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BOUNDS ON SOME NONLINEAR DISCRETE INEQUALITIES IN TWO INDEPENDENT VARIABLES

JUNXIAN ZHANG[1], WEI NIAN LI[2,3] and LIGUO HUANG[2]

[1]Department of Industrial Engineering
Binzhou Vocational College
Shandong 256603
P. R. China

[2]Department of Mathematics
Binzhou University
Shandong 256603
P. R. China

[3]Department of Applied Mathematics
Shanghai Normal University
Shanghai 200234
P. R. China
e-mail: wnli@263.net

Abstract

Our aim in this paper is to investigate certain nonlinear discrete inequalities involving functions of two independent variables. The inequalities given here

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can be used as tools in applications in the theory of certain finite difference and sum-difference equations.

1. Introduction

It is well known that the discrete inequalities involving functions of one and more than one independent variables play a fundamental role in the development of the theory of finite difference equations and numerical analysis. During the last few years, a number of discrete inequalities had been established by many scholars, which are motivated by certain applications. For example, we refer the reader to literatures [1-3, 5-9] and the references therein. In many cases, however, when studying the behavior of solutions of certain classes of finite difference and sum-difference equations, the bounds provided by the earlier inequalities are inadequate in applications and we need some new and specific type of finite difference inequalities.

In this paper, we investigate some new nonlinear discrete inequalities involving functions of two independent variables, which can be used as tools in applications in the theory of certain finite difference and sum-difference equations.

2. Main Results

In what follows, R denotes the set of real numbers and $R_+ = [0, \infty)$ is the given subset of R, and $N_0 = \{0, 1, 2, \ldots\}$ denotes the set of nonnegative integers. We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively. Throughout this paper, all the functions which appear in the inequalities are assumed to be real-valued and all the sums involved exist on the respective domains of their definitions, p and q are constants, and $p \geq 1$, $0 < q \leq p$.

We firstly introduce two lemmas, which are useful in our main results.

Lemma 1 (Bernoulli's inequality) [4]**.** *Let* $0 < \alpha \leq 1$ *and* $x > -1$. *Then* $(1 + x)^{\alpha} \leq 1 + \alpha x$.

Lemma 2 [6]. *Let* $u(n)$, $a(n)$, $b(n)$ *be nonnegative functions defined for* $n \in N_0$.

(i) *Assume that* $a(n)$ *is nondecreasing for* $n \in N_0$. *If*

$$u(n) \le a(n) + \sum_{s=0}^{n-1} b(s)u(s),$$

for $n \in N_0$, *then*

$$u(n) \le a(n)\prod_{s=0}^{n-1}[1 + b(s)],$$

for $n \in N_0$.

(ii) *Assume that* $a(n)$ *is nonincreasing for* $n \in N_0$. *If*

$$u(n) \le a(n) + \sum_{s=n+1}^{\infty} b(s)u(s),$$

for $n \in N_0$, *then*

$$u(n) \le a(n)\prod_{s=n+1}^{\infty}[1 + b(s)],$$

for $n \in N_0$.

Next, we establish our main results.

Theorem 3. *Assume that* $a(m, n) > 0$ *and* $x(m, n)$, $b(m, n)$, $f(m, n)$ $g(m, n)$ *are nonnegative functions defined for* $m, n \in N_0$.

(i) *If*

$$x^p(m, n) \le a(m, n) + b(m, n)\sum_{t=0}^{m-1}\sum_{s=n+1}^{\infty}\left[f(t, s)x^p(t, s) + g(t, s)x^q(t, s)\right],\ m, n \in N_0, \tag{I1}$$

then

$$x(m,n) \le a^{\frac{1}{p}}(m,n) + \frac{1}{p} a^{\frac{1}{p}-1}(m,n) b(m,n) h(m,n) \prod_{t=0}^{m-1} \left[1 + \sum_{s=n+1}^{\infty} F(t,s) \right], \; m, n \in N_0, \quad (1)$$

where

$$h(m,n) = \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} \left[f(t,s) a(t,s) + g(t,s) a^{\frac{q}{p}}(t,s) \right], \quad (2)$$

and

$$F(m,n) = b(m,n) \left(f(m,n) + \frac{q}{p} a^{\frac{q}{p}-1}(m,n) g(m,n) \right). \quad (3)$$

(ii) *If*

$$x^p(m,n) \le a(m,n) + b(m,n) \sum_{t=m+1}^{\infty} \sum_{s=n+1}^{\infty} \left[f(t,s) x^p(t,s) + g(t,s) x^q(t,s) \right], \; m, n \in N_0, \quad (I'1)$$

then

$$x(m,n) \le a^{\frac{1}{p}}(m,n) + \frac{1}{p} a^{\frac{1}{p}-1}(m,n) b(m,n) \hat{h}(m,n) \prod_{t=m+1}^{\infty} \left[1 + \sum_{s=n+1}^{\infty} F(t,s) \right], \; m, n \in N_0, \quad (1')$$

where

$$\hat{h}(m,n) = \sum_{t=m+1}^{\infty} \sum_{s=n+1}^{\infty} \left[f(t,s) a(t,s) + g(t,s) a^{\frac{q}{p}}(t,s) \right], \quad (2')$$

and $F(m,n)$ *is defind by* (3).

Proof. (i) Define a function $z(m,n)$ by

$$z(m,n) = \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} \left[f(t,s) x^p(t,s) + g(t,s) x^q(t,s) \right], \; m, n \in N_0. \quad (4)$$

Then $(I1)$ can be restated as

$$x^p(m, n) \le a(m, n) + b(m, n)z(m, n) = a(m, n)\left(1 + \frac{b(m, n)z(m, n)}{a(m, n)}\right). \quad (5)$$

Using Lemma 1, from (5), we easily obtain

$$x(m, n) \le a^{\frac{1}{p}}(m, n) + \frac{1}{p}a^{\frac{1}{p}-1}(m, n)b(m, n)z(m, n), \quad (6)$$

and

$$x^q(m, n) \le a^{\frac{q}{p}}(m, n) + \frac{q}{p}a^{\frac{q}{p}-1}(m, n)b(m, n)z(m, n). \quad (7)$$

Combining (4), (5) and (7), we have

$$\begin{aligned} z(m, n) \le & \sum_{t=0}^{m-1}\sum_{s=n+1}^{\infty}\Big[f(t, s)\Big(a(t, s) + b(t, s)z(t, s)\Big) \\ & + g(t, s)\left(a^{\frac{q}{p}}(t, s) + \frac{q}{p}a^{\frac{q}{p}-1}(t, s)b(t, s)z(t, s)\right)\Big] \quad (8) \\ = & \ h(m, n) + \sum_{t=0}^{m-1}\sum_{s=n+1}^{\infty}F(t, s)z(t, s),\ m, n \in N_0, \end{aligned}$$

where $h(m, n)$ and $F(m, n)$ are defined by (2) and (3) respectively. It is easy to see that $h(m, n)$ is nonnegative and nondecreasing in m and nonincreasing in n for $m, n \in N_0$. Therefore, using Theorem $5.4.1(a_1)$ in [8], we obtain

$$z(m, n) \le h(m, n)\prod_{t=0}^{m-1}\left[1 + \sum_{s=n+1}^{\infty}F(t, s)\right]. \quad (9)$$

The desired inequality (1) follows from (6) and (9).

(ii) Using Theorem $5.4.1(a_2)$ in [8], we can complete the proof by following the proof of part (i). The details are omitted here.

Theorem 4. *Assume that* $a(m, n) > 0$, $x(m, n)$, $b(m, n)$, $f(m, n)$ *are nonnegative functions for* $m, n \in N_0$, *and* $L : N_0^2 \times R_+ \to R_+$. *Let*

$$0 \le L(m, n, x) - L(m, n, y) \le K(m, n, y)(x - y), \tag{10}$$

for $x \ge y \ge 0$, *where* $K : N_0^2 \times R_+ \to R_+$.

(i) *If*

$$x^p(m, n) \le a(m, n) + b(m, n)\sum_{t=0}^{m-1}\sum_{s=n+1}^{\infty}\left[f(t, s)x^q(t, s) + L(t, s, x(t, s))\right], m, n \in N_0, \quad (I2)$$

then

$$x(m, n) \le a^{\frac{1}{p}}(m, n) + \frac{1}{p}a^{\frac{1}{p}-1}(m, n)b(m, n)G(m, n)\prod_{t=0}^{m-1}\left[1 + \sum_{s=n+1}^{\infty}H(t, s)\right], m, n \in N_0, \tag{11}$$

where

$$G(m, n) = \sum_{t=0}^{m-1}\sum_{s=n+1}^{\infty}\left[f(t, s)a^{\frac{q}{p}}(t, s) + L\left(t, s, a^{\frac{1}{p}}(t, s)\right)\right], \tag{12}$$

and

$$H(m, n) = \left[\frac{q}{p}f(m, n)a^{\frac{q}{p}-1}(m, n) + K\left(t, s, a^{\frac{1}{p}}(m, n)\right)\frac{1}{p}a^{\frac{1}{p}-1}(m, n)\right]b(m, n). \tag{13}$$

(ii) *If*

$$x^p(m, n) \le a(m, n) + b(m, n)\sum_{m+1}^{\infty}\sum_{s=n+1}^{\infty}\left[f(t, s)x^q(t, s) + L(t, s, x(t, s))\right], m, n \in N_0, \quad (I'2)$$

then

$$x(m, n) \le a^{\frac{1}{p}}(m, n) + \frac{1}{p}a^{\frac{1}{p}-1}(m, n)b(m, n)\hat{G}(m, n)$$

$$\times \prod_{m+1}^{\infty}\left[1 + \sum_{s=n+1}^{\infty}H(t, s)\right], m, n \in N_0, \tag{11'}$$

where

$$\hat{G}(m,\, n) = \sum_{m+1}^{\infty} \sum_{s=n+1}^{\infty} \left[f(t,\, s) a^{\frac{q}{p}}(t,\, s) + L\left(t,\, s,\, a^{\frac{1}{p}}(t,\, s) \right) \right], \tag{12′}$$

and $H(m,\, n)$ *is defined by* (13).

Proof. We only give the proof of (i). The proof of (ii) can be completed by following the proof of (i).

(i) Define a function $z(m,\, n)$ by

$$z(m,\, n) = \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} \left[f(t,\, s) x^{q}(t,\, s) + L(t,\, s,\, x(t,\, s)) \right]. \tag{14}$$

Then, as in the proof of Theorem 3, from $(I2)$, we obtain (5)-(7). Therefore, we have

$$\sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} f(t,\, s) x^{q}(t,\, s) \le \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} f(t,\, s) \left(a^{\frac{q}{p}}(t,\, s) + \frac{q}{p} a^{\frac{q}{p}-1}(t,\, s) b(t,\, s) z(t,\, s) \right) \tag{15}$$

and

$$\begin{aligned} \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} L(t,\, s, x(t,\, s)) \le & \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} \left\{ L\left(t,\, s,\, a^{\frac{1}{p}}(t,\, s) + \frac{1}{p} a^{\frac{1}{p}-1}(t,\, s) b(t,\, s) z(t,\, s) \right) \right. \\ & \left. - L\left(t,\, s,\, a^{\frac{1}{p}}(t,\, s) \right) + L\left(t,\, s,\, a^{\frac{1}{p}}(t,\, s) \right) \right\} \\ \le & \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} L\left(t,\, s,\, a^{\frac{1}{p}}(t,\, s) \right) \\ & + \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} K\left(t, s, a^{\frac{1}{p}}(t, s) \right) \frac{1}{p} a^{\frac{1}{p}-1}(t, s) b(t, s) z(t, s). \end{aligned} \tag{16}$$

It follows from (14)-(16) that

$$z(m,\ n) \le \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} \left[f(t,\ s) a^{\frac{q}{p}}(t,\ s) + L\left(t,\ s,\ a^{\frac{1}{p}}(t,\ s)\right) \right]$$

$$+ \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} \left[\frac{q}{p} f(t,\ s) a^{\frac{q}{p}-1}(t,\ s) b(t,\ s) \right. \tag{17}$$

$$\left. + K\left(t,\ s,\ a^{\frac{1}{p}}(t,\ s)\right) \frac{1}{p} a^{\frac{1}{p}-1}(t,\ s) b(t,\ s) \right] z(t,\ s)$$

$$= G(m,\ n) + \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} H(t,\ s) z(t,\ s),\ m,\ n \in N_0,$$

where $G(m,\ n)$ and $H(m,\ n)$ are defined by (12) and (13) respectively.

It is obvious that $G(m,\ n)$ is nonnegative, and nondecreasing in m and nonincreasing in n for $m,\ n \in N_0$. Using Theorem $5.4.1(a_1)$ in [8], from (17), we have

$$z(m,\ n) \le G(m,\ n) \prod_{t=0}^{m-1} \left[1 + \sum_{s=n+1}^{\infty} H(t,\ s) \right],\ m,\ n \in N_0. \tag{18}$$

Therefore, the desired inequality (11) follows from (6) and (18). This completes the proof.

Theorem 5. *Assume that* $a(m,\ n)$, $x(m,\ n)$, $L(m,\ n,\ x)$ *and* $K(m,\ n,\ x)$ *are the same as in Theorem* 4, $r(m,\ n) \ge 0$, $m,\ n \in N_0$, *and the condition* (10) *holds.*

(i) *Let* $a(m,\ n)$ *be nondecreasing in* m, $m \in N_0$. *If*

$$x^p(m,n) \le a(m,n) + \sum_{t=0}^{m-1} r(t,n) x^p(t,n) + \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} L(t,s,x^q(t,s)),\ m,n \in N_0, \tag{I3}$$

then

$$x(m,\ n) \le B^{\frac{1}{p}}(m,\ n) \left\{ a^{\frac{1}{p}}(m,\ n) \right.$$

$$+\frac{1}{p}a^{\frac{1}{p}-1}(m,n)J(m,n)\prod_{t=0}^{m-1}\left[1+\sum_{s=n+1}^{\infty}E(t,s)\right]\Bigg\},\ m,n\in N_0,\tag{19}$$

where

$$B(m,\ n)=\prod_{t=0}^{m-1}\Big(1+r(t,\ n)\Big),\tag{20}$$

$$J(m,\ n)=\sum_{t=0}^{m-1}\sum_{s=n+1}^{\infty}L\left(t,\ s,\ B^{\frac{q}{p}}(t,\ s)a^{\frac{q}{p}}(t,\ s)\right),\tag{21}$$

and

$$E(m,\ n)=K\left(m,\ n,\ B^{\frac{q}{p}}(m,\ n)a^{\frac{q}{p}}(m,\ n)\right)B^{\frac{q}{p}}(m,\ n)\frac{q}{p}a^{\frac{q}{p}-1}(m,\ n).\tag{22}$$

(ii) *Let* $a(m,\ n)$ *be nonincreasing in* $m,\ m\in N_0$. *If*

$$x^p(m,n)\le a(m,n)+\sum_{t=m+1}^{\infty}r(t,n)x^p(t,n)+\sum_{t=m+1}^{\infty}\sum_{s=n+1}^{\infty}L(t,s,x^q(t,s)),\ m,n\in N_0,\tag{$I'3$}$$

then

$$x(m,\ n)\le\widetilde{B}^{\frac{1}{p}}(m,\ n)\Bigg\{a^{\frac{1}{p}}(m,\ n)$$

$$+\frac{1}{p}a^{\frac{1}{p}-1}(m,\ n)\widetilde{J}(m,\ n)\prod_{t=m+1}^{\infty}\left[1+\sum_{s=n+1}^{\infty}\widetilde{E}(t,\ s)\right]\Bigg\},\ m,\ n\in N_0,\tag{19′}$$

where

$$\widetilde{B}(m,\ n)=\prod_{t=m+1}^{\infty}\Big(1+r(t,\ n)\Big).\tag{20′}$$

$$\widetilde{J}(m,\ n)=\sum_{t=m+1}^{\infty}\sum_{s=n+1}^{\infty}L\left(t,\ s,\ \widetilde{B}^{\frac{q}{p}}(t,\ s)a^{\frac{q}{p}}(t,\ s)\right),\tag{21′}$$

and

$$\widetilde{E}(m, n) = K\left(m, n, \widetilde{B}^{\frac{q}{p}}(m, n)a^{\frac{q}{p}}(m, n)\right)\widetilde{B}^{\frac{q}{p}}(m, n)\frac{q}{p}a^{\frac{q}{p}-1}(m, n). \quad (22')$$

Proof. (i) Define a function $z(m, n)$ by

$$z(m, n) = a(m, n) + v(m, n), \quad (23)$$

where

$$v(m, n) = \sum_{t=0}^{m-1}\sum_{s=n+1}^{\infty} L(t, s, x^q(t, s)). \quad (24)$$

Then $(I3)$ can be restated as

$$x^p(m, n) \le z(m, n) + \sum_{t=0}^{m-1} r(t, n)x^p(t, n). \quad (25)$$

Noting the assumption that $a(m, n)$ is nondecreasing in m, $m \in N_0$, we easily see that $z(m, n)$ is a nonnegative and nondecreasing function in m, $m \in N_0$. Therefore, treating n, $n \in N_0$, fixed in (25) and using part (i) of Lemma 2 to (25), we get

$$x^p(m, n) \le B(m, n)z(m, n),$$

i.e.,

$$x^p(m, n) \le B(m, n)(a(m, n) + v(m, n)) = B(m, n)a(m, n)\left(1 + \frac{v(m, n)}{a(m, n)}\right), \quad (26)$$

where $B(m, n)$ is defined by (20). Using Lemma 1, from (26) we have

$$x(m, n) \le B^{\frac{1}{p}}(m, n)\left(a^{\frac{1}{p}}(m, n) + \frac{1}{p}a^{\frac{1}{p}-1}(m, n)v(m, n)\right), \quad (27)$$

and

$$x^q(m, n) \le B^{\frac{q}{p}}(m, n)\left(a^{\frac{q}{p}}(m, n) + \frac{q}{p}a^{\frac{q}{p}-1}(m, n)v(m, n)\right). \quad (28)$$

Combining (24) and (28), and noting the hypotheses (10), we obtain

$$v(m, n) \le \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} \left\{ L\left(t, s, B^{\frac{q}{p}}(t, s) \left(a^{\frac{q}{p}}(t, s) + \frac{q}{p} a^{\frac{q}{p}-1}(t, s) v(t, s) \right) \right) \right.$$

$$\left. - L\left(t, s, B^{\frac{q}{p}}(t, s) a^{\frac{q}{p}}(t, s) \right) + L\left(t, s, B^{\frac{q}{p}}(t, s) a^{\frac{q}{p}}(t, s) \right) \right\} \qquad (29)$$

$$\le J(m, n) + \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} E(t, s) v(t, s),$$

where $J(m, n)$, $E(m, n)$ are defined by (21) and (22), respectively.

It is obvious that $J(m, n)$ is nonnegative, and nondecreasing in m and nonincreasing in n for $m, n \in N_0$. Now an application of Theorem 5.4.1 (a_1) in [8] to (29) yields

$$v(m, n) \le J(m, n) \prod_{t=0}^{m-1} \left[1 + \sum_{s=n+1}^{\infty} E(t, s) \right], \ m, n \in N_0. \qquad (30)$$

Therefore, the desired inequality (19) follows from (27) and (30).

(ii) Noting the assumption that $a(m, n)$ is nonincreasing in m, $m \in N_0$, and using the part (ii) of Lemma 2, we can complete the proof by following the proof of (i) with suitable changes. Therefore, the details are omitted here.

Using the ideas of the proofs of Theorem 5 and Theorem 3, we can obtain the following result.

Theorem 6. *Assume that* $a(m, n) > 0$ *and* $x(m, n)$, $r(m, n)$, $f(m, n)$, $g(m, n)$ *are nonnegative functions defined for* $m, n \in N_0$.

(i) *Let* $a(m, n)$ *be nondecreasing in* m, $m \in N_0$. *If*

$$x^p(m, n) \le a(m, n) + \sum_{t=0}^{m-1} r(t, n) x^p(t, n)$$

$$+ \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} \left[f(t, s)x^p(t, s) + g(t, s)x^q(t, s) \right], \; m, n \in N_0, \quad (I4)$$

then

$$x(m, n) \le B^{\frac{1}{p}}(m, n) \Bigg\{ a^{\frac{1}{p}}(m, n) + \frac{1}{p} a^{\frac{1}{p}-1}(m, n) H(m, n) \prod_{t=0}^{m-1} \left[1 + \sum_{s=n+1}^{\infty} U(t, s) \right] \Bigg\}, \; m, n \in N_0, \quad (31)$$

where

$$H(m, n) = \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} \left[f(t, s)B(t, s)a(t, s) + g(t, s)B^{\frac{q}{p}}(t, s)a^{\frac{q}{p}}(t, s) \right], \quad (32)$$

$$U(m, n) = f(m, n)B(m, n) + \frac{q}{p} a^{\frac{q}{p}-1}(m, n)g(m, n)B^{\frac{q}{p}}(m, n), \quad (33)$$

and $B(m, n)$ *is defined by* (20).

(ii) *Let* $a(m, n)$ *be nonincreasing in* m, $m \in N_0$. *If*

$$x^p(m, n) \le a(m, n) + \sum_{t=m+1}^{\infty} r(t, n)x^p(t, n) + \sum_{t=m+1}^{\infty} \sum_{s=n+1}^{\infty} \left[f(t, s)x^p(t, s) + g(t, s)x^q(t, s) \right], \; m, n \in N_0, \quad (I'4)$$

then

$$x(m, n) \le \tilde{B}^{\frac{1}{p}}(m, n) \Bigg\{ a^{\frac{1}{p}}(m, n) + \frac{1}{p} a^{\frac{1}{p}-1}(m, n) \tilde{H}(m, n) \prod_{t=m+1}^{\infty} \left[1 + \sum_{s=n+1}^{\infty} \tilde{U}(t, s) \right] \Bigg\}, \; m, n \in N_0, \quad (31')$$

where

$$\widetilde{H}(m, n) = \sum_{t=m+1}^{\infty} \sum_{s=n+1}^{\infty} \left[f(t, s)\widetilde{B}(t, s)a(t, s) + g(t, s)\widetilde{B}^{\frac{q}{p}}(t, s)a^{\frac{q}{p}}(t, s) \right]. \quad (32')$$

$$\widetilde{U}(m, n) = f(m, n)\widetilde{B}(m, n) + \frac{q}{p} a^{\frac{q}{p}-1}(m, n)g(m, n)\widetilde{B}^{\frac{q}{p}}(m, n), \quad (33')$$

and $\widetilde{B}(m, n)$ *is defined by* (20′).

Remark 7. Noting that p and q are constants, and $p \geq 1$, $0 < q \leq p$. we can obtain many special discrete inequalities by using our main results. For example, let $p = 1$, $q = \frac{1}{3}$, from Theorem 3, we obtain the following corollary.

Corollary 8. Assume that $a(m, n) > 0$ and $x(m, n)$, $b(m, n)$, $f(m, n)$, $g(m, n)$ are nonnegative functions defined for $m, n \in N_0$.

(i) *If*

$$x(m, n) \leq a(m, n) + b(m, n) \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} \left[f(t, s)x(t, s) + g(t, s)x^{\frac{1}{3}}(t, s) \right], \; m, n \in N_0, \quad (I5)$$

then

$$x(m, n) \leq a(m, n) + b(m, n)\overline{h}(m, n) \prod_{t=0}^{m-1} \left[1 + \sum_{s=n+1}^{\infty} \overline{F}(t, s) \right], \; m, n \in N_0, \quad (34)$$

where

$$\overline{h}(m, n) = \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} \left[f(t, s)a(t, s) + g(t, s)x^{\frac{1}{3}}(t, s) \right], \quad (35)$$

and

$$\overline{F}(m, n) = b(m, n) \left(f(m, n) + \frac{1}{3} a^{-\frac{2}{3}}(m, n)g(m, n) \right). \quad (36)$$

(ii) If

$$x(m,n) \le a(m,n) + b(m,n) \sum_{t=m+1}^{\infty} \sum_{s=n+1}^{\infty} \left[f(t,s)x(t,s) + g(t,s)x^{\frac{1}{3}}(t,s) \right], m, n \in N_0, \quad (I'5)$$

then

$$x(m,n) \le a(m,n) + b(m,n)\overline{\overline{h}}(m,n) \prod_{t=m+1}^{\infty} \left[1 + \sum_{s=n+1}^{\infty} \overline{F}(t,s) \right], m, n \in N_0, \quad (34')$$

where

$$\overline{\overline{h}}(m,n) \sum_{t=m+1}^{\infty} \sum_{s=n+1}^{\infty} \left[f(t,s)a(t,s) + g(t,s)a^{\frac{1}{3}}(t,s) \right], \quad (35')$$

and $\overline{F}(m,n)$ is defined by (36).

Remark 9. If we add $a(m,n) > 0$ to the assumptions of Theorems 2.6–2.8 in [6], then we easily see that Theorems 2.6–2.8 in [6] are special cases of Theorems 3, 6 and 5 respectively. Therefore, our paper gives some extensions of the results of [6] in a sense.

3. An Application

In this section, using Theorem 3, we obtain the bound on the solution of a nonlinear sum-difference equation.

Example. Consider the following sum-difference equation

$$x^p(m,n) = a(m,n) + b(m,n) \sum_{t=0}^{m-1} \sum_{s=n+1}^{\infty} J(t,s,x(t,s)), m, n \in N_0, \quad (37)$$

where $p \ge 1$ is a constant, $a, b : N_0 \times N_0 \to R$, $|a(m,n)| > 0$ and $J : N_0^2 \times R \to R$.

Assume that

$$|J(m,n,x(m,n))| \le f(m,n)|x(m,n)|^p + g(m,n)|x(m,n)|^q, m, n \in N_0, \quad (38)$$

where $f, g : N_0 \times N_0 \to R_+$, and $0 < q \le p$ is a constant. If $x(m, n)$ is a solution of Eq. (37), then

$$|x(m, n)| \le |a(m, n)|^{\frac{1}{p}} + \frac{1}{p}|a(m, n)|^{\frac{1}{p}-1}|b(m, n)|A(m, n)$$

$$\times \prod_{t=0}^{m-1}\left[1 + \sum_{s=n+1}^{\infty} I(t, s)\right], m, n \in N_0, \tag{39}$$

where

$$A(m, n) = \sum_{t=0}^{m-1}\sum_{s=n+1}^{\infty}\left[f(t, s)|a(t, s)| + g(t, s)|a(t, s)|^{\frac{q}{p}}\right], \tag{40}$$

and

$$I(m, n) = |b(m, n)|\left[f(m, n) + \frac{q}{p}|a(m, n)|^{\frac{q}{p}-1}g(m, n)\right]. \tag{41}$$

In fact, let $x(m, n)$ be a solution of Eq. (37). Then we have

$$|x(m, n)|^p \le |a(m, n)| + |b(m, n)|\sum_{t=0}^{m-1}\sum_{s=n+1}^{\infty}|J(t, s, x(t, s))|, m, n \in N_0. \tag{42}$$

Noting the assumption (38), we easily obtain

$$|x(m,n)|^p \le |a(m,n)| + |b(m,n)|\sum_{t=0}^{m-1}\sum_{s=n+1}^{\infty}\left(f(t,s)|x(t,s)|^p + g(t,s)|x(t,s)|^q\right). \tag{43}$$

Now a suitable application of Theorem 3(i) to (43) immediately yields (39).

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