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ON UNIQUENESS OF ENTIRE FUNCTIONS WITH DERIVATIVES SHARING ONE VALUE

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Abstract

In this paper, we study the uniqueness problem of derivatives of entire functions that share one value and improve the results of C. C. Yang and H. X. Yi.

1. Introduction

Let f be a nonconstant meromorphic function defined in the whole complex plane. It is assumed that the reader is familiar with the notations of the Nevanlinna theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $S(r, f)$, and so on, that can be found, for instance, in [1], [2], [7].

Let f and g be two nonconstant meromorphic functions. Let a be a finite complex number. We say that f, g share the value a CM (counting multiplicities), if f, g have the same a-points with the same multiplicities, and we say that f, g share the value a IM (ignoring multiplicities), if we

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do not consider the multiplicities. We denote by $N_{11}(r, \frac{1}{f-1})$, the counting function for common simple 1-points of f and g, where multiplicity is not counted. $\overline{N}_L(r, \frac{1}{f^{(k)}-1})$ is the counting function for 1-points of both $f^{(k)}$ and $g^{(k)}$ about which $f^{(k)}$ has larger multiplicity than $g^{(k)}$, with multiplicity being not counted. For any constant a, we define

$$\Theta(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r, f)}.$$

Let f be a nonconstant meromorphic function. Let a be a finite complex number, and k be a positive integer, we denote by $N_{k)}(r, \frac{1}{f-a})$ (or $\overline{N}_{k)}(r, \frac{1}{f-a})$, the counting function for zeros of $f-a$ with multiplicity $\leq k$ (ignoring multiplicities), and by $N_{(k}(r, \frac{1}{f-a})$ (or $\overline{N}_{(k}(r, \frac{1}{f-a})$), the counting function for zeros of $f-a$ with multiplicity at least k (ignoring multiplicities). Set

$$N_k(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_{(2}(r, \frac{1}{f-a}) + \ldots + \overline{N}_{(k}(r, \frac{1}{f-a}).$$

We further define

$$\delta_k(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_k(r, \frac{1}{f-a})}{T(r, f)}.$$

In 1976, Yang [3] posed the following question: What can be said about two nonconstant entire functions (denoted by f, g) and their first derivatives, if f and g share the value 0 CM and f' and g' share the value 1 CM. Yi answer the question in [4], [5], [6], proved the following result.

Theorem A. *Let* $f(z)$ *and* $g(z)$ *be two nonconstant entire functions,* $k \geq 1$ *be a positive integer. If f and g share the value* 0 *CM,* $f^{(k)}$ *and* $g^{(k)}$ *share the value* 1 *CM, and* $\delta(0, f) > \frac{1}{2}$, *then either* $f^{(k)} = g^{(k)}$ *or* $f = g$.

Naturally, what can we say if we only considered the uniqueness problem of derivatives of entire functions that share one value? In this paper, we answer the above question and prove the following theorem.

Theorem 1.1. *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions, k be a positive integer. If* $f^{(k)}$ *and* $g^{(k)}$ *share the value* 1 *CM, and*

$$\Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 3, \tag{1}$$

then $f^{(k)}g^{(k)} \equiv 1$ *or* $f \equiv g$.

Theorem 1.2. *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions, k be a positive integer. If* $f^{(k)}$ *and* $g^{(k)}$ *share the value* 1 *IM, and*

$$2\Theta(0, f) + 2\Theta(0, g) + 5\delta_{k+1}(0, f) + 5\delta_{k+1}(0, g) > 12, \tag{2}$$

then $f^{(k)}g^{(k)} \equiv 1$ *or* $f \equiv g$.

2. Some Lemmas

Lemma 2.1(see [1, 2])**.** *Let* $f(z)$ *be a transcendental entire function, k be a positive integer, and c be a nonzero finite complex number. Then*

$$\begin{aligned} T(r, f) &\leq N(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)} - c}) - N(r, \frac{1}{f^{(k+1)}}) + S(r, f) \\ &\leq N_{k+1}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f^{(k)} - c}) - N_0(r, \frac{1}{f^{(k+1)}}) + S(r, f). \end{aligned}$$

Here, $N_0(r, \frac{1}{f^{(k+1)}})$ *is the counting function which only counts those points such that* $f^{(k+1)} = 0$, *but* $f(f^{(k)} - c) \neq 0$.

Lemma 2.2 (see [1, 2]). *Let $f(z)$ be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r, a_i) = S(r, f)$, $i = 1, 2$. Then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f - a_1}) + \overline{N}(r, \frac{1}{f - a_2}) + S(r, f).$$

3. Proof of Theorems

We prove Theorem 1.2 only because Theorem 1.1 can be proved using the same method as Theorem 1.2.

Proof of Theorem 1.2

$$h(z) = \frac{f^{(k+2)}(z)}{f^{(k+1)}(z)} - 2\frac{f^{(k+1)}(z)}{f^{(k)}(z) - 1} - \frac{g^{(k+2)}(z)}{g^{(k+1)}(z)} + 2\frac{g^{(k+1)}(z)}{g^{(k)}(z) - 1}. \tag{3}$$

If z_0 is a common simple 1-point of $f^{(k)}$ and $g^{(k)}$, substituting their Taylor series at z_0 into (3), we see that z_0 is a zero of $h(z)$. Thus, we have

$$\begin{aligned} N_{11}(r, \frac{1}{f^{(k)} - 1}) = N_{11}(r, \frac{1}{g^{(k)} - 1}) &\leq \overline{N}(r, \frac{1}{h}) \\ &\leq T(r, h) + O(1) \\ &\leq N(r, h) + S(r, f) + S(r, g). \end{aligned} \tag{4}$$

By our assumptions, $h(z)$ have poles only at zeros of $f^{(k+1)}$ and $g^{(k+1)}$. Thus, we deduce from (3) that

$$\begin{aligned} N(r, h) \leq\ & \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + N_0(r, \frac{1}{f^{(k+1)}}) \\ & + N_0(r, \frac{1}{g^{(k+1)}}) + \overline{N}_L(r, \frac{1}{f^{(k)} - 1}) + \overline{N}_L(r, \frac{1}{g^{(k)} - 1}), \end{aligned} \tag{5}$$

Here, $N_0(r, \frac{1}{f^{(k+1)}})$ has the same meaning as in Lemma 2.1.

By Lemma 2.1, we have

$$T(r, f) \leq N_{k+1}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f^{(k)} - 1}) - N_0(r, \frac{1}{f^{(k+1)}}) + S(r, f). \quad (6)$$

$$T(r, g) \leq N_{k+1}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{g^{(k)} - 1}) - N_0(r, \frac{1}{g^{(k+1)}}) + S(r, g). \quad (7)$$

Thus, we deduce from (4)-(7) that

$$\begin{aligned} & N_{11}(r, \frac{1}{f^{(k)} - 1}) + T(r, f) + T(r, g) \\ & \leq \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + \overline{N}_L(r, \frac{1}{f^{(k)} - 1}) \\ & + \overline{N}_L(r, \frac{1}{g^{(k)} - 1}) + N_{k+1}(r, \frac{1}{f}) + N_{k+1}(r, \frac{1}{g}) \\ & + \overline{N}(r, \frac{1}{f^{(k)} - 1}) + \overline{N}(r, \frac{1}{g^{(k)} - 1}) + S(r, f) + S(r, g). \quad (8) \end{aligned}$$

Since $f^{(k)}$ and $g^{(k)}$ share one value 1 IM, we have

$$\begin{aligned} & \overline{N}(r, \frac{1}{f^{(k)} - 1}) + \overline{N}(r, \frac{1}{g^{(k)} - 1}) \\ & \leq N_{11}(r, \frac{1}{f^{(k)} - 1}) + \overline{N}_L(r, \frac{1}{f^{(k)} - 1}) + N(r, \frac{1}{g^{(k)} - 1}) \\ & \leq N_{11}(r, \frac{1}{f^{(k)} - 1}) + \overline{N}_L(r, \frac{1}{f^{(k)} - 1}) + T(r, g^{(k)}) + O(1) \\ & \leq N_{11}(r, \frac{1}{f^{(k)} - 1}) + \overline{N}_L(r, \frac{1}{f^{(k)} - 1}) + T(r, g) + S(r, g). \quad (9) \end{aligned}$$

Noting that

$$\begin{aligned} & \overline{N}(r, \frac{1}{f^{(k)}}) \leq \overline{N}(r, \frac{f}{f^{(k)}}) + \overline{N}(r, \frac{1}{f}) \\ & \leq T(r, \frac{f^{(k)}}{f}) + \overline{N}(r, \frac{1}{f}) \leq N_{k+1}(r, \frac{1}{f}) + S(r, f). \end{aligned}$$

$$\overline{N}_L(r, \frac{1}{f^{(k)}-1}) \le N(r, \frac{1}{f^{(k)}-1}) - \overline{N}(r, \frac{1}{f^{(k)}-1}) \le N(r, \frac{f^{(k)}}{f^{(k+1)}})$$

$$\le N(r, \frac{f^{(k+1)}}{f^{(k)}}) + S(r, f) = \overline{N}(r, \frac{1}{f^{(k)}}) + S(r, f).$$

So, we have

$$\overline{N}_L(r, \frac{1}{f^{(k)}-1}) \le \overline{N}(r, \frac{1}{f^{(k)}}) + S(r, f) \le N_{k+1}(r, \frac{1}{f}) + S(r, f). \tag{10}$$

Similarly,

$$\overline{N}_L(r, \frac{1}{g^{(k)}-1}) \le \overline{N}(r, \frac{1}{g^{(k)}}) + S(r, g) \le N_{k+1}(r, \frac{1}{g}) + S(r, g). \tag{11}$$

Thus, we deduce from (8)-(11) that

$$T(r, f) \le 3N_{k+1}(r, \frac{1}{f}) + 2N_{k+1}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{f})$$

$$+ \overline{N}(r, \frac{1}{g}) + S(r, f) + S(r, g). \tag{12}$$

Similarly,

$$T(r, g) \le 3N_{k+1}(r, \frac{1}{g}) + 2N_{k+1}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f})$$

$$+ \overline{N}(r, \frac{1}{g}) + S(r, f) + S(r, g). \tag{13}$$

We obtain from (12)-(13) that

$$T(r, f) + T(r, g) \le 5N_{k+1}(r, \frac{1}{f}) + 5N_{k+1}(r, \frac{1}{g}) + 2\overline{N}(r, \frac{1}{f})$$

$$+ 2\overline{N}(r, \frac{1}{g}) + S(r, f) + S(r, g).$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r, f) \le T(r, g)$, for $r \in I$. Then, we have

$$\{[2\Theta(0, f) + 2\Theta(0, g) + 5\delta_{k+1}(0, f) + 5\delta_{k+1}(0, g) - 12] - \varepsilon\}$$

$$\times T(r, f) \leq S(r, f), \tag{14}$$

for $r \in I$, and $0 < \varepsilon < 2\Theta(0, f) + 2\Theta(0, g) + 5\delta_{k+1}(0, f) + 5\delta_{k+1}(0, g) - 12$. Thus, we obtain from (2) and (14) that $T(r, f) \leq S(r, f)$, for $r \in I$, a contradiction.

Hence, we get $h(z) \equiv 0$; that is,

$$\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)} - 2\frac{f^{(k+1)}(z)}{f^{(k)}(z) - 1} = \frac{g^{(k+2)}(z)}{g^{(k+1)}(z)} - 2\frac{g^{(k+1)}(z)}{g^{(k)}(z) - 1}. \tag{15}$$

By solving this equation, we obtain

$$\frac{1}{f^{(k)} - 1} = \frac{bg^{(k)} + a - b}{g^{(k)} - 1}, \tag{16}$$

where a, b are two constants. Next, we consider three cases.

Case 1. $b \neq 0$ and $a = b$. From (16), we obtain that $g^{(k)} \neq 0$. Thus, there exists an entire function $\phi(z)$ such that $g^{(k)}(z) = e^{\phi(z)}$, and $f^{(k)} = 1 + \frac{1}{b} - \frac{1}{b}e^{-\phi(z)}$.

If $b = -1$, then $f^{(k)}(z)g^{(k)}(z) \equiv 1$. If $b \neq -1$, then $f^{(k)} - (1 + \frac{1}{b}) = -\frac{1}{b}e^{-\phi(z)} \neq 0$, and thus , we deduce from Lemma 2.1 that

$$T(r, f) \leq N_{k+1}(r, \frac{1}{f}) + S(r, f) \leq [1 - \delta_{k+1}(0, f)]T(r, f) + S(r, f),$$

that is,

$$\delta_{k+1}(0, f)T(r, f) \leq S(r, f).$$

Hence, by (2), we deduce that $T(r, f) \leq S(r, f)$, a contradiction.

Case 2. $b \neq 0$ and $a \neq b$. Then from (16), we have $g^{(k)} + \frac{a-b}{b} \neq 0$. From Lemma 2.1, we deduce

$$T(r, g) \leq N_{k+1}(r, \frac{1}{g}) + S(r, g).$$

Next, by using the argument as in Case 1, we get a contradiction.

Case 3. $b = 0$ and $a \neq 0$. From (16), we obtain

$$f = \frac{1}{a} g + p(z), \tag{17}$$

where $p(z)$ is a polynomial. If $p(z) \neq 0$, then by Lemma 2.2, we have

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f-p}) + S(r, f) \\ &\leq \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + S(r, f) \\ &\leq [1 - \Theta(0, f)] T(r, f) + [1 - \Theta(0, g)] T(r, g) + S(r, f). \end{aligned} \tag{18}$$

Obviously, by (17), we have $T(r, f) = T(r, g) + S(r, f)$. Hence, substituting this into (18), we get

$$[\Theta(0, f) + \Theta(0, g) - 1] T(r, f) \leq S(r, f). \tag{19}$$

Thus, by (2) and (19), we deduce that $T(r, f) \leq S(r, f)$, a contradiction. Therefore, we deduce that $p(z) \equiv 0$, that is,

$$f = \frac{1}{a} g. \tag{20}$$

If $a \neq 1$, then by $f^{(k)}$ and $g^{(k)}$ sharing the value 1 IM, we deduce from (20) that $g^{(k)} \neq 1$. Next, we can deduce a contradiction as in Case 2. Thus, we get $a = 1$, that is, $f \equiv g$. We complete the proof of the Theorem 1.2.

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