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APPLICATIONS OF *Q*-DIFFERENTIAL OPERATOR

JIAN-PING FANG

Department of Mathematics
Huaiyin Teachers College
Huaian Jiangsu 223300
P. R. China
e-mail: fjp7402@163.com

Abstract

In this paper, we use q-differential operator to give an extension of the non-terminating very-well-poised ${}_6\Phi_5$ series summation formula and a new proof of q-analog Barnes' contour integral formulas.

1. Introduction and Notation

As is customary, we employ the common conventions and notations on q-series. So we always assume $0 < |q| < 1$ and use the following notation of the q-shifted factorial [5]:

$$(a;\, q)_0 = 1, \quad (a;\, q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \quad (a;\, q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k),$$

and

$$(a;\, q)_\alpha = (a;\, q)_\infty / (aq^\alpha;\, q)_\infty, \tag{1}$$

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where a and α are complex variables.

We also adopt the following compact notation:

$$(a_1, a_2, \cdots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,$$

where n is an integer or ∞.

We will frequently use the following property:

$$(q/a; q)_n = (-a)^{-n} q^{n(n+1)/2} (q^{-n}a; q)_\infty / (a; q)_\infty. \tag{2}$$

The q-binomial coefficient and the q-binomial theorem are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}},$$

and

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |x| < 1, \tag{3}$$

respectively.

The basic hypergeometric series ${}_{r+1}\Phi_r$ is defined as follows:

$${}_{r+1}\Phi_r \left(\begin{matrix} a_1, & \cdots, & a_{r+1} \\ b_1, & \cdots, & b_r \end{matrix}; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_{r+1}; q)_n}{(q, b_1, \cdots, b_r; q)_n} x^n.$$

The q-differential operator D_q and q-shifted operator η (cf. [1-3, 7, 8, 12]), acting on the variable a are defined by:

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a} \text{ and } \eta\{f(a)\} = \{f(aq)\}.$$

In [2] and [3], we have constructed the following q-exponential operator:

$${}_1\Phi_0 \left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta \right) = \sum_{n=0}^{\infty} \frac{(b; q)_n (-c\theta)^n}{(q; q)_n}, \tag{4}$$

where b and c are variables, $\theta = \eta^{-1} D_q$.

We, then easily obtain the following basic relations (cf. [1-3, 7-10, 12, 13]):

$$\theta^n\left\{\frac{(as;\ q)_\infty}{(at;\ q)_\infty}\right\} = q^{-n(n-1)/2}t^n\,\frac{(s/t;\ q)_n(as;\ q)_\infty}{(at/q^n;\ q)_\infty}, \tag{5}$$

$$\theta^n\{f(a)g(a)\} = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\theta^k\{f(a)\}\theta^{n-k}\{g(aq^{-k})\}. \tag{6}$$

We suppose that throughout the paper the operators act on the variable a except for special state. Using the identities above, it is straightforward to verify the following identities (cf. [2])

Lemma 1.1. *If* $s/w = q^{-n}$, $|cst/w| < 1$, *where* n *is an integer. We have*

$$\begin{aligned} &{}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix};\ q,\ -c\theta\right)\left\{\frac{(as,\ at;\ q)_\infty}{(aw;\ q)_\infty}\right\} \\ &= \frac{(as,\ at,\ bct;\ q)_\infty}{(aw,\ ct;\ q)_\infty}\,{}_3\Phi_2\left(\begin{matrix} b, & s/w, & q/at \\ & q/ct, & q/aw \end{matrix};\ q,\ q\right). \end{aligned} \tag{7}$$

Lemma 1.2. *If* $|qc/a| < 1$, *we have*

$${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix};\ q,\ -c\theta\right)\left\{\frac{(as;\ q)_\infty}{(aw;\ q)_\infty}\right\} = \frac{(as;\ q)_\infty}{(aw;\ q)_\infty}\,{}_2\Phi_1\left(\begin{matrix} b, & s/w \\ & q/qw \end{matrix};\ q,\ qc/a\right). \tag{8}$$

Lemma 1.3. *If* $|sc| < 1$, *we have*

$${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix};\ q,\ -c\theta\right)\{(as;\ q)_\infty\} = \frac{(as,\ bcs;\ q)_\infty}{(cs;\ q)_\infty}. \tag{9}$$

2. A Formal Extension of Non-terminating ${}_6\Phi_5$ Series Formula

The non-terminating very-well-poised ${}_6\Phi_5$ ([5, p. 36, Equation 2.7.1]) series summation formula is one of fundamental formulas in the theory of basic hypergeometric series. There are many diverse methods presented in a large number of literatures to prove this formula. In this section, we

derive a formal extension of the non-terminating ${}_6\Phi_5$ summation from the special case of the Rogers-Fine identity by our operator method.

Theorem 2.1. *We have*

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n-1)/2}\left(\frac{\alpha q}{bc}\right)^n \frac{(\alpha,\ q\sqrt{\alpha},\ -q\sqrt{\alpha},\ b,\ c,\ fb\alpha;\ q)_n}{(q,\ \sqrt{\alpha},\ -\sqrt{\alpha},\ \alpha q/b,\ \alpha q/c,\ efb\alpha;\ q)_n}$$

$$\times {}_3\Phi_2\left(\begin{matrix} e, & q^{-n}, & q^{-n}c/\alpha \\ c, & q^{1-n}/bf\alpha \end{matrix};\ q,\ q\right) = \frac{(\alpha q,\ \alpha q/bc,\ ef\alpha,\ bf\alpha;\ q)_\infty}{(\alpha q/b,\ \alpha q/c,\ f\alpha,\ efb\alpha;\ q)_\infty}. \quad (10)$$

Proof. To prove this identity we need the Rogers-Fine identity ([4, p. 15, Equation 14.31], [10, p. 129, Equation 4.2]):

$$(1-t)\sum_{n=0}^{\infty}\frac{(x;\ q)_n}{(y;\ q)_n}t^n = \sum_{n=0}^{\infty}\frac{(x,\ qxt/y;\ q)_n}{(y,\ qt;\ q)_n}(yt)^n(1-xtq^{2n})q^{n^2-n}. \quad (11)$$

We start our proof from the special case of (11):

$$\sum_{n=0}^{\infty}\frac{(q/a,\ \alpha;\ q)_n}{(q,\ \alpha a;\ q)_n}(\alpha a)^n(1-\alpha q^{2n})q^{n^2-n} = \frac{(\alpha;\ q)_\infty}{(\alpha a;\ q)_\infty}. \quad (12)$$

It can be obtained by setting $x = q/a$, $t = a\alpha/q$ and $y = q$ in (11).

Using (1) and (2), we rewrite (12) as follows:

$$\sum_{n=0}^{\infty}A_n\frac{(aq^{-n},\ a\alpha q^n;\ q)_\infty}{(a;\ q)_\infty} = (\alpha;\ q)_\infty, \quad (13)$$

where

$$A_n = (-1)^n\frac{(1-\alpha q^{2n})(\alpha;\ q)_n}{(q;\ q)_n}\alpha^n q^{n(3n-1)/2}.$$

Applying the operator ${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix};\ q,\ -c\theta\right)$ to both sides of (13) with respect to the variable a. We have

$$\sum_{n=0}^{\infty} A_n \, {}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right)\left\{\frac{(aq^{-n}, a\alpha q^n; q)_\infty}{(a; q)_\infty}\right\} = (\alpha; q)_\infty \, {}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right)\{1\}. \quad (14)$$

From (7), we have

$${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right)\{1\} = 1,$$

and

$${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right)\left\{\frac{(aq^{-n}, a\alpha q^n; q)_\infty}{(a; q)_\infty}\right\}$$

$$= \frac{(aq^{-n}, a\alpha q^n, bc\alpha q^n; q)_\infty}{(a, c\alpha q^n; q)_\infty} \, {}_3\Phi_2\left(\begin{matrix} b, & q^{-n}, & q^{1-n}/a\alpha \\ & q/a, & q^{1-n}/c\alpha \end{matrix}; q, q\right).$$

Combined with (14), we can obtain:

$$\sum_{n=0}^{\infty} A_n \frac{(aq^{-n}, a\alpha q^n, bc\alpha q^n; q)_\infty}{(a, c\alpha q^n; q)_\infty} \, {}_3\Phi_2\left(\begin{matrix} b, & q^{-n}, & q^{1-n}/a\alpha \\ & q/a, & q^{1-n}/c\alpha \end{matrix}; q, q\right) = (\alpha; q)_\infty. \quad (15)$$

Let $b = q/a$ in (15), then using q-Chu Vandermonde's identity and simplifying, we, then get the following identity

$$\sum_{n=0}^{\infty} A_n \frac{(aq^{-n}; q)_n}{(a\alpha; q)_n} \frac{(cq^{1-n}/a, cb\alpha q^n; q)_\infty}{(cq/a; q)_\infty} = \frac{(\alpha; q)_\infty}{(a\alpha; q)_\infty} (c\alpha; q)_\infty. \quad (16)$$

Further applying the operator ${}_1\Phi_0\left(\begin{matrix} e \\ - \end{matrix}; q, -f\theta\right)$ to the both sides of (16) with respect to the variable c, and using (7), (9) we obtain

$$\sum_{n=0}^{\infty} A_n \frac{(aq^{-n}; q)_n}{(a\alpha; q)_n} \frac{(cq^{1-n}/a, cb\alpha q^n, efb\alpha q^n; q)_\infty}{(cq/a, fb\alpha q^n; q)_\infty}$$

$$\times {}_3\Phi_2\left(\begin{matrix} e, & q^{-n}, & q^{1-n}/bc\alpha \\ & a/c, & q^{1-n}/bf\alpha \end{matrix}; q, q\right) = \frac{(\alpha; q)_\infty}{(a\alpha; q)_\infty} \frac{(c\alpha, ef\alpha; q)_\infty}{(f\alpha; q)_\infty}. \quad (17)$$

Using (1) and (2), noting that $a = q/b$, we get

$$\sum_{n=0}^{\infty} A_n q^{-n^2} c^n \frac{(b,\, q/bc,\, fb\alpha;\, q)_n}{(\alpha q/b,\, \alpha bc,\, efb\alpha;\, q)_n} {}_3\Phi_2\left(\begin{matrix} e, & q^{-n}, & q^{1-n}/bc\alpha \\ & q/bc, & q^{1-n}/bf\alpha \end{matrix};\, q,\, q\right)$$

$$= \frac{(\alpha,\, c\alpha,\, ef\alpha,\, bf\alpha;\, q)_\infty}{(\alpha q/b,\, bc\alpha,\, f\alpha,\, efb\alpha;\, q)_\infty},$$

replacing q/bc by c, we have

$$\sum_{n=0}^{\infty} A_n q^{-n^2} \left(\frac{q}{bc}\right)^n \frac{(b,\, c,\, fb\alpha;\, q)_n}{(\alpha q/b,\, \alpha q/c,\, efb\alpha;\, q)_n}$$

$$\times {}_3\Phi_2\left(\begin{matrix} e, & q^{-n}, & q^{-n}c/\alpha \\ & c, & q^{1-n}/bf\alpha \end{matrix};\, q,\, q\right) = \frac{(\alpha,\, \alpha q/bc,\, ef\alpha,\, bf\alpha;\, q)_\infty}{(\alpha q/b,\, \alpha q/c,\, f\alpha,\, efb\alpha;\, q)_\infty}.$$

Hence

$$\sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2} \left(\frac{\alpha q}{bc}\right)^n \frac{(\alpha,\, q\sqrt{\alpha},\, -q\sqrt{\alpha},\, b,\, c,\, fb\alpha;\, q)_n}{(q,\, \sqrt{\alpha},\, -\sqrt{\alpha},\, \alpha q/b,\, \alpha q/c,\, efb\alpha;\, q)_n}$$

$$\times {}_3\Phi_2\left(\begin{matrix} e, & q^{-n}, & q^{-n}c/\alpha \\ & c, & q^{1-n}/bf\alpha \end{matrix};\, q,\, q\right) = \frac{(\alpha q,\, \alpha q/bc,\, ef\alpha,\, bf\alpha;\, q)_\infty}{(\alpha q/b,\, \alpha q/c,\, f\alpha,\, efb\alpha;\, q)_\infty},$$

as desired.

Remark. If we set $e = c$ in (10), then applying q-Chu Vandermonde's identity and setting $f = q/bcd$, we, then obtain a non-terminating ${}_6\Phi_5$ series summation:

$$ {}_6\Phi_5\left(\begin{matrix} \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & b, & c, & d \\ & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/b, & \alpha q/c, & \alpha q/d \end{matrix};\, q,\, \alpha q/bcd\right)$$

$$= \frac{(\alpha q,\, \alpha q/bc,\, \alpha q/bd,\, \alpha q/cd;\, q)_\infty}{(\alpha q/b,\, \alpha q/c,\, \alpha q/d,\, \alpha q/bcd;\, q)_\infty}. \tag{18}$$

3. *q*-Analogue Barnes' Contour Integral

Gasper and Rahman [5] and Watson [14] gave a q-analogue of Barnes' contour integral (also cf. [6, 11]):

$$ {}_2\Phi_1\left(\begin{matrix} a, & b \\ & c \end{matrix}; q, z\right) = \frac{(a, b; q)_\infty}{(q, c; q)_\infty} \left(\frac{-1}{2\pi i}\right) \int_{-i\infty}^{i\infty} \frac{(q^{1+s}, cq^s; q)_\infty}{(aq^s, bq^s; q)_\infty} \frac{\pi(-z)^s}{\sin \pi s} ds, \quad (19) $$

where $|z| < 1$ and $|arg(-z)| < \pi$. The following extension was given in [5]

Theorem 3.1. *If* $|z| < 1$, $|arg(-z)| < \pi$, *we have*

$$ {}_{r+1}\Phi_r\left(\begin{matrix} a_1, & a_2, & a_3, & \cdots, a_{r+1} \\ & b_1, & b_2, & \cdots, b_r \end{matrix}; q, z\right) $$

$$ = \frac{(a_1, a_2, \cdots, a_{r+1}; q)_\infty}{(q, b_1, \cdots, b_r; q)_\infty} \left(\frac{-1}{2\pi i}\right) \int_{-i\infty}^{i\infty} \frac{(q^{1+s}, b_1q^s, \cdots, b_rq^s; q)_\infty}{(a_1q^s, a_2q^s, \cdots, a_{r+1}q^s; q)_\infty} \frac{\pi(-z)^s}{\sin \pi s} ds. $$

(20)

Gasper and Rahman [5] applied Cauchy's residue theorem to prove (20). Here, we can recover it with our operator method which only require Watson's q-analogue of Barnes' contour integral formula (19).

Proof. We first prove

$$ {}_3\Phi_2\left(\begin{matrix} a_1, & a_2, & a_3 \\ & b_1, & b_2 \end{matrix}; q, z\right) $$

$$ = \frac{(a_1, a_2, a_3; q)_\infty}{(q, b_1, b_2; q)_\infty} \left(\frac{-1}{2\pi i}\right) \int_{-i\infty}^{i\infty} \frac{(q^{1+s}, b_1q^s, b_2q^s; q)_\infty}{(a_1q^s, a_2q^s, a_3q^s; q)_\infty} \frac{\pi(-z)^s}{\sin \pi s}. \quad (21) $$

From (19), we obtain

$$ \sum_{m=0}^{\infty} \frac{(a, b; q)_m}{(q; q)_m} (cq^m; q)_\infty z^m = \frac{(a, b; q)_\infty}{(q; q)_\infty} \left(\frac{-1}{2\pi i}\right) \int_{-i\infty}^{i\infty} \frac{(q^{1+s}, cq^s; q)_\infty}{(aq^s, bq^s; q)_\infty} \frac{\pi(-z)^s}{\sin \pi s} ds. $$

(22)

Applying the operator ${}_1\Phi_0\left(\begin{matrix} g \\ - \end{matrix}; q, -h\theta\right)$ to both sides of (22) with respect to the variable c, we have

$$ \sum_{m=0}^{\infty} \frac{(a, b; q)_m}{(q; q)_m} {}_1\Phi_0\left(\begin{matrix} g \\ - \end{matrix}; q, -h\theta\right)\{(cq^m; q)_\infty\} z^m = \frac{(a, b; q)_\infty}{(q; q)_\infty} \left(\frac{-1}{2\pi i}\right) $$

$$\times \int_{-i\infty}^{i\infty} \frac{(q^{1+s};\, q)_\infty}{(aq^s,\, bq^s;\, q)_\infty} \frac{\pi(-z)^s}{\sin \pi s}\, {}_1\Phi_0\Big(\begin{matrix} g \\ - \end{matrix};\, q,\, -h\theta\Big)\{(cq^s;\, q)_\infty\} ds. \tag{23}$$

By (9), we immediately get:

$${}_3\Phi_2\Big(\begin{matrix} a, & b, & h \\ c, & gh \end{matrix};\, q,\, z\Big) = \frac{(a,\, b,\, h;\, q)_\infty}{(q,\, c,\, gh;\, q)_\infty} \Big(\frac{-1}{2\pi i}\Big) \int_{-i\infty}^{i\infty} \frac{(q^{1+s},\, cq^s,\, ghq^s;\, q)_\infty}{(aq^s,\, bq^s,\, hq^s;\, q)_\infty} \frac{\pi(-z)^s}{\sin \pi s}\, ds. \tag{24}$$

Replacing a by a_1, b by a_2, h by a_3, c by b_1, and g by b_2 / a_3 respectively, we, then complete the proof of (21).

Using (1), we rewrite (21) as follows:

$$\sum_{m=0}^{\infty} \frac{(a_1,\, a_2,\, a_3;\, q)_m}{(q,\, b_1;\, q)_m} (b_2 q^m;\, q)_\infty z^m$$

$$= \frac{(a_1,\, a_2,\, a_3;\, q)_\infty}{(q,\, b_1;\, q)_\infty} \Big(\frac{-1}{2\pi i}\Big) \int_{-i\infty}^{i\infty} \frac{(q^{1+s},\, b_1 q^s,\, b_2 q^s;\, q)_\infty}{(a_1 q^s,\, a_2 q^s,\, a_3 q^s;\, q)_\infty} \frac{\pi(-z)^s}{\sin \pi s}\, ds. \tag{25}$$

Applying the operator ${}_1\Phi_0\Big(\begin{matrix} g_1 \\ - \end{matrix};\, q,\, -a_4\theta\Big)$ to both sides of (25) with respect to the variable b_2, we have

$$\sum_{m=0}^{\infty} \frac{(a_1,\, a_2,\, a_3;\, q)_m}{(q,\, b_1;\, q)_m}\, {}_1\Phi_0\Big(\begin{matrix} g_1 \\ - \end{matrix};\, q,\, -a_4\theta\Big)\{(b_2 q^m;\, q)_\infty\} z^m = \frac{(a_1,\, a_2,\, a_3;\, q)_\infty}{(q,\, b_1;\, q)_\infty} \Big(\frac{-1}{2\pi i}\Big)$$

$$\times \int_{-i\infty}^{i\infty} \frac{\pi(-z)^s}{\sin \pi s} \frac{(q^{1+s},\, b_1 q^s;\, q)_\infty}{(a_1 q^s,\, a_2 q^s,\, a_3 q^s;\, q)_\infty}\, {}_1\Phi_0\Big(\begin{matrix} g_1 \\ - \end{matrix};\, q,\, -a_4\theta\Big)\{(b_2 q^s;\, q)_\infty\} ds. \tag{26}$$

From (9), we obtain

$${}_4\Phi_3\Big(\begin{matrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & g_1 a_4 \end{matrix};\, q,\, z\Big)$$

$$= \frac{(a_1, a_2, a_3, a_4;\, q)_\infty}{(q, b_1, b_2, g_1 a_4;\, q)_\infty} \Big(\frac{-1}{2\pi i}\Big) \int_{-i\infty}^{i\infty} \frac{(q^{1+s}, b_1 q^s, b_2 q^s, g_1 a_4 q^s;\, q)_\infty}{(a_1 q^s, a_2 q^s, a_3 q^s, a_4 q^s;\, q)_\infty} \frac{\pi(-z)^s}{\sin \pi s}\, ds. \tag{27}$$

Replacing g_1 by b_3 / a_4 in (27), we get

$$
{}_4\Phi_3\left(\begin{matrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3 \end{matrix}; q, z\right)
$$

$$
= \frac{(a_1, a_2, a_3, a_4; q)_\infty}{(q, b_1, b_2, b_3; q)_\infty} \left(\frac{-1}{2\pi i}\right) \int_{-i\infty}^{i\infty} \frac{(q^{1+s}, b_1 q^s, b_2 q^s, b_3 q^s; q)_\infty}{(a_1 q^s, a_2 q^s, a_3 q^s, a_4 q^s; q)_\infty} \frac{\pi(-z)^s}{\sin \pi s} ds. \tag{28}
$$

Repeating the arguments above and applying induction, we, then obtain (20).

Remark. Equation (20) can be found in [5, p. 109, Equation 4.5.1].

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