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PERIODIC SOLUTION OF GENERALIZED DIFFERENTIAL EQUATIONS OF HIGH ORDERS

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Abstract

Generalized differential equations with high orders are discussed in this paper. By using Schauder's fixed point theorem with some known results, the existence and uniqueness of the periodic solution for the generalized differential equations with high orders are proved.

1. Introduction

Owing to the importance in both theory and practice, especially for the problem of periodic solutions have attracted much attention [6, 12, 13]. As the continuous development of mathematics, physics and other fields, various nonlinear methods have become significant implements for solving this problem. Some basic methods occur, including degree method, variation principle, partial order method, monotone iterative method, critical points method, diffeomorphism method and so on. Until now, many researches and literatures about second order differential

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equations are published [3, 5, 7, 9], but these studies about the solution of high order equation is still scarce.

With the aid of diffeomorphism method, in the paper [16], author attains the uniqueness and the existence about a kind of high order equation. Document [2] studies the problem of the even order differential equation in 2000. In 2003, author obtains the existence and uniqueness of the periodic solutions about the $2k$ differential equations in [4].

In this paper, authors discuss periodic solution to a more generalized differential equation with high orders. By using the some known results, the integral mean-value theorem for vector functions, and Schauder's fixed point theorem, the existence and uniqueness of high order differential equations are proved.

2. Preliminary Results

Lemma 1 [3]**.** *Let V be a vector space such that for subspaces Y and Z, $V = Y \oplus Z$. If Z is finite dimension and X is a subspace of V such that $X \cap Y = \{0\}$ and dimension X = dimension Z, then $V = X \oplus Y$.*

Lemma 2 [3]**.** *Let V be a vector space and let $H : V \times V \to R$ be a real valued symmetric bilinear form on V. If V is the direct sum of subspaces X and Z, such that $\forall x \in X$, $H(x, x) \geq 0$; $\forall z \in Z$, $H(z, z) \leq 0$ if some $w \in V$, $H(v, w) = 0$, then $w = 0$.*

Lemma 3 [8]**.** *The following differential equations* (1) *have only a 2π periodic solution provided the following conditions are satisfied.*

$$\begin{cases} \sum_{j=1}^{k} \alpha_j u^{(2j)} + (-1)^{k+1} f(t, u) = 0; \\ u^{(i)}(0) = u^{(i)}(2\pi), \\ i = 0, 1, 2, \ldots, 2k-1. \end{cases} \tag{1}$$

(3.1) *$f(t, u)$ is a continuous function and $f(t + 2\pi, u) = f(t, u)$.*

(3.2) *There exists two constants* α, β such that $\tau(N_i) < \alpha \le f_u(t, u) \le \beta < \tau(N_i + 1)$.

(3.3) $\tau(N_i) = \sum_{j=1}^{k} (-1)^{k-j} \alpha_j N_i^{(2j)}$, $\tau(N_i + 1) = \sum_{j=1}^{k} (-1)^{k-j} \alpha_j (N_i + 1)^{(2j)}$

$i = 0, 1, 2, \ldots, n$.

(3.4) $s = \{\tau(N) | \tau(N) = \sum_{j=1}^{k} (-1)^{k-j} \alpha_j N^{2j}, N = 0, 1, 2, \ldots, \infty\}$,

(*where* $\tau(N_i) = \sum_{j=1}^{k} (-1)^{k+1} \alpha_j N_i^{(2j)}$, $\tau(N_i + 1) = \sum_{j=1}^{k} (-1)^{k+1} \alpha_j (N_i + 1)^{(2j)}$,

$(i = 0, 1, 2, \ldots, n)$).

Lemma 4. *The following differential equations* (2) *have only a* 2π *periodic solution provided the following conditions are satisfied.*

$$\begin{cases} \sum_{j=1}^{k} \alpha_j u^{(2j)} + \sum_{j=1}^{k} \beta_j u^{(2j-1)} + (-1)^{k+1} f(t) u = 0, \\ u^{(i)}(0) = u^{(i)}(2\pi), \\ i = 0, 1, 2, \ldots, 2k - 1. \end{cases} \tag{2}$$

(4.1) *$f(t, u)$ is a continuous function and $f(t + 2\pi, u) = f(t, u)$.*

(4.2) *There exists two constants* α, β *such that* $\tau(N_i) < \alpha \le f_u(t, u) \le \beta < \tau(N_i + 1)$.

(4.3) $\tau(N_i) = \sum_{j=1}^{k} (-1)^{k-j} \alpha_j N_i^{(2j)}$, $\tau(N_i + 1) = \sum_{j=1}^{k} (-1)^{k-j} \alpha_j (N_i + 1)^{(2j)}$,

$i = 0, 1, 2, \ldots, n$.

(4.4) $s = \{\tau(N) | \tau(N) = \sum_{j=1}^{k} (-1)^{k-j} \alpha_j N^{2j},\ N = 0, 1, 2, \ldots, \infty\}$,

$(where\ \tau(N_i) = \sum_{j=1}^{k} (-1)^{k+1} \alpha_j N_i^{(2j)},\ \tau(N_i + 1) = \sum_{j=1}^{k} (-1)^{k+1} \alpha_j (N_i + 1)^{(2j)},$
$(i = 0, 1, 2, \ldots, n))$.

3. The Proneness of Lemma 4

Proof.

Define $\left\{ v(t)=(v_1(t),v_2(t),\ldots,v_n(t)) \middle| \begin{array}{l} v^i(0)=v^{(i)}(2\pi), i=0,1,2,\ldots,2k-1; \\ v^{(2k-1)}(t) \text{is absolutely continuous} [0,2\pi] \\ v^{(2k)}(t) \in L^2[0,2\pi] \end{array} \right\}$.

Then V is a Hilbert space with inner product (3)

$$\langle u, v \rangle = \int_0^{2\pi} \left\{ \sum_{j=1}^{k} (-1)^{k-j} a_j (u^{(j)})^T (t) v^{(j)}(t) + u(t)^T v(t) \right\} dt. \tag{3}$$

We shall denote the corresponding norm by $\|\cdot\|$.

Define the subspace of V :

$$X = \left\{ x(t) = \sum_{i=1}^{n} f_i(t)\alpha_i \middle| f_i(t) = c_{i0} + \sum_{m=1}^{N_i} (c_{im} \cos mt + d_{im} \sin mt) \right\}, \tag{4}$$

$$Y = \left\{ y(t) = \sum_{i=1}^{n} g_i(t)\alpha_i \middle| g_i(t) = \sum_{m=N_i+1}^{\infty} (c_{im} \cos mt + d_{im} \sin mt) \right\}, \tag{5}$$

$$Z = \left\{ z(t) = \sum_{i=1}^{n} h_i(t)b_i \middle| h_i(t) = p_{i0} + \sum_{m=1}^{N_i} (p_{im} \cos mt + q_{im} \sin mt) \right\}, \tag{6}$$

where N_i as well as before, c_{im}, d_{im}, p_{im}, q_{im} are constants. According to the definition of inner product (3), we can get easily that X and Y are two closed subspace of V and $V = X \oplus Y$.

$$\text{Let } H(u,v)=\int_0^{2\pi}\left\{\sum_{j=1}^{k}(-1)^{k-j}\alpha_j u^{(j)}\cdot v^{(j)}+\sum_{j=1}^{k}(-1)^{k-j}\beta_j u^{(j-1)}\cdot v^{(j)}-f(t)u.v\right\}dt. \tag{7}$$

Then $H : V \times V \to R$ is a real valued symmetric bilinear from on V.

If $y \in Y$, then

$$H(y,y)=\int_0^{2\pi}\left\{\sum_{j=1}^{k}(-1)^{k-j}\alpha_j y^{(j)}\cdot y^{(j)}+\sum_{j=1}^{k}(-1)^{k-j}\beta_j y^{(j-1)}\cdot y^{(j)}-f(t)y\cdot y\right\}dt. \tag{8}$$

If $z \in Z$, then

$$H(z,z)=\int_0^{2\pi}\left\{\sum_{j=1}^{k}(-1)^{k-j}\alpha_j z^{(j)}\cdot z^{(j)}+\sum_{j=1}^{k}(-1)^{k-j}\beta_j z^{(j-1)}\cdot z^{(j)}-f(t)z\cdot z\right\}dt. \tag{9}$$

By application of elementary properties of Fourier series and the *parseval* formula. So, we have $H(y, y) \le 0$, $H(z, z) \ge 0$.

According to the definition of X, Y and Z, it is straightforward to prove $Y \cap Z = \{0\}$, $\dim X = \dim Z = \sum_{i=1}^{n}(2N_i + 1)$ and $V = X \oplus Y$.

From Lemma 1, we have got the equality $V = Z \oplus Y$.

According to Lemma 2, we know that $\forall v \in V$,

$$H(u,v)=\int_0^{2\pi}\left\{\sum_{j=1}^{k}(-1)^{k-j}\alpha_j u^{(j)}\cdot v^{(j)}+\sum_{j=1}^{k}(-1)^{k-j}\beta_j u^{(j-1)}\cdot v^{(j)}-f_u(t)u.v\right\}dt=0.$$

So, we can get that $H(u, v) = 0$ has a zero-periodic solution $u = 0$, i.e., $\sum_{j=1}^{k}\alpha_j u^{(2j)} + \sum_{j=1}^{k}\beta_j u^{(2j-1)} + (-1)^{k+1} f(t)u = 0$ has a zero -periodic solution.

4. Main Results

Theorem. *The following equations* (10) *has only a zero-periodic solution provided the following conditions are satisfied*

$$\begin{cases} \sum_{j=1}^{k} \alpha_j u^{(2j)} + \sum_{j=1}^{k} \beta_j u^{(2j-1)} + (-1)^{k+1} f(t, u) = 0; \\ u^{(i)}(0) = u^{(i)}(2\pi); \\ i = 0, 1, 2, \ldots, 2k-1. \end{cases} \tag{10}$$

(5.1) $f(t, u)$ *is a continuous function and* $f(t + 2\pi)u = f(t)u$, $\lim_{u\to\infty} f(t, u) = A$.

(5.2) *there exist two constants* α, β *such that.* $\tau(N_i) < \alpha \le f(t) \le \beta < \tau(N_i + 1)$.

(5.3) $\tau(N_i) = \sum_{j=1}^{k} (-1)^{k-j} \alpha_j N_i^{(2j)}$, $\tau(N_i + 1) = \sum_{j=1}^{k} (-1)^{k-j} \alpha_j (N_i + 1)^{(2j)}$, $i = 0, 1, 2, \ldots, n$.

(5.4) $s = \{\tau(N) | \tau(N) = \sum_{j=1}^{k} (-1)^{k-j} \alpha_j N^{2j},\ N = 0, 1, 2, \ldots, \infty\}$,

where $\tau(N_i) = \sum_{j=1}^{k} (-1)^{k+1} \alpha_j N_i^{(2j)}$, $\tau(N_i + 1) = \sum_{j=1}^{k} (-1)^{k+1} \alpha_j (N_i + 1)^{(2j)}$, $(i = 0, 1, 2, \ldots, n)$.

Proof.

1. The uniqueness of periodic solution

If u_1, u_2 are two solutions of equation, by the integral mean-value theorem for vector functions, we obtain $f(t, u_1) - f(t, u_2) = g(t)(u_1 - u_2)$, where $g(t) = \int_0^{2\pi} f_u(t, u_2 + \theta(u_1 - u_2))dt$.

Let $u = u_1 - u_2$, according to Lemma 3, we have $u = u_1 - u_2 = 0$, i.e., $u_1 = u_2$.

2. The existence of periodic solution

Without loss of generality, we assume that $u^{(i)}(0) = u^{(i)}(2\pi) = 0$, $(i = 0, 1, 2, \ldots, 2k-1)$. (11)

Let $w \in C_n^{2k}[0, 2\pi] \cap E$ (12)

$$C_n^{2k}[0, 2\pi] = \{u : R \to R^n \,|\, u_i(t) \in C^{2k}[0, 2\pi],\ i = 1, 2, \ldots, n\}, \quad (13)$$

$$E = \{u \in W^{2k-1,2}([0, 2\pi], R^n) \,|\, u^{(i)}(0) = u^{(i)}(2\pi), (i = 0, 1, 2, \ldots, 2k-1)\}. \quad (14)$$

Next, consider the problem of periodic boundary

$$\begin{cases} \sum_{j=1}^{k} \alpha_j u_w^{(2j)} + (-1)^{k+1} f(t, u_w) = -\sum_{j=1}^{k} \beta_j w^{(2j-1)}, \\ u_w^{(i)}(0) = u_w^{(i)}(2\pi), \\ i = 0, 1, 2, \ldots, 2k-1. \end{cases} \quad (15)$$

By Lemma 4, according to Theorem 1, the problem of (15) has only a 2π periodic solution for a given $w \in C_n^{2k}[0, 2\pi] \cap E$.

Moreover, the mapping Γ defined by $\Gamma_w = u_w$, clearly, Γ is continuous. We can also show that Γ is compact in $E \subset W^{2k-1,2}$.

In fact, by making inner product for (15), we have

$$\left\langle \sum_{j=1}^{k} \alpha_j u_w^{(2j)},\ u_w \right\rangle + \left\langle (-1)^{k+1} f(t,\ u_w),\ u_w \right\rangle = \left\langle -\sum_{j=1}^{k} \beta_j w^{(2j-1)},\ u_w \right\rangle, \quad (16)$$

$$\text{i.e.,}\ \sum_{j=1}^{k} \alpha_j \left\langle u_w^{(2j)},\ u_w \right\rangle + (-1)^{k+1} \langle f(t,\ u_w),\ u_w \rangle = -\sum_{j=1}^{k} \beta_j \left\langle w^{(2j-1)},\ u_w \right\rangle, \quad (17)$$

$$\sum_{j=1}^{k} \alpha_j \left\langle u_w^{(j)},\ u_w^{(j)} \right\rangle = (-1)^k \langle f(t,\ u_w),\ u_w \rangle - \sum_{j=1}^{k} \beta_j \left\langle w^{(2j-1)},\ u_w \right\rangle. \quad (18)$$

Therefore, we obtain:

$$\sum_{j=1}^{k} |\alpha_j| \left\| u_w^{(j)} \right\| \le^2 \le |f(t,\ u_w)| \|u_w\| + \sum_{j=1}^{k} |\beta_j| \left\| w^{(2j-1)} \right\| \|u_w\|. \quad (19)$$

By virtue of *Wirtinger* inequality, we get:

$$|\alpha| \left\| \dot{u}_w \right\|^2 \le \sum_{j=1}^{k} |\alpha_j| \left\| \dot{u}_w \right\|^2 \le \sum_{j=1}^{k} |\alpha_j| \left\| u_w^{(j)} \right\|^2, \quad (20)$$

(where $|\alpha| = \min\limits_{1 \le j \le k} \{|\alpha_j|\}$).

We consider it in $W^{1,2}([0,\ 2\pi],\ R^n)$.

So, we attain:

$$\| u \|^2 \le 2\pi \left\| \dot{u} \right\|^2. \quad (21)$$

According to (16)-(21), we have

$$\|u_w\|^2 \le \frac{2\pi}{|\alpha|} |f(t,\ u_w)| \|u_w\| + \beta \|w'\| \|u_w\|. \quad (22)$$

Associate with the condition (5.1), we can get the conclusion that f is bounded.

Let $|f(t, u_w)| \le A_1$, (23)

and because $\left\|w^{(2j-1)}\right\| \le A_2$. (24)

So $\|u_w\| \le \frac{2\pi}{|\alpha|}|f(t, u_w)| + \sum_{j=1}^{k}|\beta_j|\|w'\| \le \frac{2\pi}{|\alpha|}(A_1 + \sum_{j=1}^{k}|\beta_j|A_2)$. (25)

With (22)-(25), form that we know that $\|u_w\|$ is bounded, i.e., the family $\{u_w\}$ is uniformly bounded.

We know that $W^{1,2}([0, 2\pi], R^n)$ norm defined on E is equivalent to the following norm

$$\left\|\overset{\bullet}{u}\right\| = (\int_0^{2\pi}\left|\overset{\bullet}{u}\right|^2 dt)^{\frac{1}{2}}, \ \forall u \in E. \tag{26}$$

In fact, for all $\forall u \in E$, we have

$$\int_0^{2\pi}\left|\overset{\bullet}{u}\right|^2 dt \le \int_0^{2\pi}(\left|\overset{\bullet}{u}\right|^2 + |u|^2)dt \le (1 + 2\pi)\int_0^{2\pi}\left|\overset{\bullet}{u}\right|^2, \ \forall u \in E. \tag{27}$$

By (25), (26) and (27), it is clear that the family $\{u_w\}$ is equicontinuous。. According to the *Ascoli Arzela* theorem, $\{u_w\}$ is relatively compact in $C_n^{2k}[0, 2\pi]$, i.e., the mapping Γ is compact.

Now choose $r > 0$, such that:

$$\frac{\sqrt{2\pi + 4\pi^2}}{|\alpha|}(|f(t, u_w)| + \sum_{j=1}^{k}|\beta_j|\left\|w^{(2j-1)}\right\|) \le r, \ (\text{where } \|w\|_E \le r).$$

By condition (2),

we can obtain $\|u_w\|_E \le \sqrt{1 + 2\pi}\left\|\overset{\bullet}{u_w}\right\|_{L^2} \le \frac{\sqrt{2\pi + 4\pi^2}}{|\alpha|}(|f(t, u_w)| + \sum_{j=1}^{k}|\beta_j|$ $\left\|w^{(2j-1)}\right\|) \le r$, which implies that the mapping Γ maps a suitable ball

$B = B(0, r) \subset C_n^{2k}[0, 2\pi] \cap E$ into itself, i.e., $\Gamma B \subset B$. By virtue of *Schauder's* fixed point theorem, Γ has a fixed point $\overline{W}$, i.e., $u = u_{\overline{W}}$, so $u = u_{\overline{W}} \in C_n^{2k}[0, 2\pi]$ is the solution of (10).

Then, we complete the proof.

5. Examples

Considering the following second order equations: $u^{(n)} + \nabla G(u, t) = \sin t$

Proof.

Because $\nabla^2 G(u,t)=\begin{pmatrix} \frac{5}{2}(1+\frac{4}{5}\sin^2 t)+\frac{1}{\sqrt{1+u_1^2}} & \frac{3}{2}(1+\frac{2}{3}\sin^2(2t)) \\ \frac{3}{2}(1+\frac{2}{3}\sin^2(2t)) & \frac{5}{2}(1+\frac{4}{5}\sin^2 t)+\frac{1}{\sqrt{1+u_2^2}} \end{pmatrix}.$

So

$$\begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{5}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{1+u_1^2}} & 0 \\ 0 & \frac{1}{\sqrt{1+u_1^2}} \end{pmatrix} \le \nabla^2 G(u, t) \le \begin{pmatrix} \frac{13}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{13}{2} \end{pmatrix} - \begin{pmatrix} 1-\frac{1}{\sqrt{1+u_2^2}} & 0 \\ 0 & 1-\frac{1}{\sqrt{1+u_2^2}} \end{pmatrix}.$$

By the theorem, we can get $u^{(n)} + \nabla G(u, t) = \sin t$ has a unique periodic solution, where

$$G(u, t) = \frac{5}{4}(1 + \frac{4}{5}\sin^2 t)(u_1^2 + u_2^2) + \frac{3}{2}(1 + \frac{2}{3}\sin^2(2t))u_1u_2 +$$

$$u_1 \ln(u_1 + \sqrt{1+u_1^2}) + u_2 \ln(u_2 + \sqrt{1+u_2^2}) - \sqrt{1+u_1^2} - \sqrt{1+u_2^2} + c_1u_1 + c_2u_2.$$

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