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CONVERGENCE RATES OF WAVELET ESTIMATORS IN SEMIPARAMETRIC REGRESSION MODELS WITH MARTINGALE DIFFERENCE ERROR

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Abstract

Under a martingale difference error sequence, a semiparametric regression model is considered by wavelet method. Under some general conditions, strong convergence rates of parametric and nonparametric estimators are $O\left(n^{-1/3}\log n\right)$. Hence our results are extensions of Qian and Cai [12] and Hu and Hu [5].

1. Introduction

Considering the following semiparametric regression model

$$y_i = X_i^T\beta + g(t_i) + \varepsilon_i,\ 1 \le i \le n, \tag{1}$$

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where $\{y_i\}$ are scalar response variables, β is an unknown d -dimensional parameter vector, $g(\cdot)$ is an unknown function on $[0, 1]$, $\{t_i\}$ is a constant sequence on $[0, 1]$, X_i are known design points and each is a fixed d-dimensional vector with $d \le n$, random errors $\{\varepsilon_i, F_i, i \le n\}$ is a martingale difference sequence.

Following Speckman [13], denote

$$x_{ir} = f_r(t_i) + \eta_{ir}, \quad 1 \le i \le n, \quad 1 \le r \le d, \tag{2}$$

where $f_r(\cdot)$ is some unknown smooth function on $[0, 1]$, $\{\overline{\eta}_i\}$ is a stochastic sequence with $\overline{\eta}_i = (\eta_{i1}, \cdots, \eta_{id})^T$ i.i.d. and

$$E\overline{\eta}_i = 0, \quad Var(\overline{\eta}_i) = V, \tag{3}$$

here $V = (V_{ij})\,(i, j = 1, 2, \cdots, d)$ is a positive definite matrix with d -order. Moreover, $\{\eta_{ir}\}$ and $\{\varepsilon_i\}$ are independent each other.

Since a semiparametric regression model contains linear components and a nonparametric component, it is more flexible than the usual standard linear models and attractive in some applications (See [2, 4, 6, 8, 11, 12, 14]). When the error sequence $\{\varepsilon_i\}$ is independent and identically distributed, using wavelet method, the semiparametric regression model has been studied by Qian and Cai [12], Qian et al. [11], Xue [14] and Chai and Xu [2]. When the error sequence $\{\varepsilon_i\}$ is a martingale difference, the semiparametric regression model with $\beta \in R^1$ and $X_i \in R^1$ is studied by using near neighbour method (See Yan et al. [15]), and the strong consistencies of wavelet estimators are investigated by wavelet method (See Hu and Hu [5]).

In this paper, using wavelet method, we continue to discuss the semiparametric regression model with a martingale difference error sequence, and obtain the strong convergence rates of wavelet estimators. The organization of this paper is as follows. In Section 2, the wavelet estimators of parameter β and nonparameter $g(t)$ are given by wavelet smoothing method. Under some general conditions, the strong convergence rates of the wavelet estimators are investigated in Section 3. The main proofs are presented in Section 4.

2. Wavelet Method

In the following, we apply the wavelet technique to estimate β and $g(t)$. Suppose that there exists a scaling function $\phi(x)$ in the Schwartz space S_l and a multiresolution analysis $\{V_m\}$ in the concomitant Hilbert space $L^2(R)$, with its reproducing kernel $E_m(t, s)$ given by

$$E_m(t, s) = 2^m E_0\left(2^m t, 2^m s\right) = 2^m \sum_{k \in Z} \phi\left(2^m t - k\right)\phi\left(2^m s - k\right). \tag{4}$$

Let $A_i = [s_{i-1}, s_i]$ denote intervals that partition $[0, 1]$ with $t_i \in A_i$ and $1 \le i \le n$. Then we can define some wavelet estimators as the following:

Firstly, suppose that β is known, we define estimator of $g(\cdot)$ by

$$\hat{g}_0(t, \beta) = \sum_{i=1}^{n}\left(y_i - X_i^T \beta\right)\int_{A_i} E_m(t, s)ds. \tag{5}$$

In succession, we define wavelet estimator $\hat{\beta}_n$ by minimizing

$$\hat{\beta}_n = \arg\min_{\beta} \sum_{i=1}^{n}\left(y_i - X_i^T \beta - \hat{g}_0(t_i, \beta)\right)^2 = \left(\tilde{X}^T \tilde{X}\right)^{-1} \tilde{X}^T \tilde{Y}, \tag{6}$$

where

$$X = (X_{ir})_{n \times d},\ Y = (y_1, \cdots, y_n)^T,\ S = \left(S_{ij}\right)_{n \times n},\ S_{ij} = \int_{A_j} E_m(t_i, s)ds,$$

$$\tilde{X} = (I - S)X,\ \tilde{Y} = (I - S)Y.$$

Finally, we define linear wavelet estimator of $g(\cdot)$ by

$$\hat{g}_n(t) = \hat{g}_0\left(t, \hat{\beta}_n\right) = \sum_{i=1}^{n}\left(y_i - X_i^T \hat{\beta}_n\right)\int_{A_i} E_m(t, s)ds. \tag{7}$$

To obtain our results, the following four conditions are sufficient.

(A_1) $g(\cdot), f_r(\cdot) \in H^a$ (Sobolev space), for some $\alpha > 1/2, 1 \le r \le d$;

(A_2) $g(\cdot)$ and $f_r(\cdot)$ are Lipschitz functions of order $\gamma > 0$, $1 \le r \le d$;

(A_3) $\phi(\cdot)$ belongs to S_l, which is a Schwartz space for $l \ge \alpha$. ϕ is a Lipschitz function of order 1 and have compact support, in addition to $\left|\hat{\phi}(\xi) - 1\right| = O(\xi)$ as $\xi \to 0$, where $\hat{\phi}$ denotes Fourier transform of ϕ;

(A_4) $S_i (i = 1, \cdots, n)$ and m satisfy $\max\limits_{1 \le i \le n}(s_i - s_{i-1}) = O\left(n^{-1}\right)$ and $2^m = O\left(n^{1/3}\right)$, respectively.

3. Statements of the Results

Theorem 3.1. *Suppose that conditions* $(A_1) - (A_4)$ *hold. If* $\sup\limits_{i \ge 1} \left\| E\left(|\varepsilon_i|^\theta \,\middle|\, F_{i-1}\right)\right\| \le C < \infty$ *for some* $\theta > 3$, $E\eta_{1j}^2 < \infty (j = 1, 2, \cdots, d)$ *and* $\sup\limits_h E|\varepsilon_h|^2 < \infty$, *then for* $\gamma \ge 1/3$ *and* $\alpha > 3/2$,

$$\sup_{1 \le i \le d} \left|\hat{\beta}_{ni} - \beta_i\right| = O\left(n^{-1/3} \log n\right) \quad a.s.,$$

where $\|\xi\| = \inf\{c > 0,\ P(|\xi| \le c) = 1\}$, $\hat{\beta}_{ni}$ *is the ith component of* $\hat{\beta}_n$, β_i *is the ith component of* β.

Theorem 3.2. *Under the same assumptions as in Theorem* 3.1. *Then*

$$\sup_{0 \le t \le 1} \left|\hat{g}_n(t) - g(t)\right| = O\left(n^{-1/3} \log n\right) \quad a.s.,\ n \to \infty.$$

Remark 1. Our results are the same as in Qian and Cai [12], where the errors are i.i.d. random variables. In the paper, however, the errors are martingale difference sequences. Hence our results are extensions of Qian and Cai [12]. Moreover, it is illustrated that the wavelet method can be successfully applied to semiparametric regression models with the martingale difference errors. In addition, we easily obtain strong consistencies of the wavelet estimators by the above results. Hence our results are extensions of Hu and Hu [5].

4. Proofs of Theorems

Throughout this paper, let C denote a generic positive constant which could take different value at each occurrence. To prove the main results, we first introduce some lemmas.

Lemma 4.1 (Antoniads et al. [1]). *If condition* (A_3) *holds, then*

(I) $|E_0(t, s)| \le \dfrac{C_k}{(1+|t-s|)^k}$ *and* $|E_m(t, s)| \le \dfrac{2^m C_k}{(1+2^m|t-s|)^k}$, *where* k *is a positive integer and* C_k *is a constant depending on* k *only.*

(II) $\sup\limits_{0\le s\le 1}|E_m(t, s)| = O\left(2^m\right)$.

(III) $\sup\limits_t \int_0^1 |E_m(t, s)|ds \le C$, *where* C *is a positive constant.*

Lemma 4.2 (Hu and Hu [5]). *Let* $\tau_m = 2^{-m(\alpha-1/2)}$ *when* $1/2 < \alpha < 3/2$, $\tau_m = \sqrt{m}\cdot 2^{-m}$ *when* $\alpha = 3/2$, $\tau_m = 2^{-m}$ *when* $\alpha > 3/2$. *If conditions* $(A_1)-(A_4)$ *hold, then*

$$\sup_t \left| f_j(t) - \sum_{k=1}^n \left(\int_{A_k} E_m(t, s)ds\right) f_j(t_k)\right| = O\left(n^{-\gamma}\right) + O(\tau_m),$$

$$\sup_t \left| g(t) - \sum_{k=1}^n \left(\int_{A_k} E_m(t, s)ds\right) g(t_k)\right| = O\left(n^{-\gamma}\right) + O(\tau_m).$$

Lemma 4.3. *Let* $\{\varepsilon_i, F_i, i \ge 1\}$ *be a martingale sequence with* $\sup\limits_{i\ge 1}\left\|E\left(|\varepsilon_i|^\theta \,\middle|\, F_{i-1}\right)\right\| \le C$ *for some* $\theta > 3$. *Assume that* $a_{ni}(t)$ *is a nonnegative function on* $[0, 1]$ *such that* $\sum\limits_{i=1}^n a_{ni}(t) \le c$ *and* $\max\limits_{1\le i\le n} a_{ni}(t)\, n^{2/3} \le c$ *for any* $t \in [0, 1]$, *and* $\max\limits_{1\le i\le n}|a_{ni}(t_1) - a_{ni}(t_2)| \le c|t_1 - t_2|$ *for any* $t_1, t_2 \in [0, 1]$. *Then*

$$\sup_{0\le t\le 1}\left|\sum_{i=1}^{n} a_{ni}(t)\varepsilon_i\right| = O\left(n^{-1/3}\log n\right) \quad a.s.$$

The proof is easy by using Lemma 3.3 of Li and Liu [9], and hence omitted here.

Lemma 4.4. *Let* $\{\varepsilon_i, F_i, i \ge 1\}$ *be a martingale sequence. Suppose that conditions* (A_3) *and* (A_4) *hold, and* $\sup_{i\ge 1}\left\|E\left(|\varepsilon_i|^{\theta}\,\middle|\, F_{i-1}\right)\right\| \le C < \infty$ *for some* $\theta > 3$. *Then*

$$\sup_{0\le t\le 1}\left|\sum_{i=1}^{n}\varepsilon_i\int_{A_i} E_m(t, s)ds\right| = O\left(n^{-1/3}\log n\right) \quad a.s.$$

Proof. Let $a_i(t) = \int_{A_i} E_m(t, s)ds$. Then by Lemma 4.3,

$$\sup_{0\le t\le 1}\sum_{i=1}^{n}\left|\int_{A_i} E_m(t, s)ds\right| \le \sup_{0\le t\le 1}\int_0^1 |E_m(t, s)|ds \le c,$$

$$\sup_{0\le t\le 1}\max_{1\le i\le n}\left|\int_{A_i} E_m(t, s)ds\right| n^{2/3} \le cn^{-1}2^m n^{2/3} \le c.$$

Since $E_0(t, s)$ satisfies Lipschitz condition with order 1 for any t, we obtain

$$\begin{aligned}\max_{1\le i\le n}\left|\int_{A_i} E_m(t_1, s)ds - \int_{A_i} E_m(t_2, s)ds\right| &\le \int_{A_i}|E_m(t_1, s) - E_m(t_2, s)|ds\\ &\le cn^{-1}2^m\left|E_0\left(2^m t_1, 2^m s\right) - E_0\left(2^m t_2, 2^m s\right)\right|\\ &\le cn^{-1}2^m\left|2^m t_1 - 2^m t_2\right| \le c|t_1 - t_2|.\end{aligned}$$

By Lemma 4.3, we obtain the Lemma 4.4.

Lemma 4.5 (Yang [16]). *Let* $S_i = \sum_{j=1}^{i} X_j$. *If* $\{S_i, F_i, i \ge 1\}$ *is a martingale sequence, then for any* $r > 2$,

$$E \max_{1\le i\le n}|S_i|^r \le 2C_{r_2}\left(\sum_{i=1}^{n}\left(E|X_i|^r\right)^{\frac{2}{r}}\right)^{\frac{r}{2}}. \tag{8}$$

Moreover, if there exist some α *with* $0 < \alpha \le 2$ *and a positive constant sequence* $\{M_n, n \ge 1\}$ *such that* $\sum_{i=1}^{n} E\left(|X_i|^{\alpha}|F_{i-1}\right) \le M_n$ *a.s. hold, then for any* $r > 2$,

$$E \max_{1\le i\le n}|S_i|^r \le \frac{2+r^2}{\alpha} C_{r_1}\sum_{i=1}^{n} E|X_i|^r + C_{r_2}^{\frac{2}{r}} M_n^{\frac{r}{\alpha}}, \tag{9}$$

where $C_{r_1} = \left(r/(r-1)\right)^r 2^{r-3} r$, $C_{r_2} = \left(C_{r_1} r\right)^{r/2}$.

Lemma 4.6. *Suppose that conditions* (A_3) *and* (A_4) *hold, and* $E\eta_{1j}^2 < \infty (j = 1, 2, \cdots, d)$. *Then*

$$\sup_t \left|\sum_{k=1}^{n} \eta_{kj} \int_{A_k} E_m(t, s)ds\right| = O(n^{-1/3}\log n) \quad a.s.$$

Proof. Note that

$$\sup_{0\le t\le 1}\left|\sum_{k=1}^{n}\eta_{kj}\int_{A_k} E_m(t, s)ds\right| \le \sup_{0\le t\le 1}\left|\sum_{k=1}^{n}\eta_{kj}\int_{A_k}\sum_{i=1}^{n}\left(E_m(t, s) - E_m(s_j, s)\right)I(s_{i-1} < t \le s_i)ds\right|$$

$$+ \sup_{0\le t\le 1}\left|\sum_{k=1}^{n}\eta_{kj}\int_{A_k}\sum_{i=1}^{n}E_m(s_i, s)I(s_{i-1} < t \le s_i)ds\right| \triangleq T_1 + T_2. \tag{10}$$

By (2.10) of Qian and Cai [12], we obtain

$$T_1 = o\left(n^{-1/3}\log n\right) \quad a.s. \tag{11}$$

By lemma A.3 of Hardle et al. [3], we obtain

$$T_2 = O\left(n^{-1/3}\log n\right) \quad a.s. \tag{12}$$

Therefore, combining (10)-(12), we obtain the Lemma 4.6.

Remark 2. The above results are sharper than $O\left(n^{-1/6}(\log n)^{1/2}\right)$ in Qian and Cai [12].

Proof of Theorem 3.1. Let $\tilde{\varepsilon} = (I - S)\varepsilon$, $\tilde{g} = (I - S)g$ and $g = \left(g(t_1), \cdots, g(t_n)\right)^T$. Then

$$\hat{\beta}_n - \beta = \left(n^{-1}\tilde{X}^T\tilde{X}\right)^{-1}\left(n^{-1}\tilde{X}^T\tilde{g} + n^{-1}\tilde{X}^T\tilde{\varepsilon}\right). \tag{13}$$

By the proof of Theorem 2.1 of Hu and Hu [5], we obtain

$$n^{-1}\tilde{X}^T\tilde{X} \xrightarrow{a.s.} V \quad (n \to \infty). \tag{14}$$

Next we shall prove that

$$\left(n^{-1}\tilde{X}^T\tilde{g}\right)_i = O\left(n^{-1/3}\log n\right) \quad a.s., \tag{15}$$

where $\left(n^{-1}\tilde{X}^T\tilde{g}\right)_i$ is the ith component of $n^{-1}\tilde{X}^T\tilde{g}$. In fact,

$$\left(n^{-1}\tilde{X}^T\tilde{g}\right)_i = \frac{1}{n}\sum_{h=1}^{n}\eta_{hi}\left(g(t_h) - \sum_{r=1}^{n}S_{hr}g(t_r)\right) - \frac{1}{n}\sum_{h=1}^{n}\sum_{k=1}^{n}S_{hk}\eta_{ki}\left(g(t_h) - \sum_{r=1}^{n}S_{hr}g(t_r)\right)$$

$$+ \frac{1}{n}\sum_{h=1}^{n}\left(f_i(t_h) - \sum_{k=1}^{n}S_{hk}f_i(t_k)\right)\left(g(t_h) - \sum_{r=1}^{n}S_{hr}g(t_r)\right) \triangleq J_1 + J_2 + J_3. \tag{16}$$

Using Markov inequality and Lemma 3.2, we obtain

$$\sum_{n=1}^{\infty}P\left(|J_1| \geq n^{-1/3}\log n\right) \leq \sum_{n=1}^{\infty}\left(EJ_1^2 \Big/ \left(n^{-1/3}\log n\right)^2\right) \leq C\sum_{n=1}^{\infty}n^{-1}(\log n)^{-2} < \infty.$$

Hence by Borel-Cantelli lemma,

$$J_1 = O\left(n^{-1/3}\log n\right) \quad a.s. \tag{17}$$

By Lemma 4.2 and Lemma 4.6, it is easy to get that

$$J_2 = o\left(n^{-1/3}\log n\right) \quad a.s. \tag{18}$$

By Lemma 4.2, we obtain

$$J_3 = O\left(n^{-2\gamma}\right) + O\left(\tau_m^2\right) = o\left(n^{-1/3}\log n\right). \tag{19}$$

Hence the (15) follows from (17)-(19).

The end, we shall show that

$$\left(n^{-1}\widetilde{X}^T\widetilde{\varepsilon}\right)_i = O\left(n^{-1/3}\log n\right) \quad a.s. \tag{20}$$

In fact,

$$\left(n^{-1}\widetilde{X}^T\widetilde{\varepsilon}\right)_i = \frac{1}{n}\sum_{h=1}^{n}\left(\eta_{hi} - \sum_{k=1}^{n} S_{hk}\eta_{ki}\right)\left(\varepsilon_h - \sum_{r=1}^{n} S_{hr}\varepsilon_r\right)$$

$$+ \frac{1}{n}\sum_{h=1}^{n}\left(f_i(t_h) - \sum_{k=1}^{n} S_{hk} f_i(t_k)\right)\left(\varepsilon_h - \sum_{r=1}^{n} S_{hr}\varepsilon_r\right) \hat{=} T_1 + T_2. \tag{21}$$

$$T_1 = \frac{1}{n}\sum_{h=1}^{n}\left(\eta_{hi}\varepsilon_h - \left(\sum_{k=1}^{n} S_{hk}\eta_{ki}\right)\varepsilon_h - \left(\sum_{r=1}^{n} S_{hr}\varepsilon_r\right)\eta_{hi} + \left(\sum_{k=1}^{n} S_{hk}\eta_{ki}\right)\left(\sum_{r=1}^{n} S_{hr}\varepsilon_r\right)\right)$$

$$\hat{=} T_1^{(1)} - T_1^{(2)} - T_1^{(3)} + T_1^{(4)}. \tag{22}$$

Note that $\{\eta_{hi}\}$ and $\{\varepsilon_h\}$ are independent each other, and $\sup_h E|\varepsilon_h|^2 < \infty$ and $E\eta_{1j}^2 < \infty$. Thus, by Lemma 3.3 of Yang [16], we can easily obtain

$$T_1^{(1)} = o\left(n^{-1/3}\log n\right) \quad a.s.,\ n \to \infty. \tag{23}$$

By Lemma 4.6 and strong law of large numbers, we obtain

$$\left|T_1^{(2)}\right| \le \max_h\left|\left(\sum_{k=1}^{n} S_{hk}\eta_{ki}\right)\right| \cdot \left(\frac{1}{n}\sum_{h=1}^{n}|\varepsilon_h|\right) = O\left(n^{-1/3}\log n\right) \quad a.s. \tag{24}$$

By Lemma 4.4 and strong law of large numbers, we obtain

$$\left|T_1^{(3)}\right| \le \max_h\left|\left(\sum_{r=1}^{n} S_{hr}\varepsilon_r\right)\right| \cdot \frac{1}{n}\sum_{h=1}^{n}|\eta_{hi}| = O\left(n^{-1/3}\log n\right) \quad a.s. \tag{25}$$

By Lemmas 4.4 and 4.6, we obtain

$$\left|T_1^{(4)}\right| \le \max_h \left|\left(\sum_{k=1}^n S_{hk}\eta_{ki}\right)\left(\sum_{r=1}^n S_{hr}\varepsilon_r\right)\right| = o\left(n^{-1/3}\log n\right) \quad a.s. \tag{26}$$

Hence from (22)-(26), we obtain

$$T_1 = O\left(n^{-1/3}\log n\right) \quad a.s.,\ n \to \infty. \tag{27}$$

Let $\varepsilon_{ni} = \varepsilon_i I\left(|\varepsilon_i| \le n^{1/q}\right)$, $\tilde{\eta}_{ni} = \varepsilon_i I\left(|\varepsilon_i| > n^{1/q}\right)(q > 1)$, $y_{ni} = \varepsilon_{ni} - E\left(\varepsilon_{ni}\middle| F_{i-1}\right)$, $z_{ni} = \tilde{\eta}_{ni} - E\left(\tilde{\eta}_{ni}\middle| F_{i-1}\right)$, $a_{hi} \hat{=} f_i(t_h) - \sum_{k=1}^n S_{hk} f_i(t_k)$. Then

$$T_2 = \frac{1}{n}\sum_{h=1}^n a_{hi}\left(\varepsilon_h - \sum_{r=1}^n S_{hr}\varepsilon_r\right) = \frac{1}{n}\sum_{h=1}^n a_{hi}y_{nh} + \frac{1}{n}\sum_{h=1}^n a_{hi}z_{nh}$$

$$- \frac{1}{n}\sum_{h=1}^n a_{hi}\sum_{r=1}^n S_{hr}\varepsilon_r \hat{=} T_2^{(1)} + T_2^{(2)} + T_2^{(3)}. \tag{28}$$

Note that $E\left((a_{hi}y_{nh}/n)^2\middle| F_{i-1}\right) \le 2Cn^{1/q}(a_{hi}/n)^2$ and $\left\{\frac{1}{n}\sum_{h=1}^m a_{hi}y_{nh},\ m \le n\right\}$ is a martingale sequence, and by Lemma 4.5, we obtain

$$E\left|T_2^{(1)}\right|^r \le C\left\{\sum_{h=1}^n |a_{hi}/n|^r E|y_{nh}|^r + \left(\sum_{h=1}^n |a_{hi}/n|^2 n^{1/q}\right)^{r/2}\right\}$$

$$\le C\left\{n^{-r(1/2-1/(2q)+\gamma)} + n^{-r(1/2-1/(2q))}\tau_m^r\right\} \le Cn^{-r(5/6-1/(2q))}.$$

Hence for all large r,

$$\sum_{n=1}^\infty P\left(\left|T_2^{(1)}\right| \ge n^{-1/3}\log n\right) \le \sum_{n=1}^\infty n^{r/3}(\log n)^{-r} E\left|T_2^{(1)}\right|^r$$

$$\le \sum_{n=1}^\infty n^{-r(1/2-1/(2q))}(\log n)^{-r} < \infty.$$

Thus

$$T_2^{(1)} = O\left(n^{-1/3} \log n\right) \quad a.s. \tag{29}$$

Let $\xi_i = \varepsilon_i I\left(|\varepsilon_i| > i^{1/q}\right), \zeta_i = |\xi_i| + E\left(|\xi_i| \mid F_{i-1}\right), S_n = \sum_{i=1}^{n} \zeta_i \Big/ i^{\rho}$ (where $\rho = 2/3$). Then there exists a random variable S such that $S_n \xrightarrow{a.s.} S$ and $ES < \infty$. Thus

$$\left|T_2^{(2)}\right| \le \sum_{h=1}^{n} \frac{1}{n} |a_{hi}| |z_{nh}| \le C \cdot \frac{1}{n} \cdot \left(n^{-\gamma} + \tau_m\right) \sum_{h=1}^{n} \zeta_h$$

$$\le C n^{-1/3} \left(n^{-\gamma} + \tau_m\right) \sum_{h=1}^{n} \frac{\zeta_h}{h^{\rho}}. \tag{30}$$

Hence

$$T_2^{(2)} = o\left(n^{-1/3} \log n\right) \quad a.s. \tag{31}$$

By Lemma 4.2 and Lemma 4.4, we obtain

$$T_2^{(3)} = \frac{1}{n} \sum_{h=1}^{n} a_{hi} \sum_{r=1}^{n} S_{hr} \varepsilon_r = \max_h \left(a_{hi} \sum_{r=1}^{n} S_{hr} \varepsilon_r \right) = o\left(n^{-1/3} \log n\right) \quad a.s. \tag{32}$$

Hence by (28), (29), (31) and (32),

$$T_2 = o\left(n^{-1/3} \log n\right) \quad a.s. \tag{33}$$

Thus, the (20) follows from (21), (27) and (33).

By (13), we obtain

$$\sup_{1 \le i \le d} \left|\hat{\beta}_{ni} - \beta_i\right| \le d \cdot \sup_{1 \le j \le d} \left| \left(n^{-1} \tilde{X}^T \tilde{X}\right)_{ij}^{-1} \right| \sup_{1 \le j \le d} \left| \left(n^{-1} \tilde{X}^T \tilde{g}\right)_j + \left(n^{-1} \tilde{X}^T \tilde{\varepsilon}\right)_j \right|. \tag{34}$$

Hence the Theorem 3.1 follows from (14), (15), (20) and (34).

Proof of Theorem 3.2. Note that

$$\sup_t |\hat{g}_n(t) - g(t)| \le \sup_t |\hat{g}_0(t, \beta) - g(t)| + \sup_t \left| \sum_{j=1}^{n} X_i^T \left(\beta - \hat{\beta}_n\right) \int_{A_j} E_m(t, s) ds \right|$$

$$\le \sup_t \left| \sum_{j=1}^{n} g(t_j) \int_{A_j} E_m(t,\, s)ds - g(t) \right| + \sup_t \left| \sum_{j=1}^{n} \varepsilon_j \int_{A_j} E_m(t,\, s)ds \right|$$

$$+ \sum_{j=1}^{d} \left(\left| \hat{\beta}_{nj} - \beta_j \right| \sup_t \left| \sum_{i=1}^{n} f_j(t_i) \int_{A_i} E_m(t, s)ds \right| \right) + \sum_{j=1}^{d} \left(\left| \hat{\beta}_{nj} - \beta_j \right| \sup_t \left| \sum_{i=1}^{n} \eta_{ij} \int_{A_i} E_m(t, s)ds \right| \right)$$

$$\hat{=}\ K_1 + K_2 + K_3 + K_4. \tag{35}$$

By Lemma 4.2, we obtain

$$K_1 = O\left(n^{-\gamma}\right) + O(\tau_m) = o\left(n^{-1/3} \log n\right). \tag{36}$$

By Lemma 4.4, we obtain

$$K_2 = O\left(n^{-1/3} \log n\right) \quad a.s. \tag{37}$$

Let $M = \sup_{j, t_i} \left| f_j(t_i) \right|$. Then by Lemma 4.1 and Theorem 3.1, we obtain

$$K_3 \le d \cdot \sup_{j, t_i} \left| f_j(t_i) \right| \cdot \sup_t \int_0^1 \left| E_m(t,\, s) \right| ds \cdot \max_j \left| \hat{\beta}_{nj} - \beta_j \right|$$

$$\le C \max_{1 \le j \le d} \left| \hat{\beta}_{nj} - \beta_j \right| = O\left(n^{-1/3} \log n\right). \tag{38}$$

By Lemma 4.6 and Theorem 3.1, we obtain

$$K_4 \le d \sup_{1 \le j \le d} \left| \hat{\beta}_{nj} - \beta_j \right| \max_{1 \le j \le d} \sup_t \left| \sum_{i=1}^{n} \eta_{ij} \int_{A_i} E_m(t,\, s)ds \right| = o\left(n^{-1/3} \log n\right). \tag{39}$$

From (35)-(39), the desired conclusion follows.

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