - \alpha}, \frac{1}{k_n}\})$ *be such that* $\lim_{n\to\infty} t_n = 1$, $\sum_{n=1}^{\infty} t_n(1 - t_n) = \infty$ *and* $\lim_{n\to\infty} \frac{k_n - 1}{k_n - t_n} = 0$. *And* $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ *for any bounded subset* B *of* K. $f : K \to K$ *a contraction with constant* $\alpha \in [0, \frac{1}{2})$. *Let* $\{x_n\}$ *be a sequence in* K *defined by* $x_1 \in K$ *and*

$$x_{n+1} = (1 - \frac{t_n}{k_n})f(x_n) + \frac{t_n}{k_n} T_n x_n.$$

Then

(i) x_n *is bounded ;*

(ii) *Let C is a bounded subset of K containing $\{x_n\}$, T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in K$ and suppose that*

$$F(T) = \bigcap_{n=1}^{\infty} F(T_n).$$

Then $\{x_n\}$ converges strongly to p, such that p is the unique solution in $F(T)$ to the inequality (3.2).

Proof. Firstly, we show that $\{x_n\}$ is bounded. Take $u \in F$. It follows that

$$\begin{aligned}
\|x_{n+1} - u\| &\le (1 - \frac{t_n}{k_n})\|f(x_n) - f(u) + f(u) - u\| + \frac{t_n}{k_n}\|T_{n+1}x_n - u\| \\
&\le (1 - \frac{t_n}{k_n})(\alpha\|x_n - u\| + \|f(u) - u\|) + \frac{t_n}{k_n}\|x_n - u\| \\
&= [1 - (1 - \frac{(k_n - t_n)\alpha}{k_n} - t_n)]\|x_n - u\| + (1 - \frac{t_n}{k_n})\|f(u) - u\| \\
&\le \max\{\|x_n - u\|, \frac{1}{\alpha}\|f(u) - u\|\}.
\end{aligned}$$

By induction, $\|x_n - u\| \le \max\{\|x_0 - u\|, \frac{1}{\alpha}\|f(u) - u\|\}$, $n \ge 0$, and $\{x_n\}$ is bounded, which leads to the boundedness of $\{f(x_n)\}$ and $\{T_n x_n\}$.

Using the assumption that $\lim_{n\to\infty} t_n = 1$, we get that

$\|x_{n+1} - T_n x_n\| = (1 - \frac{t_n}{k_n})\|f(x_n) - T_n x_n\| \to 0$. Then

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(1 - \frac{t_n}{k_n})f(x_n) + \frac{t_n}{k_n}T_n x_n - (1 - \frac{t_n}{k_n})f(x_{n-1}) - \frac{t_{n-1}}{k_{n-1}}T_{n-1}x_{n-1}\| \\
&= \|(1 - \frac{t_n}{k_n})[f(x_n) - f(x_{n-1})] + \frac{t_n}{k_n}[T_n x_n - T_{n-1}x_{n-1}]
\end{aligned}$$

$$+ \left(\frac{t_{n-1}}{k_{n-1}} - \frac{t_n}{k_n}\right)[f(x_{n-1}) - T_{n-1}x_{n-1}]\|$$

$$\leq [(1 - \frac{t_n}{k_n})\alpha + t_n]\|x_n - x_{n-1}\| + |\frac{t_{n-1}}{k_{n-1}} - \frac{t_n}{k_n}|\|f(x_{n-1}) - T_{n-1}x_{n-1}\|$$

$$+ \frac{t_n}{k_n}\|T_n x_n - T_{n-1}x_{n-1}\|.$$

By Lemma 2.3, Let $\omega_n = 1 - [(1 - \frac{t_n}{k_n})\alpha + t_n] = 1 - \alpha - t_n(1 - \frac{\alpha}{k_n})$, thus

$$\omega_n > 1 - \alpha - \frac{(1 - 2\alpha)k_n}{k_n - \alpha}\frac{k_n - \alpha}{k_n} = \alpha, \sum_{n=0}^{\infty}\omega_n > \sum_{n=0}^{\infty}\alpha = \infty.$$

Let $b_n = |\frac{t_{n-1}}{k_{n-1}} - \frac{t_n}{k_n}|\|f(x_{n-1}) - T_{n-1}x_{n-1}\|$. Since $\{f(x_n)\}$ and $\{T_n x_n\}$ are bounded, and $\omega_n > \alpha$, then we have $\frac{b_n}{\omega_n} < \frac{|\frac{t_{n-1}}{k_{n-1}} - \frac{t_n}{k_n}|M}{\alpha} \to 0$, as $n \to \infty$, where

$$M > \sup\{|\{f(x_n)\}|, |\{T_n x_n\}|, n = 1, 2, \ldots\}.$$

Let

$$c_n = \frac{t_n}{k_n}\|T_n x_n - T_{n-1}x_{n-1}\|,$$

using the assumption $\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C, thus $\sum_{n=1}^{\infty} c_n < \infty$. Now,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Then

$$\|x_n - T_n x_n\| \to 0, \quad \|x_n - Tx_n\| \to 0.$$

Define $z_n = (1 - \frac{t_n}{k_n})f(z_n) + \frac{t_n}{k_n} T_n z_n$. Set $\alpha_n = \frac{t_n}{k_n}$, let $n < m$, we have

$$\begin{aligned}
\|z_n - x_m\|^2 &= \|\alpha_n(Tz_n - x_m) + (1 - \alpha_n)(f(z_n) - x_m)\|^2 \\
&\leq \alpha_n^2\|Tz_n - x_m\|^2 + 2(1 - \alpha_n)\langle f(z_n) - x_m, j(z_n - x_m)\rangle \\
&\leq \alpha_n^2(\|Tz_n - Tx_m\| + \|Tx_m - x_m\|)^2 + 2(1 - \alpha_n) \\
&\quad (\langle f(z_n) - z_n, j(z_n - x_m)\rangle + \|z_n - x_m\|^2) \\
&\leq \alpha_n^2(\|z_n - x_m\| + \|Tx_m - x_m\|)^2 + 2(1 - \alpha_n)(\langle f(z_n) - z_n, \\
&\quad j(z_n - x_m)\rangle + \|z_n - x_m\|^2) \\
&= (1 - (1 - \alpha_n))^2\|z_n - x_m\|^2 + \|Tx_m - x_m\|(2\alpha_n\|z_n - x_m\| \\
&\quad + \|Tx_m - x_m\|) + 2(1 - \alpha_n)(\langle f(z_n) - z_n, j(z_n - x_m)\rangle \\
&\quad + \|z_n - x_m\|^2) \\
&\leq (1 + (1 - \alpha_n))^2\|z_n - x_m\|^2 + \|Tx_m - x_m\|(2\alpha_n\|z_n - x_m\| \\
&\quad + \|Tx_m - x_m\|) + 2(1 - \alpha_n)\langle f(z_n) - z_n, j(z_n - x_m)\rangle.
\end{aligned}$$

Since C is bounded, for some constant $N > 0$, it follows that

$$\limsup_{m\to\infty}\langle f(z_n) - z_n, j(x_m - z_n)\rangle \leq (1 - \alpha_n)N + \limsup_{m\to\infty}\frac{\|Tx_m - x_m\|N}{1 - \alpha_n}.$$

By Theorem 3.1, $z_n \to p \in F(T)$, which solve the variational inequality (3.2). Since j is norm to weak* continuous on bounded sets, in the limit as $n \to \infty$, we obtain that

$$\limsup_{m\to\infty}\langle f(p) - p, j(x_m - p)\rangle \leq 0, \tag{c}$$

there exists a sequence $\{\varepsilon_n\}$, $\varepsilon_n \geq 0$ for all $n \geq 0$ such that

$\langle f(p) - p,\ j(x_{m+1} - p)\rangle \le \varepsilon_n$ with $\varepsilon_n \to \infty$.

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n^2\|T_n x_n - p\|^2 + 2(1-\alpha_n)\langle f(x_n) - p,\ j(x_{n+1} - p)\rangle \\ &\le t_n^2\|x_n - p\|^2 + 2(1-\alpha_n)\|f(x_n) - f(p)\|\|x_{n+1} - p\| \\ &\quad +2(1-\alpha_n)\langle f(p) - p,\ j(x_{n+1} - p)\rangle \\ &\le t_n^2\|x_n - p\|^2 + \alpha(1-\alpha_n)(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) \\ &\quad +2(1-\alpha_n)\langle f(p) - p,\ j(x_{n+1} - p)\rangle,\end{aligned}$$

so that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \frac{[t_n^2 + \alpha(1-\alpha_n)]}{1-\alpha(1-\alpha_n)}\|x_n - p\|^2 \\ &\quad + 2\frac{1-\alpha_n}{1-\alpha(1-\alpha_n)}\langle f(p) - p,\ j(x_{n+1} - p)\rangle \\ &= \left(1 - \frac{1-\alpha(1-\alpha_n) - t_n^2}{1-\alpha(1-\alpha_n)}\right)\|x_n - p\|^2 \\ &\quad + 2\frac{1-\alpha_n}{1-\alpha(1-\alpha_n)}\langle f(p) - p,\ j(x_{n+1} - p)\rangle.\end{aligned}$$

Let $\gamma_n := \dfrac{1-\alpha(1-\alpha_n) - t_n^2}{1-\alpha(1-\alpha_n)}$.

For some $N > 0$, $\gamma_n \to 0$ as $n \to 0$ and $\sum_{n=1}^{\infty}\gamma_n = \sum_{n=1}^{\infty}\left(1 - \dfrac{t_n^2}{1-\alpha(1-\dfrac{t_n}{k_n})}\right)$

$$\ge \sum_{n=1}^{\infty}\left(1 - \frac{t_n^2}{1-\alpha(1-t_n^2)}\right) = \sum_{n=1}^{\infty}\frac{(1-\alpha)(1-t_n^2)}{1-\alpha(1-t_n^2)} \ge \sum_{n=1}^{\infty}\frac{(1-\alpha)t_n(1-t_n)}{1-\alpha(1-t_n^2)} = \infty$$

set $\lambda_n = 2\dfrac{1-\alpha_n}{1-\alpha(1-\alpha_n)}$, then $\{\dfrac{\lambda_n}{\gamma_n}\}$ is a bounded sequence. Let $M_1 > 0$ be a constant such that $\dfrac{\lambda_n}{\gamma_n} \le M_1$. Then, we have

$$\|x_{n+1} - p\|^2 \leq (1 - \gamma_n)\|x_n - p\|^2 + \frac{\lambda_n}{\gamma_n}\gamma_n \varepsilon_n$$

$$\leq (1 - \gamma_n)\|x_n - p\|^2 + M_1 \gamma_n \varepsilon_n.$$

Using Lemma 2.5, we conclude that $x_n \to p$. This completes the proof of the theorem.

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