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THE DISTRIBUTION AND MOMENTS OF THE RUN LENGTH DISTRIBUTION OF A GENERALISED CONTROL CHART

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Abstract

Control charts are widely employed in quality control to monitor a process or to evaluate historic data. These charts are designed to exhibit acceptable average run lengths both when the process is in and out of control. It is conventional to report these values. This paper examines the moments of the underlying distribution and its probability density function. These values are used to examine claims concerning the skewness of this distribution and provide guides to adoption of appropriate parameters. A number of disparate papers are described by using a consistent notation. The approach is developed to cover generalised control charts.

1. Introduction

Control charts are often employed to detect changes in a process mean over time. In the traditional approach, a sample is drawn, and the

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sample mean ($\bar{x}$) is calculated and plotted on a Shewhart [18] $\bar{x}$-chart having control limits, which depict the extremes of pure chance fluctuations. A point outside the limits suggests that the process is off target. While a Shewhart $\bar{x}$-chart is relatively easy to use and interpret, a cumulative sum (Page [16] and Woodall [19]) (CUSUM) chart is more capable of detecting small changes in the process mean, as well as pinpointing the exact time when the production line goes "out of control." Faster detection of significant changes means tighter control for the operator, if corrective action is to be taken promptly.

Like Shewhart and CUSUM control schemes, an exponentially weighted moving average (EWMA) (Lucas and Saccucci [13]) control scheme is easy to implement and interpret. The ability of the EWMA chart to detect small shifts in the process mean is on a par with the CUSUM chart and superior to the Shewhart $\bar{x}$-chart. It has been argued (Lucas and Saccucci [13]) that the EWMA chart is simpler to explain to the lay user than the CUSUM chart, by noting its similarity to the classical Shewhart $\bar{x}$-chart. Both the CUSUM and EWMA charts are more suitable for single sampling schemes.

Many papers have discussed elements of the material covered here. The aim is to pull together a number of disparate papers employing a consistent notation. The approach is generalised to a wider family of control charts with CUSUM, EWMA, and Shewhart charts as special cases.

An early work plus related computer programme examining the ARL for a CUSUM chart was presented by Gan [10]. While this work described a numerical method for addressing the problem no discussion of the results was presented. For the EWMA chart, first and second moments of the run length distribution were reported by Crowder [7]. Both these authors solved a system of integral equations by using Gaussian quadrature. The results for the EWMA chart were extended by Lucas and Saccucci [13], who employed a continuous-state Markov chain approach (Brook and Evans [3]). They point out that the ARL characterizes the run length distribution for both an in/out of control process. The equivalence of their approach to the integral equation adopted here was demonstrated by Calzada and Scariano [4], to within numerical uncertainties, the methods produce identical results. For a recent paper that reviews the area of run length distributions for control charts see Fu et al. [9], however, the authors concentrate on normally distributed observations.

A generalised control chart has been proposed for which the Shewhart $\bar{x}$-chart, the cumulative sum chart, and the exponentially weighted moving average chart are special cases. The procedures for implementing these charts have been described by Champ et al. [5]; the notation employed here closely follows that previously introduced.

Assume that the individual random variables are normally distributed with mean μ and variance σ^2, that is, $x_i \sim \phi(\mu, \sigma^2)$. Let $\bar{x}_t$ be the mean of a random sample of n observations during the time interval, t. Then $\bar{x}_t$ is normally distributed $\bar{x}_t \sim \phi\left(\mu, \frac{\sigma^2}{n}\right)$. For simplicity, all charts will be constructed for a standard normal variable $z_t = \frac{\bar{x}_t - \mu}{\sigma / \sqrt{n}}$; therefore $z_t \sim \phi(0, 1)$.

The generalised control procedure is based on the cumulative values.

$$U_t = \max\{-a_0, a_1 U_{t-1} + a_2 z_t - a_3\}, \quad U_0 = a_4,$$

$$L_t = \min\{b_0, b_1 L_{t-1} + b_2 z_t + b_3\}, \quad L_0 = -b_4.$$

The two-sided generalised control chart gives an out-of-control signal as soon as either $U_t \geq a_5$ or $L_t \leq -b_5$, where the values of $a_0, \ldots, a_5$ and $b_0, \ldots, b_5$ are usually chosen to be non-negative, obviously, a_2 and b_2 cannot vanish.

The head start values are required to satisfy $-a_0 \leq a_4 \leq a_5$ and $-b_5 \leq -b_4 \leq b_0$, respectively. This notation is particularly convenient for the problem addressed here. It only differs from the original in the adoption of a sign change for $\{a_0, a_3, b_4, b_5\}$.

The average run length (ARL) gives the number of batches sampled till one is rejected. Initially interest centres on the moments of the run length distribution, the mean being the value usually desired. However, the comparison and design of quality control charts is usually based on the properties of the run length distribution. Moments of the run length distribution can be used to design and study the performance of quality control charts. It is reported by Woodall [19] that the run length distribution is often right skew. This statement will be investigated. It would appear a much stronger proposition might be supported.

Initially, attention concentrates on the first four moments of the run length distribution. In a typical application, trying to identify an appropriate two-parameter distribution that best describes the run length distribution, the first two moments could be used to estimate the parameters of the assumed distribution, while the third and fourth moments would then be used to assess the fit. (Alternative approaches would be maximum likelihood or least squares estimates obtained by fitting a regression line to the points in a probability plot.) The third moment may be scaled to give the skewness, which is a measure of symmetry, or more precisely, the lack of symmetry. The skewness for a normal distribution is zero, and any symmetric distribution should have skewness near zero. Negative values are associated with a left skew; the left tail is long relative to the right tail. While positive values imply the converse. Some measurements have a natural lower bound and are skewed right by definition. For example, in reliability studies, failure times cannot be negative. The fourth moment may be scaled to give the kurtosis. A high kurtosis distribution has a sharper peak and longer, fatter tails, while a low kurtosis distribution has a more rounded peak and shorter thinner tails. It is a measure of whether the data are peaked or flat, relative to a normal distribution. That is, distributions with high kurtosis tend to have a distinct peak near the mean, decline rather rapidly, and have heavy tails. Distributions with low kurtosis tend to have a flat top near the mean rather than a sharp peak. The kurtosis for a standard normal distribution is three. These measures are formally defined later.

The sections of the paper are briefly outlined. Having presented integral equations for the central moments of the run length distribution, their approximate numerical solution is derived. Results are then obtained for typical CUSUM and EWMA charts, in particular, an evaluation of the skewness. Often, it is the density of the run length that is of interest; here its exact distribution is approximated by employing numerical quadrature to the integral that arises in the formulation. Finally, some general conclusions are drawn.

2. Equations for the Central Moments of the Run Length Distribution

Following Champ et al. [5] for a general one sided control chart, let M_k denote the k-th central moment of the run length distribution about the origin. Then for a head start value of u, in the following discussion various values of u are employed, although we are primarily interested in $u = a_4$.

$$M_1(u) = 1 + M_1(-a_0)F_x\left(\frac{-a_0 - a_1 u + a_3}{a_2}\right) + \frac{1}{a_2}\int_{-a_0}^{a_5} M_1(y) f_x\left(\frac{y - a_1 u + a_3}{a_2}\right) dy, \qquad (1)$$

where f_x is the density function of the quality measure x and cumulative density F_x.

$$M_2(u) = 1 + 2M_1(-a_0)F_x\left(\frac{-a_0 - a_1 u + a_3}{a_2}\right) + \frac{2}{a_2}\int_{-a_0}^{a_5} M_1(y) f_x\left(\frac{y - a_1 u + a_3}{a_2}\right) dy + M_2(-a_0)F_x\left(\frac{-a_0 - a_1 u + a_3}{a_2}\right)$$

$$+\frac{1}{a_2}\int_{-a_0}^{a_5} M_2(y)f_x\left(\frac{y-a_1u+a_3}{a_2}\right)dy, \tag{2}$$

$$M_3(u) = 1 + 3M_1(-a_0)F_x\left(\frac{-a_0-a_1u+a_3}{a_2}\right)$$

$$+\frac{3}{a_2}\int_{-a_0}^{a_5} M_1(y)f_x\left(\frac{y-a_1u+a_3}{a_2}\right)dy$$

$$+3M_2(-a_0)F_x\left(\frac{-a_0-a_1u+a_3}{a_2}\right)$$

$$+\frac{3}{a_2}\int_{-a_0}^{a_5} M_2(y)f_x\left(\frac{y-a_1u+a_3}{a_2}\right)dy$$

$$+M_3(-a_0)F_x\left(\frac{-a_0-a_1u+a_3}{a_2}\right)$$

$$+\frac{1}{a_2}\int_{-a_0}^{a_5} M_3(y)f_x\left(\frac{y-a_1u+a_3}{a_2}\right)dy, \tag{3}$$

$$M_4(u) = 1 + 4M_1(-a_0)F_x\left(\frac{-a_0-a_1u+a_3}{a_2}\right)$$

$$+\frac{4}{a_2}\int_{-a_0}^{a_5} M_1(y)f_x\left(\frac{y-a_1u+a_3}{a_2}\right)dy$$

$$+6M_2(-a_0)F_x\left(\frac{-a_0-a_1u+a_3}{a_2}\right)$$

$$+\frac{6}{a_2}\int_{-a_0}^{a_5} M_2(y)f_x\left(\frac{y-a_1u+a_3}{a_2}\right)dy$$

$$+4M_3(-a_0)F_x\left(\frac{-a_0-a_1u+a_3}{a_2}\right)$$

$$+ \frac{4}{a_2} \int_{-a_0}^{a_5} M_3(y) f_x\left(\frac{y - a_1 u + a_3}{a_2}\right) dy$$

$$+ M_4(-a_0) F_x\left(\frac{-a_0 - a_1 u + a_3}{a_2}\right)$$

$$+ \frac{1}{a_2} \int_{-a_0}^{a_5} M_4(y) f_x\left(\frac{y - a_1 u + a_3}{a_2}\right) dy. \tag{4}$$

3. Numerical Solution of the Integral Equations

The method of Gaussian quadrature is employed to approximate the integrals, see, for example, Luceño and Puig-Pey [14]. The standard quadrature problem is the evaluation of the integral of $f(t)$ in the interval $[-1, 1]$, employing an approximation of degree m. The best estimate of the integral is

$$\int_{-1}^{1} f(t) dt = \sum_{j=1}^{m} w_j f(t_j),$$

where t_i is a designated evaluation point and w_i is the weight of that point in the sum. If the number of points at which the function $f(t)$ is evaluated is m, the resulting value of the integral is of the same accuracy as a simple polynomial method (such as Simpson's rule) of about twice the degree ($2m$). Algorithm's to calculate the weights and abscissa are readily available (Press et al. [17]).

It is noted that the integrals in the Equations (1)-(4) all take the same range, which need not be $[-1, 1]$. Hence a change of variables must be adopted.

$$\int_{-a_0}^{a_5} f(y) dy = b \int_{-1}^{1} f(a + bt) dt = b \sum_{j=1}^{m} w_j f(a + bt_j) = b \sum_{j=1}^{m} w_j f(y_j),$$

where the linear transformation

$$y = a + bt \text{ with } a = \frac{a_5 - a_0}{2} \text{ and } b = \frac{a_5 + a_0}{2}$$

has been employed.

A series of equations will be developed, which can be solved for $M_1(y_j)$. If $-a_0$ is not a knot point $(-a_0 \notin \{y_1, \ldots, y_m\})$, then define $y_0 = -a_0$ and $w_0 = 0$. It is this slightly more complex case that is considered below.

The $m+1$ equations are derived from Equation (1) on setting $u = y_j : j = 0, \ldots, m$. This may be represented in matrix notation as $A\underline{M_1} = \underline{1}$,

where

$$A_{ij} = \delta_{ij} - \delta_{i0} F_x\left(\frac{-a_0 - a_1 a_4 + a_3}{a_2}\right) - \frac{b}{a_2} w_j f_x\left(\frac{y_j - a_1 y_i + a_3}{a_2}\right).$$

Then $\underline{M_1} = A^{-1}\underline{1}$.

Similarly, Equation (2) for $M_2(y_j)$. Here $A\underline{M_2} = \underline{b}$, where

$$b_i = 1 + 2M_1(-a_0) F_x\left(\frac{-a_0 - a_1 y_i + a_3}{a_2}\right) + \frac{2b}{a_2}\sum_{j=1}^{m} w_j M_1(y_j) f_x\left(\frac{y_j - a_1 y_i + a_3}{a_2}\right).$$

The values of $M_1(y_j)$ having previously been derived.

It is very convenient that the same matrix (A) is required to be inverted. Then $\underline{M_2} = A^{-1}\underline{b}$.

Similarly, Equation (3) for $M_3(y_j)$. Here $A\underline{M_3} = \underline{c}$, where

$$c_i = 1 + 3M_1(-a_0) F_x\left(\frac{-a_0 - a_1 y_i + a_3}{a_2}\right)$$

$$+ \frac{3b}{a_2}\sum_{j=1}^{m} w_j M_1(y_j) f_x\left(\frac{y_j - a_1 y_i + a_3}{a_2}\right)$$

$$+ 3M_2(-a_0)F_x\left(\frac{-a_0 - a_1 y_i + a_3}{a_2}\right)$$

$$+ \frac{3b}{a_2}\sum_{j=1}^{m} w_j M_2(y_j) f_x\left(\frac{y_j - a_1 y_i + a_3}{a_2}\right).$$

The values of $M_1(y_j)$ and $M_2(y_j)$ having previously been derived.

Again, the same matrix (A) is required to be inverted. Then $\underline{M_3} = A^{-1}\underline{c}$.

Similarly, Equation (4) for $M_4(y_j)$. Here $A\underline{M_4} = \underline{d}$, where

$$d_i = 1 + 4M_1(-a_0)F_x\left(\frac{-a_0 - a_1 y_i + a_3}{a_2}\right)$$

$$+ \frac{4b}{a_2}\sum_{j=1}^{m} w_j M_1(y_j) f_x\left(\frac{y_j - a_1 y_i + a_3}{a_2}\right)$$

$$+ 6M_2(-a_0)F_x\left(\frac{-a_0 - a_1 y_i + a_3}{a_2}\right)$$

$$+ \frac{6b}{a_2}\sum_{j=1}^{m} w_j M_2(y_j) f_x\left(\frac{y_j - a_1 y_i + a_3}{a_2}\right)$$

$$+ 4M_3(-a_0)F_x\left(\frac{-a_0 - a_1 y_i + a_3}{a_2}\right)$$

$$+ \frac{4b}{a_2}\sum_{j=1}^{m} w_j M_3(y_j) f_x\left(\frac{y_j - a_1 y_i + a_3}{a_2}\right).$$

The values of $M_1(y_j)$, $M_2(y_j)$, and $M_3(y_j)$ having previously been derived. Where A^{-1} is already available. Then $\underline{M_4} = A^{-1}\underline{d}$.

If desired, using the following equations, the moments may be interpolated for any intermediate value of x.

$$M_1(x) = 1 + M_1(-a_0)F_x\left(\frac{-a_0 - a_1x + a_3}{a_2}\right)$$

$$+ \frac{b}{a_2}\sum_{j=1}^{m} w_j M_1(y_j) f_x\left(\frac{y_j - a_1x + a_3}{a_2}\right),$$

$$M_2(x) = 1 + 2M_1(-a_0)F_x\left(\frac{-a_0 - a_1x + a_3}{a_2}\right)$$

$$+ \frac{2b}{a_2}\sum_{j=1}^{m} w_j M_1(y_j) f_x\left(\frac{y_j - a_1x + a_3}{a_2}\right)$$

$$+ M_2(-a_0)F_x\left(\frac{-a_0 - a_1x + a_3}{a_2}\right)$$

$$+ \frac{b}{a_2}\sum_{j=1}^{m} w_j M_2(y_j) f_x\left(\frac{y_j - a_1x + a_3}{a_2}\right),$$

$$M_3(x) = 1 + 3M_1(-a_0)F_x\left(\frac{-a_0 - a_1x + a_3}{a_2}\right)$$

$$+ \frac{3b}{a_2}\sum_{j=1}^{m} w_j M_1(y_j) f_x\left(\frac{y_j - a_1x + a_3}{a_2}\right)$$

$$+ 3M_2(-a_0)F_x\left(\frac{-a_0 - a_1x + a_3}{a_2}\right)$$

$$+ \frac{3b}{a_2}\sum_{j=1}^{m} w_j M_2(y_j) f_x\left(\frac{y_j - a_1x + a_3}{a_2}\right)$$

$$+ M_3(-a_0)F_x\left(\frac{-a_0 - a_1x + a_3}{a_2}\right)$$

$$+ \frac{b}{a_2}\sum_{j=1}^{m} w_j M_3(y_j) f_x\left(\frac{y_j - a_1x + a_3}{a_2}\right),$$

$$M_4(x) = 1 + 4M_1(-a_0)F_x\left(\frac{-a_0 - a_1x + a_3}{a_2}\right)$$

$$+ \frac{4b}{a_2}\sum_{j=1}^{m} w_j M_1(y_j) f_x\left(\frac{y_j - a_1x + a_3}{a_2}\right)$$

$$+ 6M_2(-a_0)F_x\left(\frac{-a_0 - a_1x + a_3}{a_2}\right)$$

$$+ \frac{6b}{a_2}\sum_{j=1}^{m} w_j M_2(y_j) f_x\left(\frac{y_j - a_1x + a_3}{a_2}\right)$$

$$+ 4M_3(-a_0)F_x\left(\frac{-a_0 - a_1x + a_3}{a_2}\right)$$

$$+ \frac{4b}{a_2}\sum_{j=1}^{m} w_j M_3(y_j) f_x\left(\frac{y_j - a_1x + a_3}{a_2}\right)$$

$$+ M_4(-a_0)F_x\left(\frac{-a_0 - a_1x + a_3}{a_2}\right)$$

$$+ \frac{b}{a_2}\sum_{j=1}^{m} w_j M_4(y_j) f_x\left(\frac{y_j - a_1x + a_3}{a_2}\right).$$

Finally, the standard deviation may also be obtained

$$Std(x) = \sqrt{M_2(x) - M_1(x)^2}.$$

The skewness is

$$Skew(x) = \frac{M_3(x) - 3M_2(x)M_1(x) + 2M_1^3(x)}{Std^3(x)},$$

with kurtosis

$$Kurt(x) = \frac{M_4(x) - 4M_3(x)M_1(x) + 6M_2(x)M_1^2(x) - 3M_1^4(x)}{Std^4(x)}.$$

The claim made by Woodall [20], concerning the right skewness of the run length distribution will be investigated for typical CUSUM and EWMA charts.

4. Results for Typical CUSUM Charts

For a CUSUM chart, the parameterisation is $a_0 = 0, a_1 = 1, a_2 = 1,$ $a_3 = k, a_4 = 0,$ and $a_5 = h$ defines a chart with reference value k and signal level h. To provide a thorough investigation, the following ranges for the variables were adopted, $2 \leq h \leq 5$ in steps of 0.5 and $0.2 \leq k \leq 1.2$ in steps of 0.2. These values match the range of the usual nomogram used to select chart parameters (Ewan and Kemp [8] and Kemp [12]). For the quadrature, calculations presented here 40-knot points were employed.

It should be noted that the matrix inversion step might lead to numerical problems, if the system of equations is effectively non-singular. In the calculations presented here, the determinant of the matrix tends to be quite small, as a rule of thumb a cut off at $5e^{-8}$ was employed. The determinant should be positive. The precise cut off will depend on the precision of the computer employed and the numerical algorithm adopted to invert the matrix.

For the range of parameters considered Figure 1 shows the standard deviation (denoted Std), which is plotted against the mean values (denoted ARL). As outlined in the introduction, normality is assumed for the underlying density function.

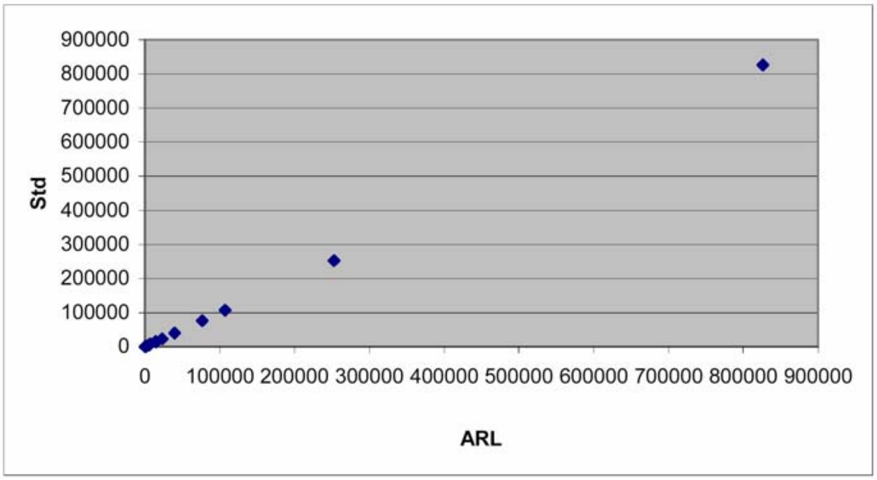

Figure 1. The average against the standard deviation for the run length for a CUSUM chart.

A high ARL is desired, when the process mean is at an acceptable target level, and a low ARL is desired, when the process mean moves off to a specified rejection level Δ. The acceptable ARL is usually fixed at several hundreds, while that for rejection is generally between 3 and 10. In the previous equations, the density functions may be replaced by

$$f_x(y) \to \phi(y - \Delta) \text{ and } F_x(y) \to \Phi(y - \Delta).$$

Support for the apparent linearity is demonstrated by calculating the correlation between the ARL and the Std, which is 1.000. While the result appears convincing, the wide range of observations may be misleading, a logarithmic (base 10) transformation is employed to address this point, the data is presented in Figure 2.

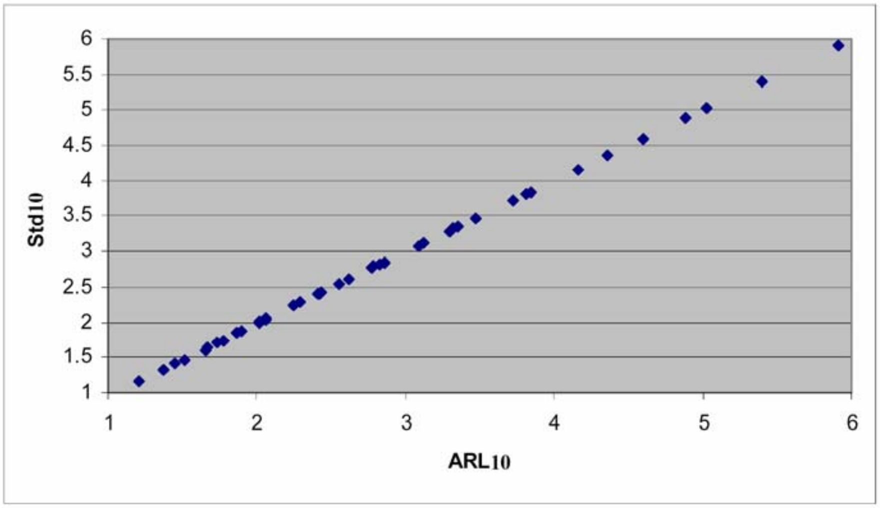

Figure 2. The logarithmic transformations of the average and the standard deviation for the run length for a CUSUM chart.

Support for the apparent linearity of the logarithmic variables is again demonstrated by calculating the correlation between the ARL_{10} and the Std_{10}, which is 1.000.

5. Evaluation of the Skewness for Typical CUSUM Charts

Having investigated the mean and standard deviation, the skewness is evaluated for the same range of parameters. The mean value of the skewness is 1.9980 with a standard deviation of 0.0040. The values are illustrated in a boxplot, Figure 3.

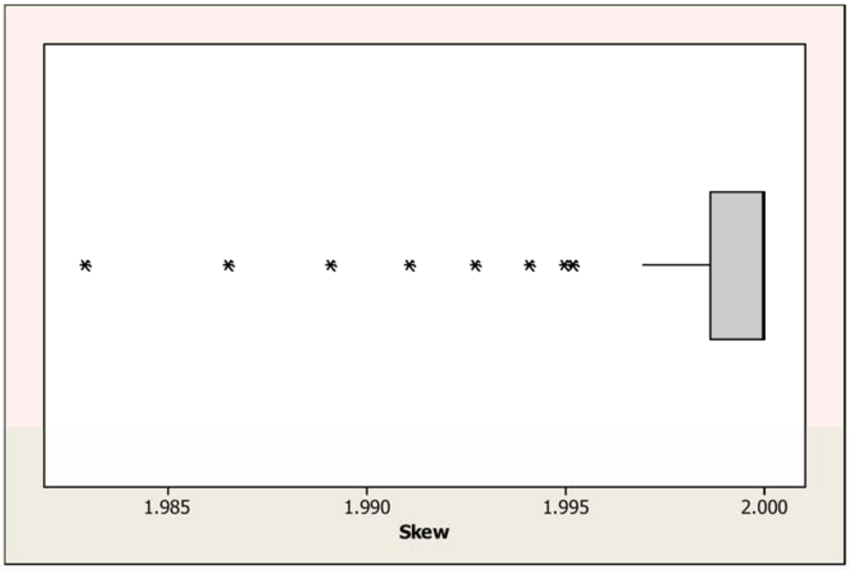

Figure 3. A boxplot of the skewness for CUSUM charts.

As predicted the data exhibits a strong right skew, the majority of values being 2. While it is tempting to conjecture that the skewness equals precisely two, any deviations being numerical errors, extensive calculations suggest that this is not quite the case. Extending the range of application to $h = 1.5, k = 0.2$ gives a skewness of 1.9784, while $h = 0.5, k = 0.2$ gives a skewness of 2.0026. A very tight confidence interval could be constructed, however, the numerical accuracy employed appears to discount an exact value of two.

For completeness the kurtosis is also examined, its mean value is 12.321 with a standard deviation of 0.466, a boxplot of the data is presented in Figure 4.

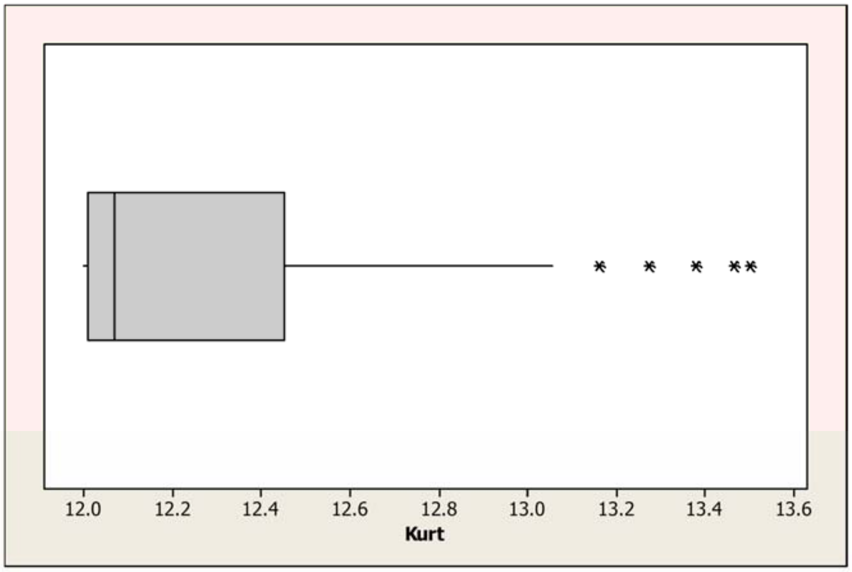

Figure 4. A boxplot of the kurtosis for CUSUM charts.

Similar calculations are now performed for typical EWMA charts.

6. Results for Typical EWMA Charts

For an EWMA chart, the parameterisation is $a_0 = 0$, $a_1 = 1 - \lambda$, $a_2 = \lambda$, $a_3 = 0$, $a_4 = 0$, and $a_5 = \Phi^{-1}(0.999)\sqrt{\frac{\lambda}{2 - \lambda}}$. For the special case $\lambda = 1$, a Shewhart chart is obtained, in this instance, exact results may be derived, these are presented in the Appendix. To examine a broad variety of charts, the following ranges of parameters are adopted, $0.01 \leq \lambda \leq 1$ in steps of 0.01. The results are broadly similar to those obtained for the CUSUM chart, and are summarised in Figures (5)-(8).

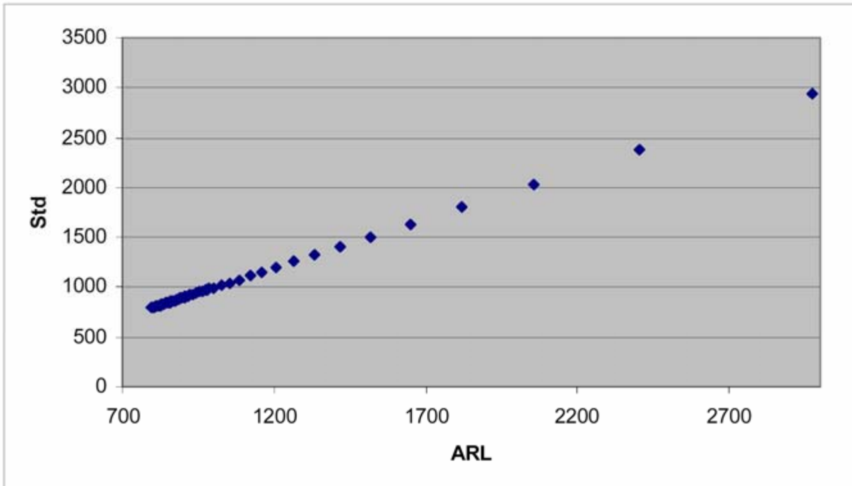

Figure 5. The average against the standard deviation for the run length for an EWMA chart.

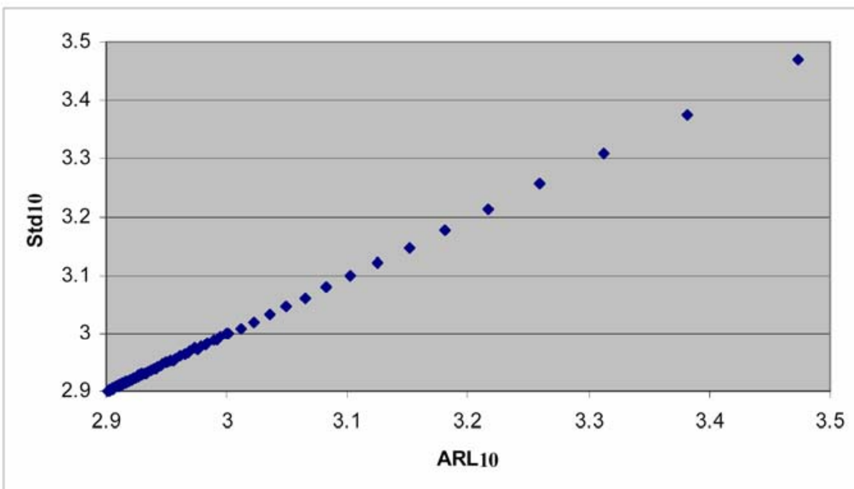

Figure 6. The logarithmic transformations of the average and the standard deviation for the run length for an EWMA chart.

Having evaluated the mean and standard deviation, the skewness is evaluated for the same range of parameters. The mean value of the skewness is 2.0000 with a standard deviation of 0.0000187 the data is illustrated in a boxplot, Figure 7.

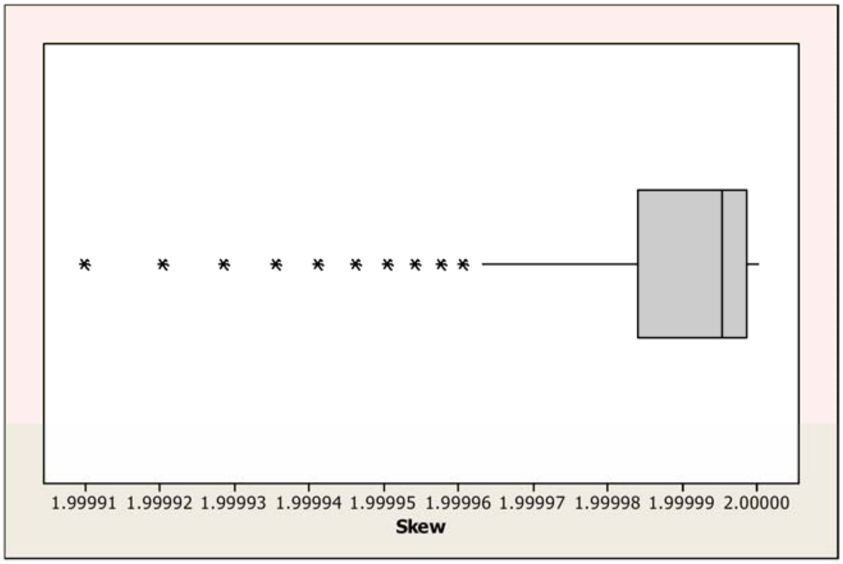

Figure 7. A boxplot of the skewness for EWMA charts.

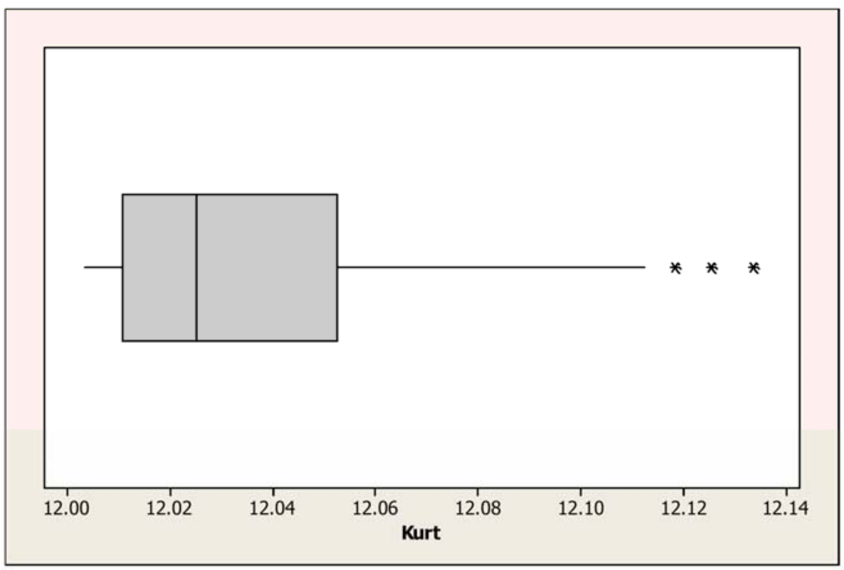

Figure 8. A boxplot of the kurtosis for EWMA charts.

It is important to now investigate the precise distribution of the run length in detail.

7. Exact Distribution of the Run Length

For the exact distribution, Equation (6) is taken from Champ et al. [5] (with a minor correction); Equation (5) is required to initialise the scheme.

$$\text{Prob}(1;\, u) = 1 - F_x\left(\frac{a_5 - a_1 u + a_3}{a_2}\right), \tag{5}$$

$$\text{Prob}(i;\, u) = \text{Prob}(i-1;\, -a_0)F_x\left(\frac{-a_0 - a_1 u + a_3}{a_2}\right)$$

$$+ \frac{1}{a_2}\int_{-a_0}^{a_5} \text{Prob}(i-1;\, y) f_x\left(\frac{y - a_1 u + a_3}{a_2}\right)dy. \tag{6}$$

Using the method of Gaussian quadrature (Luceño and Puig-Pey [14]) and assuming a normal distribution results in the following equations:

$$\text{Prob}(1;\, u) = 1 - \Phi\left(\frac{a_5 - a_1 u + a_3}{a_2} - \Delta\right),$$

$$\text{Prob}(i;\, u) = \text{Prob}(i-1;\, -a_0)\Phi\left(\frac{-a_0 - a_1 u + a_3}{a_2} - \Delta\right)$$

$$+ \frac{b}{a_2}\sum_{j=1}^{m} w_j \text{Prob}(i-1;\, y_j)\phi\left(\frac{y_j - a_1 u + a_3}{a_2} - \Delta\right).$$

These may be evaluated for $u = \{-a_0,\, y_1,\, \ldots,\, y_m\}$.

Note that it is not necessary to store all the probabilities, since $\text{Prob}(i;\, u)$ only depends on $\text{Prob}(i-1;\, u)$, however, it is simpler to store all the values. On a small computer, an array of 100,000 by 51 $(1 \le i \le 100{,}000$ and $0 \le m \le 50)$ caused no problems. Also efficiencies may be made on noting that the normal probabilities are independent of i. Hence, they may be evaluated initially and stored. Terms are generated until

$$\sum_{i=1}^{N} \text{Prob}(i; -a_0) \geq 1 - \varepsilon,$$

for some selected ε. For example, a CUSUM chart with $h = 4$ and $k = 0.2$ with 100 knot points and $\varepsilon = 1e^{-8}$ resulted in $N = 791$. So, the density may be readily generated and output. If required percentiles and moments may also be evaluated.

The 100α percentile is found from the previously evaluated probabilities and denoted by $N(\alpha, a_0)$. These values can be quite useful (Barnard [1] and Bissell [2]).

$$\sum_{i=1}^{N(\alpha, a_0)-1} \text{Prob}(i; -a_0) \leq \alpha \text{ and } \sum_{i=1}^{N(\alpha, a_0)} \text{Prob}(i; -a_0) > \alpha.$$

The moments generated can be evaluated from the density and checked against those previously calculated. The distribution is shown in Figure 9, in this case, the maximum run length observed was 1021.

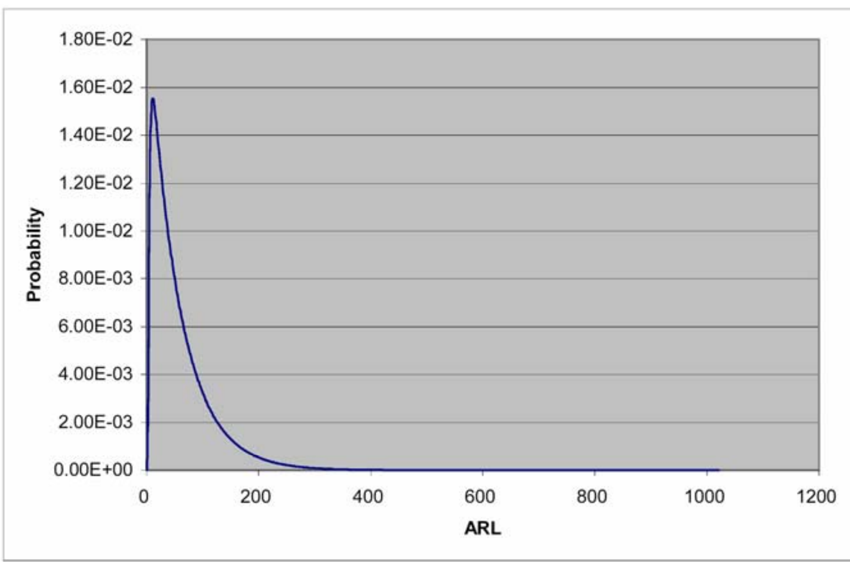

Figure 9. The density function for a CUSUM chart with $h = 4$ and $k = 0.2$.

It is interesting to examine the section of the graph for small values of the run length, see Figure 10.

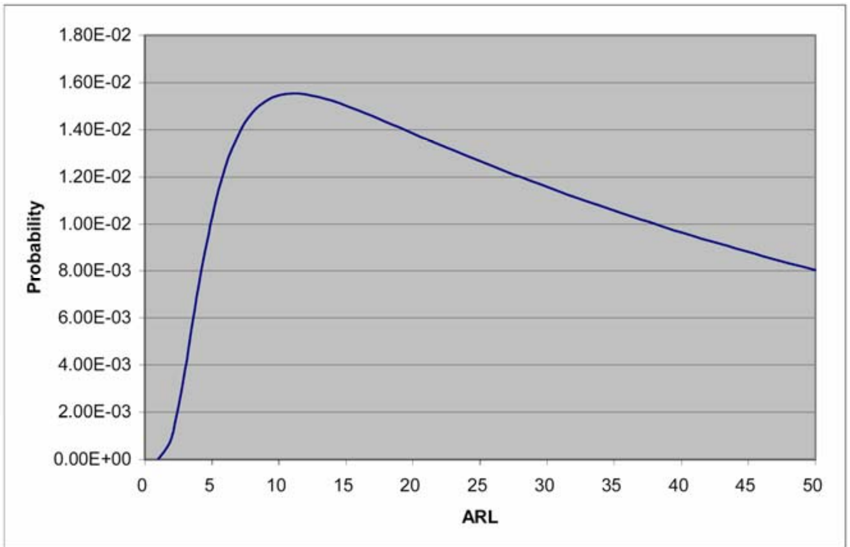

Figure 10. The lower region of the density function for a CUSUM chart with $h = 4$ and $k = 0.2$.

Some percentiles are displayed in Table 1.

Table 1. Percentiles for a CUSUM chart with $h = 4$ and $k = 0.2$

α	**.001**	**.01**	**.05**	**.1**	**.2**	**.3**	**.4**	**.5**	**.6**	**.8**
$N(\alpha, 0)$	3	4	8	11	18	25	33	43	56	94

This approach has the advantage of generating additional material (the actual density and, for instance, the quartiles), however, it takes longer to compute than direct evaluation of the moments and does require extensive computer storage. It is necessarily an approximation since the procedure is terminated, when the cumulative probabilities approximate unity. Also, there is the limitation imposed by employing quadrature with a fixed number of knot points. Figures 11 and 12 demonstrate how the key measures (quartiles, moments and central moments) vary with the head start values for the traditional charts considered here. These results being obtained employing an approximation to the integral by using 25 knot points.

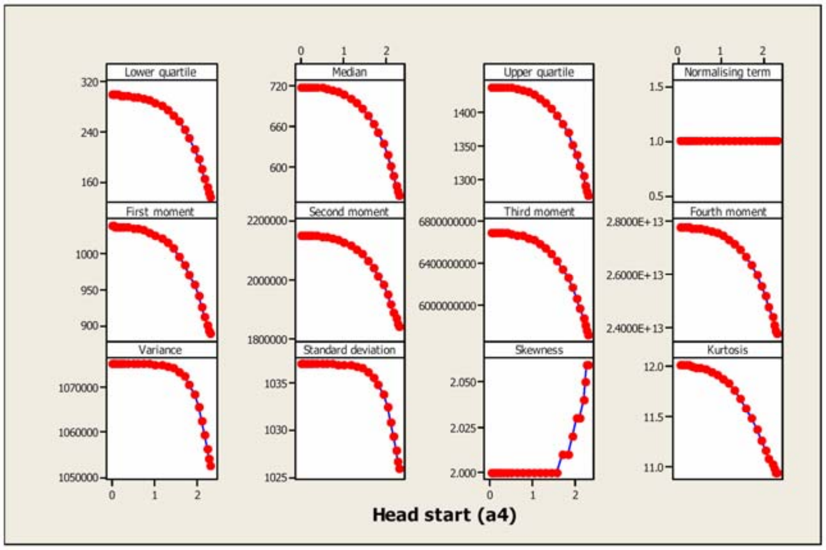

Figure 11. The descriptive moments for a CUSUM chart with $h = 4$ and $k = 0.2$.

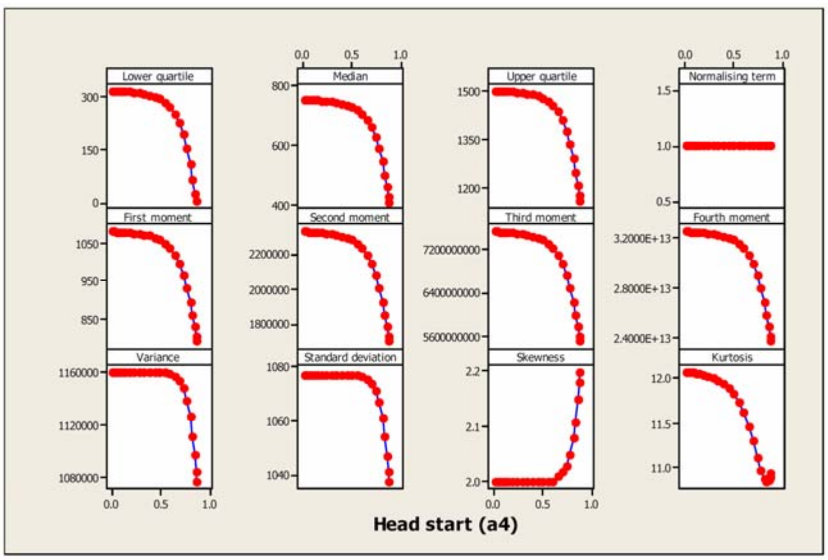

Figure 12. The descriptive moments for an EWMA chart with $\lambda = 0.15$.

There is a slight aberration in the kurtosis in Figure 12. This behaviour was still present when the number of knot points for the numerical integration was increased to 50. While the effect may be genuine, it is relatively small and hence of little practical interest. In both cases, there is an effectively perfect correlation between the median, quartiles and the moments as the head start value is varied. Indicating that the distribution profile is invariant under this change.

8. Example

As a typical example, consider a hybrid chart (Champ et al. [5], Table 1) with parameters $a_0 = 0, a_1 = 0.85, a_2 = 0.15, a_3 = -0.08, a_4 = 0,$ and $a_5 = 1.2867$ with in control ARL = 500. The effect of increasing the rejection level Δ on the moments is displayed in Table 2.

Table 2. ARL and measures for a hybrid chart against Δ

Δ	ARL	Std	Kurt	Skew
0.0	500.43	487.80	2.00	12.31
0.1	224.74	212.34	2.00	12.74
0.2	115.35	103.46	1.99	13.56
0.3	67.04	55.84	1.98	15.04
0.4	43.37	32.95	1.95	17.57
0.5	30.60	20.98	1.90	21.70
0.6	23.10	14.24	1.83	28.12
0.7	18.35	10.20	1.75	37.62
0.8	15.15	7.64	1.65	50.93
0.9	12.89	5.93	1.56	68.66
1.0	11.21	4.75	1.46	91.25
2.0	5.01	1.28	0.84	574.43
3.0	3.36	0.67	0.62	1330.37
4.0	2.58	0.53	0.18	1068.53
5.0	2.10	0.30	2.70	3738.20

These values would appear to be satisfactory, exhibiting a large ARL when the process is in control and rapidly detecting significant deviations.

9. Conclusions

Moments of the run length distribution are often used to design and study the performance of quality control charts (Champ et al. [6]). Early work for a CUSUM chart (Brook and Evans [3]) employed a Markov chain. The transition probability matrix was used to determine not only the average run length, but also moments, percentage points of the run length distribution, and exact probabilities of the run length. The run length distribution of the EWMA chart with estimated parameters was derived by Jones et al. [11]. The effect of estimation on the performance of the chart was discussed in a variety of practical scenarios. Control charts are extensively used in many real world applications. Since process parameters are rarely known, common practice is to estimate them. Then, the control limits are modified and become actually random variables. Maravelakis et al. [15] dealt with univariate control charts in this case. They studied the effect of estimating the process parameters of these charts based on the first two moments of the run length distribution. The work presented here generalises these previous results.

General procedures have been described, which will generate the moments of a one-sided universal control chart. A similar procedure has been employed to produce the associated probability density function. These methods were used to investigate a variety of CUSUM, EWMA, and Shewhart charts under assumptions of normality. In all these cases, the distributions are right skew, the majority of examples having a skewness of 2. For no head start, the minimum skewness for conventional charts is 1.9829. In general, a skewness of 2 may be expected, this is particularly true for an EWMA chart. In extreme cases, a deviation from this value of at most $\pm 1\%$ may be expected. When a head start value was introduced into some specimen charts, a skewness as large as 2.2 was observed. Thus, a rule of thumb for assessing the skewness of acceptable run length distributions is available. An introduction of an extreme head start value can increase the skewness. The kurtosis does not exhibit similar consistency properties.

Following Lucas and Saccucci [13], the following scheme is suggested when designing a control chart. Firstly, the parameters are selected for a desired in-control ARL and a shift in the process mean that is to be detected quickly. The parameters are selected that will result in a minimum ARL for the specified shift in the process mean. Finally, the entire ARL(Δ) profile should be considered to ensure the scheme gives sufficient protection against other shifts. These values may be readily derived employing the methods described in this paper.

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Appendix

Exact results for a Shewhart chart

For a Shewhart chart, the selected parameters are $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, $a_3 = 0$, $a_4 = 0$, and $a_5 = \Phi^{-1}(0.999)\frac{\sigma}{\sqrt{n}}$, giving a conventional chart. A first principles approach will be used to derive the first two moments (the mean and the variance). This cumbersome approach could, in principle, be used to derive higher moments, however, the approach adopted here is more straightforward, although less transparent.

Let P_α denote the probability of failing to identify a significant change in the process level with a single sample, after a shift has occurred. That is $P_\alpha = F_x(a_5)$. In the case of a shift in the process average, the probability of detecting the shift with only one sample is

$1 - P_\alpha$. If the shift is not detected with the first subgroup, it may be detected with the second subgroup and so on. Thus, we may have a run of points falling within the control limits and then a shift may be detected, with a point falling outside the limits. These probabilities are summarised in Table 3.

Table 3. Run length distribution for a Shewhart chart

Run length	Probability
1	$(1 - P_\alpha)$
2	$P_\alpha(1 - P_\alpha)$
3	$P_\alpha^2(1 - P_\alpha)$
4	$P_\alpha^3(1 - P_\alpha)$
...	...
n	$P_\alpha^{n-1}(1 - P_\alpha)$
...	...

As a first step, it is wise to ensure the distribution is correctly normalised,

$$\sum_{n=1}^{\infty} P_\alpha^{n-1}(1 - P_\alpha) = (1 - P_\alpha)\sum_{n=1}^{\infty} P_\alpha^{n-1} = 1,$$

so it is safe to proceed. For the first moment, we require

$$M_1 = \sum_{n=1}^{\infty} nP_\alpha^{n-1}(1 - P_\alpha),$$

for simplicity consider

$$\frac{\partial}{\partial P_\alpha}\left(P_\alpha^n(1 - P_\alpha)\right) = nP_\alpha^{n-1}(1 - P_\alpha) - P_\alpha^n. \tag{7}$$

Now, summing over n requires the following results:

$$\sum_{n=1}^{\infty} P_\alpha^n(1 - P_\alpha) = P_\alpha \text{ and } \sum_{n=1}^{\infty} P_\alpha^n = \frac{P_\alpha}{(1 - P_\alpha)},$$

substituting into Equation (7) gives the required result

$$1 = M_1 - \frac{P_\alpha}{(1 - P_\alpha)}, \text{ that is, } M_1 = \frac{1}{(1 - P_\alpha)}.$$

A similar approach may be adopted for the second moment

$$M_2 = \sum_{n=1}^{\infty} n^2 P_\alpha^{n-1}(1 - P_\alpha),$$

for simplicity consider

$$\frac{\partial^2}{\partial P_\alpha^2}\left(P_\alpha^{n+1}(1 - P_\alpha)\right) = n^2 P_\alpha^{n-1}(1 - P_\alpha) + nP_\alpha^{n-1}(1 - P_\alpha) - 2(n+1)P_\alpha^n. \quad (8)$$

Now, summing over n requires the following additional results

$$\sum_{n=1}^{\infty} P_\alpha^{n+1}(1 - P_\alpha) = P_\alpha^2 \text{ and } \sum_{n=1}^{\infty}(n+1)P_\alpha^n = \frac{P_\alpha(M_1 + 1)}{(1 - P_\alpha)},$$

substituting into Equation (8) gives the desired result.

$$2 = M_2 + M_1 - 2\frac{P_\alpha(M_1 + 1)}{(1 - P_\alpha)}, \text{ that is, } M_2 = \frac{(1 + P_\alpha)}{(1 - P_\alpha)^2}.$$

If the variance is desired, then

$$Variance = M_2 - M_1^2 = \frac{P_\alpha}{(1 - P_\alpha)^2}.$$

A similar approach may be adopted for the higher moments, however, evaluating the sums becomes convoluted.

Returning to the approach adopted here. On comparing successive rows of A, used in deriving the solutions to Equations (1), (2), and (3), are identical. Then the moments are constants in this case, being independent of u. So that Equation (1) reduces to

$$M_1 = 1 + M_1 F_x(a_5) \text{ as expected } M_1 = \frac{1}{(1 - P_\alpha)}.$$

While from Equation (2), we find

$$M_2 = 1 + 2M_1F_x(a_5) + M_2F_x(a_5) \text{ as expected } M_2 = \frac{(1 + P_\alpha)}{(1 - P_\alpha)^2}.$$

Equation (3) gives

$$M_3 = 1 + 3M_1F_x(a_5) + 3M_2F_x(a_5) + M_3F_x(a_5) \text{ giving } M_3 = \frac{1 + 4P_\alpha + P_\alpha^2}{(1 - P_\alpha)^3}.$$

From Equation (4), we find

$$M_4 = 1 + 4M_1F_x(a_5) + 6M_2F_x(a_5) + 4M_3F_x(a_5) + M_4F_x(a_5)$$

giving

$$M_4 = \frac{1 + 11P_\alpha + 11P_\alpha^2 + P_\alpha^3}{(1 - P_\alpha)^4}.$$

Thus, the first four moments have been derived directly. As a simple check direct evaluation of the moments can be compared to the functions, for example, if $P_\alpha = 0.2$, then $M_1 = 1.250$, $M_2 = 1.875$, $M_3 = 3.594$, and $M_4 = 8.906$.

■