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ASYMPTOTIC NORMALITY OF THE KERNEL POVERTY MEASURE ESTIMATE

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Abstract

In this paper we study the kernel estimator of Foster, Greer and Thorbecke class of measures $P(z,\alpha)=\int_0^z(\frac{z-x}{z})^\alpha f(x)dx$, where z is the poverty line, $f(x)$ the density of the income distribution and α the so-called poverty aversion [1]. The asymptotic normality of the estimator is established.

1. Introduction. The Kernel Estimate of The Poverty Index

Let F be the cumulative distribution function of the income variable X from a population, $(X_1, \ldots, X_n)$ a sample of this distribution. The F.G.T class of poverty measures is defined by $P(z, \alpha) = \int_0^z (\frac{z-x}{z})^\alpha dF(x)$, $\alpha \geq 0$. If $z \leq 0$ we set $P(z,\alpha)=0$. These poverty measures are commonly

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estimated by $\hat{P}_n(z, \alpha) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{X_i}{z}\right)_+^{\alpha}$ where $x_+ = \max\ (x, 0)$ [7]. This empirical estimator is unbiased, consistent and has a limit normal law with zero mean and variance equal to $n^{-1}(P(z, 2\alpha) - (P(z, \alpha))^2)$. Therefore, the statistic $T = \sqrt{n}\,\frac{\hat{P}_n(z, \alpha) - P(z, \alpha)}{\sqrt{(\hat{P}_n(z, 2\alpha - (\hat{P}_n(z, \alpha))^2)}}$ can be used to built confidence interval for the poverty measure. So, asymptotic normal estimator with smaller variance will be in generally preferable to $\hat{P}_n(z, \alpha)$.

In a previous work [2] we proposed a new method of kernel estimate based on Riemann sum, namely the following estimator :

$$P_n(z, \alpha) = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=0}^{\left[\frac{z}{h}\right]}(1 - \frac{ih}{z})^{\alpha} K(\frac{X_j - ih}{h}) \quad \text{for} \quad \alpha = 0 \quad \text{or} \quad \alpha \geq 1,$$

where $\left[\frac{*}{h}\right]$ stands for the integer part of $\frac{*}{h}$, h is a positive function of the sample size n such that $h \to 0$ as $n \to +\infty$ and K a Riemann integrable kernel. We note that if $\alpha = 0$ then $P(z, \alpha) = F(z)$ the distribution funtion of X, therefore the earlier estimator is new compared with the one proposed by Sing et al. [8].

Two hypotheses are made about the density f we suppose bounded. That is :

$\mathbf{C_1}$: f is uniformly continuous

$\mathbf{C_2}$: f admits almost everywhere a derivative f' belonging to $L_1(\mathbf{R})$.

For each case, the convergence of the above estimator was established.

The following hypotheses are considered about the kernel K, that is :

$$\mathbf{H_1}) \sup_{-\infty<x<+\infty}|K(x)| < +\infty, \quad \mathbf{H_2}) \int_{\mathbf{R}} K(x)\,dx = 1, \quad \mathbf{H_3}) \lim_{x\to\pm\infty}|xK(x)| = 0.$$

$\mathbf{H_4}$) K is of bounded variation $V_{-\infty}^{u}K$ on $\mathbf{R}$.

$\mathbf{H_5}$) $\int_{\mathbf{R}} |uK(u)|\,du < +\infty$.

$\mathbf{H_6}$) There exists a nonincreasing function λ, such that for all x, x' we have $|K(x) - K(x')| \le \lambda(|x - x'|)$ and $\lambda(u) \to 0$, $u \to 0$, $u \ge 0$. Moreover $\lambda(\frac{u}{h}) = 0(h)$ on bounded intervals.

$\mathbf{H_7}$) $\frac{|x|}{|h|^{1+\varepsilon}} |K(\frac{x}{h})| \to 0$, $0 < \varepsilon < \frac{1}{2}$, as $\frac{|x|}{h} \to +\infty$.

We showed that the asymptotic efficientcy of $P_n(z, \alpha)$ relatively to $\hat{P}_n(z, \alpha)$ is

$$e^K(z, \alpha) = \left((\int_{\mathbf{R}} K^2(y)\,dy) P(z, 2\alpha) - (P(z, \alpha))^2 \right) / P(z, 2\alpha) - (P(z, \alpha))^2.$$

We have $e^K(z, \alpha) \le \int_{\mathbf{R}} K^2(y)\,dy < 1$ for the usual kernels [6] p. 1068, see also Remark1 below.

Throughout this paper the hypotheses $\mathbf{H_1}$, $\mathbf{H_2}$, $\mathbf{H_3}$ are supposed verified. We consider a family of kernels K_v, $v \in \Gamma \subset \mathbf{R}_+^*$, $\mathbf{R}_+^*$ being the set of strictly positive real numbers, about which we make respectively the same hypotheses $\mathbf{H_1} - \mathbf{H_7}$. Denote by P_n^v the estimator of $P(z, \alpha)$ when we replace K by K_v in P_n. We suppose $\int_{\mathbf{R}} K_v^2(y)\,dy < 1$ for all v.

2. The Limit Law

Our main results are the following :

Theorem 2.1. *Assume that*

(1) *f satisfies a C-Lipschitz condition of order* γ, $0 < \gamma \le 1$,

(2) $\int_{\mathbf{R}} |y^\gamma K_v(y)|\,dy < +\infty$ *for all* $v > 0$ *and hypotheses* $\mathbf{H_6}$ *and* $\mathbf{H_7}$

hold. If $nh^{2\gamma} \to 0$ *as* $n \to +\infty$,

then

$$\frac{P_n^v(z,\,\alpha) - P(z,\,\alpha)}{\sqrt{Var(P_n^v(z,\,\alpha))}} \to N(0,\,1), \quad n \to +\infty$$

provided $(\int_{\mathbf{R}} K_v^2(y)\,dy)P(z,\,2\alpha) - (P(z,\,\alpha))^2 > 0.$

Theorem 2.2. *Assume that*

(1) *hypotheses* $\mathbf{C_2}$, $\mathbf{H_5} - \mathbf{H_7}$ *hold*,

(2) $nh^2 \to 0$ *as* $n \to +\infty$,

then

$$\frac{P_n^v(z,\,\alpha) - P(z,\,\alpha)}{\sqrt{Var(P_n^v(z,\,\alpha))}} \to N(0,\,1), \quad n \to +\infty$$

provided $(\int_{\mathbf{R}} K_v^2(y)\,dy)P(z,\,2\alpha) - (P(z,\,\alpha))^2 > 0.$

Var stands for variance.

Evidently assumption (1) of Theorem 2.1 implies hypothesis $\mathbf{C_1}$ is satisfied.

We need the following lemma which is valid if hypothesis $\mathbf{C_1}$ or $\mathbf{C_2}$ is verified :

Lemma 2.1. *Let* $0 \le \theta_i \le 1$; $i = 1, 2, 3$. *Then for all* x, y, t *pairwise different we have* :

$$\lim_{n\to+\infty} \sup_{(\theta_1,\theta_2,\theta_3)\in[0,\,1]\times[0,\,1]\times[0,\,1]} \left(h^{-3} \int_{\mathbf{R}} |K(\frac{u - x + \theta_1 h}{h}) K(\frac{u - y + \theta_2 h}{h}) \right.$$

$$\left. K(\frac{u - t + \theta_3 h}{h}) | f(u)\,du \right) = 0.$$

Proof. It is a generalisation of Lemma 2.3 [2].

Let $\delta > 0$. Define, as in Masry [5],

$$I_n(x, y) = h^{-3}\int_{\mathbf{R}} |K(\frac{u-x+\theta_1 h}{h})K(\frac{u-y+\theta_2 h}{h})K(\frac{u-t+\theta_3 h}{h})|f(u)\,du$$

$$= \int_R (h^{-1}K(\frac{v}{h}))|(h^{-1}K(\frac{v+x-\theta_1 h-y+\theta_2 h}{h}))$$

$$(h^{-1}K(\frac{v+x-\theta_1 h-t+\theta_3 h}{h}))|f(x+v-\theta_1 h)\,dv$$

$$= \int_{|v|\le\delta} + \int_{|v|>\delta}.$$

Since f is bounded on $I = [x-\delta, x+\delta]$, we have for h large enough

$$\int_{|v|\le\delta}| \le \sup_{|v|\le\delta} f(x+v-\theta_1 h)\int_{-\frac{\delta}{h}\le u\le\frac{\delta}{h}} |K(u)K(\frac{x-\theta_1 h-y+\theta_2 h}{h}+u)$$

$$h^{-1}K(\frac{x-\theta_1 h-t+\theta_3 h}{h}+u)h^{-1}|du \tag{1}$$

$$= \sup_{|v|\le\delta}|f(x+v-\theta_1 h)|\int_{\mathbf{R}} \chi_{(-\frac{\delta}{h}\le u\le\frac{\delta}{h})}(u)|K(u)K(\frac{x-\theta_1 h-y+\theta_2 h}{h}+u)h^{-1}$$

$$K(\frac{x-\theta_1 h-t+\theta_3 h}{h}+u)h^{-1}|du.$$

Write

$$|K(\frac{x-\theta_1 h-y+\theta_2 h}{h}+u)h^{-1} =$$

$$|(\frac{x-\theta_1 h-y+\theta_2 h}{h}+u)K(\frac{x-\theta_1 h-y+\theta_2 h}{h}+u)|\,|\frac{1}{x-\theta_1 h-y+\theta_2 h+hu}|.$$

We have:

$$|\frac{1}{x-\theta_1 h-y+\theta_2 h+hu}| = \frac{1}{|x-y|\,|1-\frac{\theta_1-\theta_2-u}{x-y}h|}.$$

Since $|u| \leq \frac{\delta}{h}$ we may choose δ small enough and n_0 such that for $n \geq n_0$ we have $|\frac{\theta_1 - \theta_2 - u}{|x-y|} h| \leq \frac{2h_{n_0} + \delta}{|x-y|} = \eta_1 < 1$. Therefore

$$|\frac{1}{x - \theta_1 h - y + \theta_2 h + hu}| \leq \frac{1}{|x-y|(1-\eta_1)}. \tag{2}$$

We deduce that

$$|K(\frac{x - \theta_1 h - y + \theta_2 h}{h} + u)h^{-1}| \leq |(\frac{x - \theta_1 h - y + \theta_2 h}{h} + u)$$

$$K(\frac{x - \theta_1 h - y + \theta_2 h}{h} + u)| \frac{1}{|x-y|(1-\eta)}.$$

Since $|\theta_2 - \theta_1| \leq 1$, for fixed u, the right-hand side of this inequality tends to zero uniformly with respect to (θ_1, θ_2) because of assumption $\mathbf{H_3}$.

On the other hand, hypothesis $\mathbf{H_3}$ implies there exists a constant B such that

$$|(\frac{x - \theta_1 h - y + \theta_2 h}{h} + u)K(\frac{x - \theta_1 h - y + \theta_2 h}{h} + u)| \leq B.$$

We then have

$$\chi_{(-\frac{\delta}{h} \leq u \leq \frac{\delta}{h})}(u)|K(\frac{x - \theta_1 h - y + \theta_2 h}{h} + u)h^{-1}|$$

is bounded by $\frac{B}{|x-y|(1-\eta_1)}$.

Similarly there exists a constant C such that

$$|(\frac{x - \theta_1 h - y + \theta_2 h}{h} + u)K(\frac{x - \theta_1 h - t + \theta_3 h}{h} + u)| \leq C.$$

and

$$\chi_{(-\frac{\delta}{h} \leq u \leq \frac{\delta}{h})}(u)|K(\frac{x - \theta_1 h - t + \theta_3 h}{h} + u)h^{-1}|$$

is bounded by $\frac{C}{|x-t|(1-\eta_2)}$.

Therefore, if δ is small enough and n sufficiently large we have, for $-\frac{\delta}{h} \le u \le \frac{\delta}{h}$

$$|K(u)K(\frac{x-\theta_1 h - y + \theta_2 h}{h} + u)h^{-1}K(\frac{x-\theta_1 h - t + \theta_3 h}{h} + u)h^{-1}$$

$$\le \frac{BC|K(u)|}{|x-y|\,|x-t|(1-\eta_1)(1-\eta_2)}.$$

$|K(u)|$ being integrable , by dominated convergence theorem, the integral in the right-hand side of (1) tends to zero as $n \to +\infty$, uniformly with respect to $(\theta_1, \theta_2, \theta_3)$. Hence $\int_{|v|\le\delta} \to 0$ as $n \to +\infty$ uniformly with respect to $(\theta_1, \theta_2, \theta_3)$.

Let $\int_{|v|>\delta}$; write it in the form

$$\int_{|v|>\delta} = \int_{|v|>\delta} |vh^{-1}K(\frac{v}{h})(h^{-1}K(\frac{v+x-\theta_1 h - y + \theta_2 h}{h}$$

$$h^{-1}K(\frac{v+x-\theta_1 h - t + \theta_3 h}{h}))|\frac{f(x+v-\theta_1 h)}{v}dv.$$

It does not exceed

$$\frac{1}{\delta}\sup_{|v|>\delta}|\frac{v}{h}K(\frac{v}{h})|\int_{|v|>\delta}|h^{-1}K(\frac{v+x-\theta_1 h - y + \theta_2 h}{h})h^{-1}$$

$$K(\frac{v+x-\theta_1 h - t + \theta_3 h}{h}))|f(x+v-\theta_1 h)\,dv. \qquad (3)$$

Making the change of variable $u = v + x - \theta_1 h$, this integral is less than

$$\int_{\mathbf{R}} |h^{-1}K(\frac{u-y+\theta_2 h}{h})||h^{-1}K(\frac{u-t+\theta_3 h}{h})|f(u)du,$$

and this quantity tends to zero as $n \to +\infty$ uniformly with respect to (θ_2, θ_3) according to the Lemma 2.3 [2] (valid under assumption $\mathbf{C_1}$ or

$\mathbf{C_2}$). Since $\sup_{|v|>\delta} |\frac{v}{h} K(\frac{v}{h})| \to 0$ as $n \to +\infty$, the proof of the lemma is complete.

Theorem 2.3. *Assume that*

hypothesis $\mathbf{C_1}$ *or* $\mathbf{C_2}$ *verified.*

Then, under hypothesis $\mathbf{H_6}$ *and* $\mathbf{H_7}$, *we have for all* $b > 0$,

$$\lim_{n\to+\infty} \sup_{z\in[0,\, b]} \sum_{0\le i\ne j\ne l\ne i\le\left[\frac{z}{h}\right]} (1 - \frac{ih}{z})^{\alpha}(1 - \frac{jh}{z})^{\alpha}$$

$$(1 - \frac{lh}{z})^{\alpha} \int_{\mathbf{R}} K(\frac{u - ih}{h}) K(\frac{u - jh}{h}) K(\frac{u - lh}{h}) f(u)\, du = 0.$$

Proof It is an extension of theorem 2.5 [2] in three dimensional space.

Let $\Delta = [0,\, b] \times [0,\, b] \times [0,\, b]$. We can write for all z in $[0,\, b]$,

$$\sum_{0\le i\ne j\ne l\ne i\le\left[\frac{z}{h}\right]} (1 - \frac{ih}{z})^{\alpha}(1 - \frac{jh}{z})^{\alpha}(1 - \frac{lh}{z})^{\alpha} \int_{\mathbf{R}} |K(\frac{u - ih}{h}) K(\frac{u - jh}{h}) K(\frac{u - lh}{h})| f(u)\, du$$

$$\le \iiint_{\{(x,\, y,\, t)\in\Delta:\, |x-y||x-t||t-y|>0\}} \Phi_n(x,\, y,\, t) dx\, dy\, dt,$$

where

$$\Phi_n(x,\, y,\, t) = \frac{1}{h^3} \sum_{0\le i\ne j\ne l\ne i\le\left[\frac{z}{h}\right]} \chi_{\Delta_{h,i}\times\Delta_{h,\,j}\times\Delta_{h,l}}(x,\, y,\, t) \int_{\mathbf{R}} |K(\frac{u - ih}{h})$$

$$K(\frac{u - jh}{h}) K(\frac{u - lh}{h})| f(u)\, du. \tag{4}$$

Let $(x,\, y,\, t) \in \Delta_{h,i} \times \Delta_{h,\,j} \times \Delta_{h,l}$, $i \ne j \ne l \ne i$.

With the representation $x = hi + \theta_1 h$, $y = hj + \theta_2 h$, $t = hl + \theta_3 h$, $0 \le \theta_i < 1$, $i = 1, 2, 3$. (4) becomes

$$\frac{1}{h^3} \int_{\mathbf{R}} |K(\frac{u - x + \theta_1 h}{h}) K(\frac{u - y + \theta_2 h}{h}) K(\frac{u - t + \theta_3 h}{h})| f(u)\, du.$$

Let $\delta = \min(\frac{|x-y|}{2}, \frac{|x-t|}{2}, \frac{|t-y|}{2})$. With the change of variable $v = u - x + \theta_1 h,$ we split the integral above in the following form

$$\frac{1}{h^3}\int_{\mathbf{R}} |K(\frac{v}{h})K(\frac{x-\theta_1 h - y + \theta_2 h}{h})K(\frac{x-\theta_1 h - t + \theta_3 h}{h})|f(u)\,du = \int_{|v|\le\delta} + \int_{|v|>\delta}.$$

We then have

$$\iiint_{\{(x,\,y,\,t)\in\Delta:\,|x-y||x-t||t-y|>0\}} \Phi_n(x,\,y,\,t)\,dx\,dy\,dt$$

$$\le \iiint_{\{\{(x,y,t)\in\Delta:\,|x-y||x-t||t-y|>0\}} \sum_{0\le i\ne j\ne l\ne i\le\left[\frac{b}{h}\right]} \chi_{\Delta_{h,i}\times\Delta_{h,\,j}\times\Delta h,\,l}(x,y,t)\left(\int_{|v|\le\delta} + \int_{|v|>\delta}\right).$$

The proof is conducted as follow:

First consider

$$\iiint_{\{(x,\,y,\,t)\,\in\,\Delta:\,|x-y||x-t||t-y|>0\}} \sum_{0\le i\ne j\ne l\ne i\le\left[\frac{b}{h}\right]} \chi_{\Delta_{h,i}\times\Delta_{h,\,j}\times\Delta_{h,l}}(x,\,y,\,t)\int_{|v|\le\delta}.$$

Let $A = \sup_{x\in\mathbf{R}} f(x)$. The notations being as in the proof of Lemma 2.1 and we suppose $h \le \frac{b}{4}$ for n large enough. We have, following inequality (1)

$$\int_{|v|\le\delta} \le A\int_{\mathbf{R}} \chi_{(-\frac{\delta}{h}\le u\le\frac{\delta}{h})}(u)|K(u)K(\frac{x-\theta_1 h - y + \theta_2 h}{h} + u)h^{-1}$$

$$K(\frac{x-\theta_1 h - t + \theta_3 h}{h} + u)h^{-1}|\,du. \tag{5}$$

Write

$$|K(\frac{x-\theta_1 h - y + \theta_2 h}{h} + u)h^{-1}| = |K(\frac{x-\theta_1 h - y + \theta_2 h}{h} + u)$$

$$- K(\frac{2b + x - \theta_1 h - y + \theta_2 h}{h} + u)$$

$$+ K\left(\frac{2b + x - \theta_1 h - y + \theta_2 h}{h} + u\right)|h^{-1}$$

$$\leq \left(\lambda\left(\frac{2b}{h}\right) + |K\left(\frac{2b + x - \theta_1 h - y + \theta_2 h}{h} + u\right)|\right)h^{-1}.$$

Express $|K\left(\frac{2b + x - \theta_1 h - y + \theta_2 h}{h} + u\right)|)h^{-1}$ as

$$\frac{|2b + x - y + hu|}{h}\left|K\left(\frac{2b + x - \theta_1 h - y + \theta_2 h}{h} + u\right)\right| \frac{1}{|2b + x - y + hu|}.$$

Let $B = \sup_{y \in \mathbf{R}}|yK(y)|$ and $C = \sup_{y \in \mathbf{R}}|K(y)|$, then we have

$$\frac{|2b + x - y + hu|}{h}\left|K\left(\frac{2b + x - \theta_1 h - y + \theta_2 h}{h} + u\right)\right| \leq B + 2C.$$

Hence

$$|K\left(\frac{2b + x - \theta_1 h - y + \theta_2 h}{h} + u\right)|)h^{-1} \leq \frac{B + 2C}{|2b + x - y + hu|}.$$

Therefore

$$\left|K\left(\frac{x - \theta_1 h - y + \theta_2 h}{h} + u\right)h^{-1}\right| \leq \lambda\left(\frac{2b}{h}\right)h^{-1} + \frac{B + 2C}{|2b + x - y + hu|}.$$

Similarly

$$\left|K\left(\frac{x - \theta_1 h - t + \theta_3 h}{h} + u\right)h^{-1}\right| \leq \lambda\left(\frac{2b}{h}\right)h^{-1} + \frac{B + 2C}{|2b + x - t + hu|}.$$

Since $-\delta \leq hu \leq +\delta$ we have $\frac{b}{4} \leq |2b + x - y + hu|$ and $\frac{b}{4} \leq |2b + x - t + hu|$. Hence

$$\int_{|v| \leq \delta} \leq \frac{16AD}{b^2} + O(h)$$

D being the finite bound of $\int_{\mathbf{R}}|K(u)|(B + 2C)\,du$, and this inequality is true for all (x, y, t) in $\{(x, y, t) \in \Delta, x \neq y \neq t \neq x\}$.

For all u, we have from the proof of the Lemma 2.1

$$\sup_{(\theta_1,\theta_2,\theta_3)\in[0,1]\times[0,1]\times[0,1]}\int_{\mathbf{R}}\chi_{(-\frac{\delta}{h}\le u\le\frac{\delta}{h})}(u)|K(u)K(\frac{x-\theta_1 h-y+\theta_2 h}{h}+u)h^{-1}$$

$$K(\frac{x-\theta_1 h-t+\theta_3 h}{h}+u)h^{-1}|du$$

tends to zero as $n\to+\infty$ except on the complement in Δ of $\{(x,\,y,\,t)\in\Delta,\,x\neq y\neq t\neq x\}$ which is $dxdydt$- measure nul.

Therefore, by Lebesgue-dominated convergence theorem

$$\lim_{n\to+\infty}\int\int\int_{\Delta}\sum_{0\le i\neq j\neq l\neq i\le\left[\frac{b}{h}\right]}\chi_{\Delta_{h,i}\times\Delta_{h,j}\times\Delta_{h,l}}(x,\,y,\,t)\int_{|v|\le\delta}dx\,dy\,dt$$

$$=\int\int\int_{\Delta}\lim_{n\to+\infty}\int_{|v|\le\delta}dx\,dy\,dt\,=\,0.$$

Consider finally

$$\int\int\int_{\{(x,\,y,\,t)\in\Delta:\,|x-y||x-t||t-y|>0\}}\sum_{0\le i\neq j\neq l\neq i\le\left[\frac{b}{h}\right]}\chi_{\Delta_{h,i}\times\Delta_{h,j}\times\Delta_{h,l}}(x,\,y,\,t)\int_{|v|>\delta}. \quad (6)$$

We use expression (3) in the second part of the proof of Lemma 2.1 by analogous reasoning.

$$\int_{|v|>\delta}\le\frac{1}{\delta}\sup_{|v|>\delta}|\frac{v}{h}K(\frac{v}{h})|\int_{|v|>\delta}|h^{-1}K(\frac{v+x-\theta_1 h-y+\theta_2 h}{h})$$

$$h^{-1}K(\frac{v+x-\theta_1 h-t+\theta_3 h}{h}))|f(x+v-\theta_1 h)\,dv. \quad (7)$$

Making the change of variable $u=v+x-\theta_1 h,$ the integral of the right-hand side of (7) does not exceed

$$\int_{\mathbf{R}}|h^{-1}K(\frac{u-y+\theta_2 h}{h})||h^{-1}K(\frac{u-t+\theta_3 h}{h})|f(u)\,du.$$

Let $\delta\ge h^{\varepsilon}$, $0<\varepsilon<\frac{1}{2}$. We have

$$\int_{|v|>\delta} \leq \sup_{|v|>h^{\varepsilon}} \frac{|v|}{h^{1+\varepsilon}} |K(\frac{v}{h})| \sup_{(\theta_2,\theta_3)\in[0,1]\times[0,1]} \int_{\mathbf{R}} |h^{-1}K(\frac{u-y+\theta_2 h}{h})| |h^{-1}$$

$$K(\frac{u-t+\theta_3 h}{h})|f(u)du. \tag{7}$$

When writing $\int_{\mathbf{R}} |h^{-1}K(\frac{u-y+\theta_2 h}{h})| |h^{-1}K(\frac{u-t+\theta_3 h}{h})|f(u)du \leq \int_{|v|\leq\bar{\delta}} +$ $\int_{v>\bar{\delta}}$ with the change of variable $v = u - t + \theta_3 h$, and $\bar{\delta} = \frac{|t-y|}{2}$ we proved in [2] that, under the hypothesis $\mathbf{C_1}$ or $\mathbf{C_2}$ we had, for $\bar{\delta} \geq h^{\varepsilon}$,

$|\int_{|v|\leq\bar{\delta}}| \leq \frac{4AD}{b} + 0(h)$, $y \neq t$ and $|\int_{|v|>\bar{\delta}}| \leq \sup_{|v|>h^{\varepsilon}} |\frac{v}{h^{1+\varepsilon}} K(\frac{v}{h})| \int_{\mathbf{R}} |h^{-1}K(\frac{u}{h})|f(u+y-\theta_2 h)\,du.$

Since $|\int_{\mathbf{R}} |h^{-1}K(\frac{u}{h})|f(u+y-\theta_2 h)\,du \leq A\int_{\mathbf{R}} |K(u)|\,du,$ we deduce that the right-hand side of (7) is bounded except on the set $\bar{\delta} = 0$. Hence $|\int_{|v|>\delta}|$ is bounded except on $\{(x, y, t) \in \Delta : \delta = 0\}$ which is dxdydt- measure nul. Δ being bounded, hypothesis $\mathbf{H_7}$ implies that the integral in (6) tends to zero as $n \to +\infty$.

The proof of the lemma is then complete.

Proof of Theorem 2.1. For sake of simplicity in the notations, K stands for K_v and P_n for P_n^v.

It is sufficient to prove :

on the one hand $1)\sqrt{n}(E(P_n(z, \alpha)) - P(z, \alpha) \to 0$ as $n \to +\infty$

and on the other hand $2)\frac{P_n(z, \alpha) - E(P_n(z, \alpha))}{\sqrt{Var(P_n(z, \alpha))}} \to N(0, 1)$ as $n \to +\infty$.

Let $x_0 \geq 0$ be the infimum of the support of f. Observe first that $\lim_{z\to 0} \frac{F(z)}{z} = 0$ if $x_0 > 0$ and $f(0)$ if $x_0 = 0$. Therefore $\frac{F(z)}{z}$ is bounded.

Let $\overline{\Delta}_{h,i} = \Delta_{h,i} \cap [0, z]$ and χ_B the indicator function of a set B. We have

$$E(P_n(z, \alpha)) = \sum_{i=0}^{\left[\frac{z}{h}\right]} (1 - \frac{ih}{z})^{\alpha} h \int_{\mathbf{R}} K(u) f(uh + ih)\, du;$$

which can be written in the following form

$$\int_0^z \sum_{i=0}^{\left[\frac{z}{h}\right]} \chi_{\overline{\Delta}_{h,i}}(x) (1 - \frac{ih}{z})^{\alpha} \int_{\mathbf{R}} K(u) f(uh + ih)\, du\, dx$$

$$+ \left(h\left(\left[\frac{z}{h}\right] + 1\right) - z\right) (1 - \frac{h\left[\frac{z}{h}\right]}{z})^{\alpha} \int_{\mathbf{R}} K(u) f(uh + \frac{h\left[\frac{z}{h}\right]}{z})\, du. \tag{8}$$

We have

$$\sup_{z \in \mathbf{R}} \left| \left(h\left(\left[\frac{z}{h}\right] + 1\right) - z\right) (1 - \frac{h\left[\frac{z}{h}\right]}{z})^{\alpha} \int_{\mathbf{R}} K(u) f(uh + \frac{h\left[\frac{z}{h}\right]}{z})\, du \right| \leq h \sup_{z \in \mathbf{R}} f(x) \int_{\mathbf{R}} |K(u)|\, du. \tag{9}$$

Because of assumption $\mathbf{H_2}$ we can write

$$P(z, \alpha) = \int_0^z (1 - \frac{x}{z})^{\alpha} \int_{\mathbf{R}} K(u)\, du f(x)\, dx. \tag{10}$$

Let $x \in \overline{\Delta}_{h,i}$. By considering the terms in (8) and (10) we get :

$$\left| (1 - \frac{ih}{z})^{\alpha} \int_{\mathbf{R}} K(u) f(uh + ih)\, du - (1 - \frac{x}{z})^{\alpha} \int_{\mathbf{R}} K(u)\, du f(x) \right|$$

$$= \left| \int_{\mathbf{R}} \left[(1 - \frac{ih}{z})^{\alpha} f(uh + ih) - (1 - \frac{x}{z})^{\alpha} f(x) \right] K(u)\, du \right|$$

$$\leq \int_{\mathbf{R}} \left| (1 - \frac{ih}{z})^{\alpha} - (1 - \frac{x}{z})^{\alpha} \right| f(x) |K(u)| du$$

$$+ \int_{\mathbf{R}} |(1 - \frac{ih}{z})^{\alpha}| f(uh + ih) - f(x) | K(u)|\, du. \tag{11}$$

If $x \in \Delta_{h,i}$ we have by the mean value theorem applied to the function

$g(x) = (1 - \frac{x}{z})^{\alpha}, |(1 - \frac{ih}{z})^{\alpha} - (1 - \frac{x}{z})^{\alpha}| \le \frac{2\alpha h}{z}$ because $\alpha = 0$ or ≥ 1.

Therefore, denoting by $I_1^i(x)$ the first integral of the right hand-side of (11) and $I_1(x) = \sum_{i=0}^{\left[\frac{z}{h}\right]} \chi_{\bar{\Delta}_{h,i}}(x) I_1^i(x)$, we have

$$\int_0^z I_1(x)dx \le \frac{2\alpha h}{z} \int_0^z \left(\int_{\mathbf{R}} f(x)|K(u)|du\right) dx = 2\alpha h \left(\int_{\mathbf{R}} |K(u)\,du|\right) \frac{F(z)}{z}. \quad (12)$$

Denoting by $I_2^i(x)$ the second integral of (11) and $I_2(x) = \sum_{i=0}^{\left[\frac{z}{h}\right]} \chi_{\bar{\Delta}_{h,i}}(x) I_2^i(x)$. We have :

$$\int_0^z I_2(x)dx \le \int_0^z \sum_{i=0}^{\left[\frac{z}{h}\right]} \chi_{\Delta_{h,i}}(x) \left(\int_{\mathbf{R}} |f(uh + x - \theta h) - f(x)|\,|K(u)|du\right) dx$$

$$\le Ch^{\gamma} z \int_0^z \left(\int_{\mathbf{R}} (|u|^{\gamma} + 1)|K(u)|du\right) dx. \quad (13)$$

Since

$$\int_0^z I_1(x)\,dx \le 2\alpha h \left(\int_{\mathbf{R}} |K(u)|du\right) \frac{F(z)}{z},$$

we get, together with the right hand side of (9)

$$\sqrt{n}(E(P_n(z, \alpha)) - P(z, \alpha)) \le h^{\gamma}\sqrt{n}\left\{Cz \int_0^z \left(\int_{\mathbf{R}} (|u|^{\gamma} + 1)|K(u)|du\right) dx\right.$$

$$\left. + h^{1-\gamma} \sup_{x \in \mathbf{R}} f(x) \int_{\mathbf{R}} |K(u)|\,du + 2\alpha h^{1-\gamma} \left(\int_{\mathbf{R}} |K(u)|\,du\right) \frac{F(z)}{z}\right\}.$$

The integrals in the braces exist. Hence the first part is proved.

For the second part, define

$$U_i = \frac{1}{n} \sum_{0 \le l \le \left[\frac{z}{h}\right]} (1 - \frac{lh}{z})^{\alpha} K(\frac{X_i - lh}{h})\ i = 1 \ldots n,\ \mu_i = E(U_i)\ et\ \beta_i = E(|U_i - \mu_i|^3).$$

Let $B_n = (\sum_{i=1}^{n} \beta_i)^{\frac{1}{3}}$. We shall obtain the statement 2) if, by Liapounov's theorem, we prove $\frac{B_n}{\sqrt{Var(P_n(z, \alpha))}} \to 0$ as $n \to +\infty$.

Consider $n^3 E(U_i)^3$. If we develop it, we have

$$n^3 E(|U_j^3|) \le \sum_{i=0}^{\left[\frac{z}{h}\right]} \int_R |K^3(\frac{u-ih}{h})| f(u)\,du + \sum_{0 \le i \ne j \le \left[\frac{z}{h}\right]} \int_R K^2(\frac{u-ih}{h}) |K(\frac{u-jh}{h})| f(u)\,du +$$

$$\sum_{0 \le i \ne j \ne l \ne i \le \left[\frac{z}{h}\right]} \int_{\mathbf{R}} |K(\frac{u-ih}{h}) K(\frac{u-jh}{h}) K(\frac{u-lh}{h})| f(u)\,du.$$

The first term of the right hand-side of this inequality tends to $\left(\int_{\mathbf{R}} |K^3(y)|dy\right) F(z)$ as $n \to +\infty$ because of Corollary 2.1 [2] (with $\alpha = 0$, K^2 replaced by $|K^3|$). The second term does not exceed $\sup_{y \in \mathbf{R}} |K(y)| \sum_{0 \le i \ne j \le \left[\frac{z}{h}\right]} \int_{\mathbf{R}} |K(\frac{u-ih}{h}) K(\frac{u-jh}{h})| f(u)\,du$ and tends to zero with the third term because of Theorem 2.5 [2] and Theorem 2.3 (with $\alpha = 0$). The remainder terms in $n^3 E(|U_i - \mu_i|^3$ are expressed in Theorem 2.3 [2] and 2.4 [2] and their limits exist. Hence, since $Var(P_n(z, \alpha)) = O(\frac{1}{n})$ [2], we have

$$\frac{B_n}{\sqrt{Var(P_n(z, \alpha))}} = O(\frac{n^{\frac{1}{3}}}{n\sqrt{Var(P_n(z, \alpha))}})$$

$$\cong O(\frac{n^{1/3}}{n^{1/2}})$$

which tends to zero as $n \to +\infty$. This completes the proof.

Proof of theorem 2.2. We have $\sqrt{n}\left(E(P_n^v(z, \alpha)) - P(z, \alpha)\right) \to 0$ as $n \to +\infty$ by Lemma 2.2 [2]. The second part remains unchanged. The proof is complete.

Remark 1. To construct a confidence interval we proceed as follow. For $0 < \beta < 1$, let $q_{1-\frac{\beta}{2}}$ be the $\beta-$quantile of the standard normal distribution; for exemple $\beta = 0.05$.

Since $Var(P_n^{\upsilon}(z, \alpha)) \geq 0$ for all z and $\lim_{n\to+\infty} nVar(P_n^{\upsilon}(z, \alpha)) = \left(\int_{\mathbf{R}} K_{\upsilon}^2(y)dy\right)P(z, 2\alpha) - (P(z, \alpha))^2$ (see Theorem 2.3 and 2.4 in [2]), we have $\lim_{n\to+\infty} nVar(P_n^{\upsilon}(z, \alpha)) = 0$ for z such that $\left(\int_{\mathbf{R}} K_{\upsilon}^2(y)dy\right)P(z, 2\alpha) - (P(z, \alpha))^2 \leq 0$. Therefore, the asymptotic efficientcy $e^{K_{\upsilon}}(z, \alpha)$ satisfies $0 \leq e^{K_{\upsilon}}(z, \alpha) = \lim_{n\to+\infty} \frac{nVar(P_n^{\upsilon}(z, \alpha))}{nVar(\hat{P}_n(z, \alpha))} \leq \int_{\mathbf{R}} K_{\upsilon}^2(y)dy.$ The $100(1-\beta)\%$ confidence interval for $P(z, \alpha)$ is then

$$CI = P_n^{\upsilon}(z, \alpha) \pm q_{1-\frac{\beta}{2}}\{(\int_{\mathbf{R}} K_{\upsilon}^2(y)\,dy)P_n^{\upsilon}(z, 2\alpha) - (P_n^{\upsilon}(z, \alpha))^2\}^{\frac{1}{2}} / \sqrt{n}$$

as long as $(\int_{\mathbf{R}} K_{\upsilon}^2(y)\,dy)P_n^{\upsilon}(z, 2\alpha) - (P_n^{\upsilon}(z, \alpha))^2 > 0$. We conclude that, the length of the confidence interval for $P(z, \alpha)$ associated with our estimator is asymptotically lower than that of the empirical estimator, the coefficient being $\{\int_{\mathbf{R}} K_{\upsilon}^2(y)\,dy\}^{\frac{1}{2}} < 1$.

Remark 2. $\int_{\mathbf{R}} K_{\upsilon}^2(y)\,dy$ may be considered as a weight. Greater weight is attached to higher poverty line. So, in order to perform a normality test or to construct a confidence interval concerning $P(z, \alpha)$, we must have the constraint $P(z, \alpha) < P(z, 0) < \int_{\mathbf{R}} K_{\upsilon}^2(y)\,dy$, $\alpha \geq 1$.

Remark 3. Let us consider the following family of kernels $K_{\upsilon}(x) = \frac{K(\upsilon x)}{\int_{\mathbf{R}} K(\upsilon x)}$, $\upsilon > 0$. It is easy to verify that they satisfy assumptions

$\mathbf{H_1} - \mathbf{H_7}$. Moreover, $\int_{\mathbf{R}} K_v^2(y)dy = v\int_{\mathbf{R}} K^2(y)dy$. Therefore Γ can be the interval $\left]0, \frac{1}{\int_{\mathbf{R}} K^2(y)\,dy}\right[$.

As an illustrative exemple for the Gauss's kernel, let v be equal to $\frac{1}{\sigma}$, then $K_v(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$. We have $\int_{\mathbf{R}} K_v^2(y)\,dy = \frac{1}{2\sigma\sqrt{\pi}}$. Let p_k, $k = 1, 2, \ldots,$ be any increasing sequence of real numbers, $0 < p_k < 1$, and converging to 1. ($p_k = \frac{k}{10}$, $k = 1, \ldots, 9$ if we are interested in the deciles or $p_k = \frac{k}{4}$, $k = 1, 2, 3$ for the quartiles). By setting $\frac{1}{2\sigma\sqrt{\pi}} = p_k$ for $k = 1, 2, \ldots,$ we obtain successive kernels corresponding to $\sigma = \frac{1}{2p_k\sqrt{\pi}}$. Therefore, the test or the confidence interval will concern all z_p quantile of the distribution $P(z, 0)$ such that $p < p_k$, $k = 1, 2, \ldots.$

This just described process can be thought of as a test or confidence intervals generated by a **moving kernel** process.

Finally, we give the following theorem which will be useful for further research on testing statistical dominance method, different from the version of Kolmogorov-Smirnov type statistic (see [3] and the references therein).

Theorem 2.4. *Assume the conditions of Theorem* 2.1 *or* 2.2 *hold. Then* $P_n^v(z, \alpha) - E(P_n^v(z, \alpha))$ *is asymptotically normal with mean zero and covariance such that, for* $0 < z_1 \le z_2$, *we have*

$$\lim_{n\to+\infty} nCov(P_n^v(z_1, \alpha), P_n^v(z_2, \alpha)) = (\int_{\mathbf{R}} K_v^2(y)\,dy)$$

$$\int_0^{z_1} (1-\frac{x}{z_1})^{\alpha}(1-\frac{x}{z_2})^{\alpha} f(x)dx - P(z_1,\alpha)P(z_2,\alpha).$$

Proof. After straightforward calculations, the only equality that remains to show is the following

$$\lim_{n\to+\infty} \sum_{i=0}^{\left[\frac{z_1}{h}\right]} (1-\frac{ih}{z_1})^{\alpha}(1-\frac{ih}{z_2})^{\alpha} \int_{\mathbf{R}} K_v^2(\frac{u-ih}{h}) f(u)du$$

$$= (\int_{\mathbf{R}} K_v^2(y)\,dy) \int_0^{z_1} (1-\frac{x}{z_1})^{\alpha}(1-\frac{x}{z_2})^{\alpha} f(x)\,dx. \tag{14}$$

Let α be an integer. As in [3] we use the binomial formula

$$(z_2-ih)^{\alpha} = ((z_1-ih)+(z_2-z_1))^{\alpha} = \sum_{l=0}^{\alpha} C_{\alpha}^{l} (z_1-ih)^{\alpha-1}(z_2-z_1)^{l}.$$

The sum in the left-hand side of (14) becomes

$$\sum_{i=0}^{\left[\frac{z_1}{h}\right]} \sum_{l=0}^{\alpha} C_{\alpha}^{l} \frac{z_1^{\alpha-l}}{z_2^{\alpha}} (1-\frac{ih}{z_1})^{2\alpha-1}(z_2-z_1)^{l} \int_{\mathbf{R}} K_v^2(\frac{u-ih}{h}) f(u)\,du.$$

which is equal to

$$(\frac{z_1}{z_2})^{\alpha} \sum_{l=0}^{\alpha} C_{\alpha}^{l} (\frac{z_2}{z_1}-1)^{l} \sum_{i=0}^{\left[\frac{z_1}{h}\right]} (1-\frac{ih}{z_1})^{2\alpha-l} \int_{\mathbf{R}} K_v^2(\frac{u-ih}{h}) f(u)\,du. \tag{15}$$

Since $\sum_{i=0}^{\left[\frac{z_1}{h}\right]} (1-\frac{ih}{z_1})^{2\alpha-l} \int_{\mathbf{R}} K_v^2(\frac{u-ih}{h}) f(u)du \to (\int_{\mathbf{R}} K_v^2(y)dy) P(z_1, 2\alpha-l)$

as $n \to +\infty$ by Corollary 2.1 [2] and the sum with respect to l in (15) is finite, (15) becomes as $n \to +\infty$

$$(\int_{\mathbf{R}} K_v^2(y)\,dy) \int_0^{z} \sum_{l=0}^{\alpha} C_{\alpha}^{l} \frac{1}{z_1^{\alpha} z_2^{\alpha}} (z_2-z_1)^{l}(z_1-x)^{\alpha-1}(z_1-x)^{\alpha} dx. \tag{16}$$

This expression is the binomial form of the right-hand side of (14).

If α is not an integer, we can obtain, in a slight general setting of our previous paper [2], that the result is again true.

The proof is then complete.

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