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SOME NOTES ON MINIMALITY IN DYNAMICAL SYSTEMS

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Abstract

In this paper, minimal systems are classified according to an isolated point existing or not. And as for a homeomorphism on a sequentially compact space, an equicontinuous homeomorphism on a compact metric space and the product of two minimal maps, necessary and sufficient conditions for the map to be minimal are given.

1. Introduction

In the theory of dynamical systems, minimality describes the irreducibility of a system from the topological point of view, which means that a minimal system has no proper subsystem. So research on

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minimality plays an important role and there have sprung up many results about it. In [1, 5] several equivalent definitions of a compact minimal system are given. For other results on minimality see [2, 3, 4]. In this paper, we will classify minimal systems by means of isolated points. And at the same time, equivalent descriptions for two kinds of homeomorphisms and the product of two minimal systems to be minimal will be given.

Now we will introduce the concept of minimal map and a useful lemma as follows.

Definition 1. Let $f : X \to X$ be a continuous map on X. A closed nonempty f-invariant subset $Y \subseteq X$ is called a minimal set for f, if it contains no proper closed non-empty f-invariant subset. If X itself is a minimal set, we say that f is minimal.

Lemma 1 [5]. *Suppose $f : X \to X$ is a continuous map on X and Y is a closed nonempty f-invariant subset. Then Y is a minimal set if and only if for every $y \in Y$, $\overline{orb(y)} = Y$.*

2. Main Results

Theorem 1. *A minimal set of system (X, d) either has no isolated point or is a periodic orbit.*

The proof is not difficult, so we omit it.

Theorem 2. *Suppose f is a homeomorphism on a sequentially compact space X. Then f is minimal if and only if for every non-empty open set $U \subset X$, there exists $n \in \mathbb{N}$ such that $\bigcup_{k=-n}^{n} f^k(U) = X$.*

Proof. The necessity is to be proved firstly. Suppose that there is a non-empty open set $U \subset X$, satisfying that for every $n \in \mathbb{N}$, there exists $x_n \in X$, such that $x_n \notin \bigcup_{k=-n}^{n} f^k(U)$. The sequential compactness of X implies that $\{x_n\}$ has a convergent subsequence, denoted by $\{x_{n_k}\}$, and $x_{n_k} \to x_0 (k \to \infty)$. As f is minimal, $\overline{orb(x_0)} = X$. So for U, there exists

$i \in \mathbb{Z}$ such that $f^i(x_0) \in U$, i.e., $x_0 \in f^{-i}(U)$. From $x_{n_k} \to x_0 (k \to \infty)$, we know that for $f^{-i}(U)$ as a neighborhood of x_0, there exists $k \in \mathbb{N}$ sufficiently large, such that $n_k > |i|$, and $x_{n_k} \in f^{-i}(U)$. So $x_{n_k} \in \bigcup_{j=-n_k}^{n_k} f^{-j}(U)$, which contradicts the choice of x_{n_k}.

Conversely, for every $x \in X$ and every non-empty open set $U \subset X$, there exists $n \in \mathbb{N}$ such that $\bigcup_{k=-n}^{n} f^k(U) = X$. So there is $-n \le k_0 \le n$, such that $x \in f^{k_0}(U)$, i.e., $f^{-k_0}(x) \in U$. The arbitrariness of U implies $orb(x)$ is dense in X. Then the minimality of f can be inferred from the arbitrariness of x and Lemma 1.

Theorem 3. *Suppose homeomorphism f of compact metric space (X, d) is equicontinuous. Then f is minimal if and only if for every $\varepsilon > 0$, there is $N = N(\varepsilon) \in \mathbb{N}$, such that for every $x \in X$, the set $\{x, f(x), \cdots, f^N(x)\}$ is ε-dense in X.*

Proof. Firstly we will show the necessity of the condition. From the equicontinuity of f, we know that for any $\varepsilon > 0$, there exists $0 < \delta < \frac{\varepsilon}{2}$, such that $d(x, y) < \delta$ implies $d(f^n(x), f^n(y)) < \frac{\varepsilon}{2}$ for all $n \in \mathbb{Z}$.

As X is compact, the open cover $\{B(x, \delta) | x \in X\}$ of X has a finite subcover which we denote by $\{B(x_i, \delta) | i = 1, 2, \cdots, k\}$. From the minimality of f, we know that $\overline{orb(x_i)} = X$, and accordingly $\bigcup_{j=0}^{\infty} B(f^j(x_i), \frac{\varepsilon}{2}) = X (i = 1, 2, \cdots, k)$. Therefore for every $i \in \{1, 2, \cdots, k\}$, the open cover $\{B(f^j(x_i), \frac{\varepsilon}{2}) | j \in \mathbb{N}\}$ has a finite subcover, in which the maximum of j is denoted by $j(i)$. Let $N = \max\{j(i) | i = 1, 2, \cdots, k\}$.

Now because $\{B(x_i, \delta) | i = 1, 2, \cdots, k\}$ is an open cover of X, for every $x \in X$, there exists $i \in \{1, 2, \cdots, k\}$, such that $x \in B(x_i, \delta)$. The equicontinuity of f implies that $d(f^t(x), f^t(x_i)) < \frac{\varepsilon}{2}(t = 0, 1, \cdots, N)$. So for every $t \in \{0, 1, \cdots, N\}$ and $y \in B(f^t(x_i), \frac{\varepsilon}{2})$, we have that

$$d(y, f^t(x)) \leq d(y, f^t(x_i)) + d(f^t(x_i), f^t(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which implies that $B(f^t(x_i), \frac{\varepsilon}{2}) \subset B(f^t(x), \varepsilon)(t = 0, 1, \cdots, N)$. Note $\bigcup_{t=0}^{N} B(f^t(x_i), \frac{\varepsilon}{2}) = X$. So $\bigcup_{t=0}^{N} B(f^t(x), \varepsilon) = X$, which means that $\{x, f(x), \cdots, f^N(x)\}$ is ε-dense in X.

Conversely, as is known, for every $x \in X$, there is a natural number N such that $\bigcup_{i=0}^{N} B(f^i(x), \varepsilon) = X$. So for every $y \in X$ and $\varepsilon > 0$, there exists $i \in \{0, 1, \cdots, N\}$, such that $y \in B(f^i(x), \varepsilon)$, which implies $f^i(x) \in B(y, \varepsilon)$. Therefore $\overline{orb(x)} = X$. So the minimality of f follows from the arbitrariness of x and Lemma 1.

Theorem 4. *Suppose $f : X \to X$ and $g : Y \to Y$ are minimal maps of compact metric spaces. Then $f \times g$ is minimal if and only if for every $x \in X$ and $y \in Y$, $(x, g(y)) \in \overline{orb^+_{f\times g}(x, y)}$ and $(f(x), y) \in \overline{orb^+_{f\times g}(x, y)}$.*

Proof. Firstly we claim that for every $x \in X$ and $y \in Y$, $\overline{orb^+_{f\times g}(x, y)} = \overline{orb^+_f(x)} \times \overline{orb^+_g(y)}$ if and only if $(x, g(y)) \in \overline{orb^+_{f\times g}(x, y)}$ and $(f(x), y) \in \overline{orb^+_{f\times g}(x, y)}$ for arbitrary $x \in X$ and $y \in Y$.

In fact, the necessity is straightforward since $(x, g(y)), (f(x), y) \in \overline{orb^+_f(x)} \times \overline{orb^+_g(y)}$. And next the sufficiency is to be proved. Obviously $\overline{orb^+_{f\times g}(x, y)} \subseteq \overline{orb^+_f(x)} \times \overline{orb^+_g(y)}$, so it suffices to prove that $\overline{orb^+_f(x)} \times \overline{orb^+_g(y)} \subseteq \overline{orb^+_{f\times g}(x, y)}$. As $(x, g(y)) \in \overline{orb^+_{f\times g}(x, y)}$, $\overline{orb^+_{f\times g}(x, g(y))} \subseteq$

$\overline{orb^{+}_{f\times g}(x,\ y)}$. Then with reference to the fact that $(x,\ g^{2}(y)) \in \overline{orb^{+}_{f\times g}(x,\ g(y))}$, $(x,\ g^{2}(y)) \in \overline{orb^{+}_{f\times g}(x,\ y)}$ holds. By mathematical induction, it can be inferred that for every $x \in X$, $y \in Y$ and $n \in \mathbb{N}$,

$$(x,\ g^{n}(y)) \in \overline{orb^{+}_{f\times g}(x,\ y)}. \tag{2.1}$$

In the same way, $(f(x),\ y) \in \overline{orb^{+}_{f\times g}(x,\ y)}$ implies that for every $x \in X$, $y \in Y$ and $n \in \mathbb{N}$,

$$(f^{n}(x),\ y) \in \overline{orb^{+}_{f\times g}(x,\ y)}. \tag{2.2}$$

(2.1) together with (2.2) demonstrates that for any $k \in \mathbb{N}$, $x \in X$ and $y \in Y$,

$$(f^{k}(x),\ g^{n}(y)) \in \overline{orb^{+}_{f\times g}(x,\ y)},\ n = 0, 1, \cdots. \tag{2.3}$$

Then $\{f^{k}(x)\} \times \overline{orb^{+}_{g}(y)} \subseteq \overline{orb^{+}_{f\times g}(x,\ y)}$. And hence $\bigcup_{k=0}^{\infty}\{f^{k}(x)\} \times \overline{orb^{+}_{g}(y)} \subseteq \overline{orb^{+}_{f\times g}(x,\ y)}$. Therefore $\overline{orb^{+}_{f}(x)} \times \overline{orb^{+}_{g}(y)} \subseteq \overline{orb^{+}_{f\times g}(x,\ y)}$.

Combination of the above claim with the minimality of f and g implies that $\overline{orb^{+}_{f\times g}(x,\ y)} = \overline{orb^{+}_{f}(x)} \times \overline{orb^{+}_{g}(y)} = X \times Y$ (for every $x \in X$ and $y \in Y$), which leads to the result that $f \times g$ is minimal. Conversely, the minimality of f, g and $f \times g$ means that $\overline{orb^{+}_{f}(x)} \times \overline{orb^{+}_{g}(y)} = \overline{orb^{+}_{f\times g}(x,\ y)}$. Then from the above claim, the conclusion that for every $x \in X$ and $y \in Y$, $(x,\ g(y)) \in \overline{orb^{+}_{f\times g}(x,\ y)}$ and $(f(x),\ y) \in \overline{orb^{+}_{f\times g}(x,\ y)}$ follows.

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