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ON THE PICARD BOUNDARY VALUE PROBLEM FOR $[h(t, x(t), x'(t))]' = f(t, x(t), x'(t))$

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Abstract

By making use of monotone operator methods, we discuss the solvability of problem $[h(t, x(t), x'(t))]' = f(t, x(t), x'(t)), x(0) = 0 = x(1)$ under some coercive conditions. The results are much more general than the previous ones.

1. Introduction

Consider the boundary value problem

$$[h(t, x(t), x'(t))]' = f(t, x(t), x'(t)), \tag{1.1}$$

$$x(0) = 0 = x(1), \tag{1.2}$$

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where $h, f : [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$. A function $x : [0, 1] \to \mathbf{R}$ is said to be a solution of (1.1)-(1.2), if and only if $x \in H_0^1(0, 1) := \{x \in H^1(0, 1) : x$ satisfies $(1.2)\}$ with the inner product

$$(u, v) = \int_0^1 u'(t)v'(t)dt,$$

such that

$$(Tx, v) = 0, \text{ for every } v \in H_0^1(0, 1), \tag{1.3}$$

where $T : H_0^1(0, 1) \to H_0^1(0, 1)$, $x \to Tx$ is defined by

$$(Tx, v) = \int_0^1 h(t, x(t), x'(t))v'(t)dt + \int_0^1 f(t, x(t), x'(t))v(t)dt, \tag{1.4}$$

for every $v \in H_0^1(0, 1)$. Problem (1.1)-(1.2) was first discussed by Ford [4] in 1972, and under the conditions: T is hemicontinuous, and there exist constants $\lambda > 0$ and γ with $\lambda\pi^2 - \gamma > 0$ such that

$$(V - v)[h(t, U, V) - h(t, u, v)] + (U - u)[f(t, U, V) - f(t, u, v)]$$

$$\geq \lambda(V - v)^2 - \gamma(U - u)^2, \tag{1.5}$$

for every $V, U, u, v \in \mathbf{R}$, and a.e. $t \in (0, 1)$, the solvability of (1.1)-(1.2) was shown. As the author pointed out, condition (1.5) contains the Lee's solvable condition [6]: $f : [0, 1] \times \mathbf{R} \to \mathbf{R}$ is continuous and,

$$f_x(t, x) \geq -\gamma > -\pi^2 \tag{1.6}$$

to the problem

$$x'' = f(t, x), x(0) = 0 = x(1). \tag{1.7}$$

In 1985, Joshi and Bose [5] considered a special case of (1.1)-(1.2) as

$$[a(t, x(t))\, x'(t)]' = f(t, x(t)), \tag{1.8}$$

$$x(0) = 0 = x(1),$$

where $f, a : [0, 1] \times \mathbf{R} \to \mathbf{R}$ are continuous. Under the conditions:

$$a(t, x) \geq \epsilon_0 > 0, |f(t, x)| \leq A_1|x| + A_2, \tag{1.9}$$

for every $(t, x) \in [0, 1] \times \mathbf{R}$, and

$$[f(t, x_1) - f(t, x_2)](x_1 - x_2) \geq 0, \text{ for every } t \in [0, 1], x_1, x_2 \in \mathbf{R}, \tag{1.10}$$

the solvability of (1.8) and (1.2) is proved. And in 1995, as a steady-state model for some one-dimensional conduction problems, Bobisud considers the problem

$$(k(t, x)x')' = f(t, x),$$

$$x(0) = \alpha, x(1) = \beta,$$

and gives some sufficient conditions for solvability. In this paper, under more generalized conditions, we discuss problem (1.1)-(1.2), and the main results contain conditions (1.5), and (1.9)-(1.10), as special cases. A function $f : [0, 1] \times \mathbf{R} \to \mathbf{R}$ is called to satisfy *Caratheodory conditions*, if and only if it is continuous with respect to x for a.e. $t \in [0, 1]$, it is measurable with respect to t for all values of x, and for every positive constant r, there exists $\rho_r \in L^2(0, 1)$ such that

$$|f(t, x)| \leq \rho_r(t), \tag{1.11}$$

for every $x \in \mathbf{R}$ with $|x| \leq r$ and a.e. $t \in [0, 1]$. Let X, Y be Hilbert spaces; an operator $A : X \to Y$ is said to be hemicontinuous, if and only if $A(x_0 + t_n x) \rightharpoonup A(x_0)$ as $t_n \to 0$, $x_0, x \in X$. For any $a, b \in L^1(0, 1)$, denote $a < b$, if and only if $a(t) \leq b(t)$, for a.e. $t \in (0, 1)$ and $a(t) < b(t)$ on a subset of $(0, 1)$ with positive measure. For an absolutely continuous function $p : [0, 1] \to (0, +\infty)$, denote $q_i \in H_i(p)$, if there exists a nontrivial solution with exactly i zeros on $(0, 1)$ for the problem

$$(p(t)x'(t))' + q_i(t)x = 0, \tag{1.12}$$

$$x(0) = 0 = x(1). \tag{1.13}$$

Now, we state the main results of this paper.

Theorem 1. *Assume that*

(1) *The operators* $x \to f(\cdot, x(\cdot), x'(\cdot))$, $x \to h(\cdot, x(\cdot), x'(\cdot))$ *from* $H_0^1(0, 1)$ *to* $L^2(0, 1)$ *are hemicontinuous.*

(2) *For every* $U, u, V, v \in \mathbf{R}$, *and* a.e. $t \in (0, 1)$, *we have*

$$(V - v)[h(t, U, V) - h(t, u, v)] + (U - u)[f(t, U, V) - f(t, u, v)]$$

$$\geq p(t)(V - v)^2 - q(t)(U - u)^2, \tag{1.14}$$

where $p : [0, 1] \to (0, +\infty)$ *is absolutely continuous and* $q \in L^\infty(0, 1)$ *such that* $q < q_0 \in H_0(p)$.

Then problem (1.1)-(1.2) *has a unique solution.*

Theorem 2. *Assume that*

(1) $a : [0, 1] \times \mathbf{R} \to \mathbf{R}$ *is continuous,* $f : [0, 1] \times \mathbf{R} \to \mathbf{R}$ *satisfies the Caratheodory conditions.*

(2) *There exist an absolutely continuous function* $p : [0, 1] \to (0, +\infty)$ *and* $q \in L^\infty(0, 1)$ *with* $q < q_0 \in H_0(p)$ *such that*

$$a(t, x) \geq p(t), \text{ for every } (t, x) \in [0, 1] \times \mathbf{R},$$

and

$$f(t, x)/x \geq -q(t), \text{ for } |x| \geq r > 0, \text{ a.e. } t \in (0, 1).$$

Then problem (1.8)-(1.2) *has at least one solution.*

The method used here is the monotone operator theory. Note that problem (1.1), (1.2) does not have variational structures, so the variational method such as the minimax methods and the Morse theory could not be used. Moreover, because of the nonlinearity of the operator

$x \to f(\cdot, x(\cdot), x'(\cdot))$, the Leray-Schauder continuation principle could not be used either. The proofs of the above two theorems will be given in Section 2. As applications, we discuss in Section 3, the existence of solutions for problem (1.7), and improve the main results of [7] by Wang and Li. In the last section, we give improvements of some main results of [1] by the method used in the proof of Theorem 2.

2. Proofs of Theorems 1 and 2

In order to prove Theorems 1 and 2, we give some lemmas at first.

Lemma 1. *Assume that* $p : [0, 1] \to (0, +\infty)$ *is absolutely continuous and* $q_0 \in H_0(p)$, *then*

$$\int_0^1 p(t)[x'(t)]^2 dt - \int_0^1 q_0(t)x^2(t)dt \geq 0, \text{ for every } x \in H_0^1(0, 1). \tag{2.1}$$

Proof. In a way similar to the proof of [2, Theorem 4.5.2], there exist a sequence $\{\lambda_j\}_{j=0}^{+\infty}$ with $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, and a sequence $\{u_j\}_{j=0}^{+\infty} \in H_0^2(0, 1)$ such that

$$(p(t)u_j'(t))' + (q_0(t) + \lambda_j)u_j(t) = 0, \text{ for a.e. } t \in (0, 1), \tag{2.2}$$

$$u_j(0) = 0 = u_j(1), \tag{2.3}$$

and $\{u_j\}$ is a complete orthonormal sequence for $L^2(0, 1)$. Set for every $x \in H_0^2(0, 1)$ that

$$x_j(t) = u_j(t)\int_0^1 x(t)u_j(t)dt \Big/ \int_0^1 u_j^2(t)dt, \ j = 0, 1, 2, \cdots,$$

we have

$$\sum_{j=0}^{n} x_j(t) \to x(t) \tag{2.4}$$

in $L^2(0, 1)$. Noticing (2.2)-(2.4), for every $x \in H_0^2(0, 1)$, we have

$$\int_0^1 p(t)[x'(t)]^2 dt - \int_0^1 q_0(t)x^2(t)dt$$

$$= -\int_0^1 [(p(t)x'(t))' + q_0(t)x(t)]x(t)dt$$

$$= -\liminf_{n\to\infty} \sum_{j=0}^{n} \int_0^1 [(p(t)x'(t))' + q_0(t)x(t)]x_j(t)dt$$

$$= \liminf_{n\to\infty} \sum_{j=0}^{n} \lambda_j \int_0^1 x_j^2(t)dt \geq 0,$$

and hence (2.1) is satisfied.

Lemma 2. *Let X be a Hilbert space, and let T be a map of X into itself. Suppose that*

(1) *T is hemicontinuous.*

(2) *T is monotone, i.e.,*

$$(Tu - Tv, u - v) \geq 0, \text{ for every } u, v \in X.$$

(3) *T is coercive, i.e.,*

$$(Tu, u)/|u|_X \to +\infty \text{ as } |u|_X \to +\infty.$$

Then $T(X) = X$.

Lemma 3. *Let X be a Hilbert space, and let $T : X \to X$ be such that*

(1) *If $x_n \rightharpoonup x$, $Tx_n \rightharpoonup y$, and*

$$\limsup(Tx_n, x_n) \leq (y, x),$$

then $Tx = y$.

(2) *For each finite-dimensional subspace X_0, the map*

$$w \to (Tw, v)$$

is continuous on X_0, for all $v \in X_0$.

(3) *T is coercive.*

Then $T(X) = X$.

Proof of Theorem 1. We will use Lemma 2. To this end, we need only to show that the operator $T : H_0^1(0, 1) \to H_0^1(0, 1)$ defined by (1.4) is monotone and coercive. By [3, Lemmas 2 and 3 and Proposition 4], there exists a positive constant ϵ such that $q + \epsilon \in H_0(p)$. From (1.14) and Lemma 1, for every $u, v \in H_0^1(0, 1)$, we have

$$\begin{aligned}(Tu - Tv, u - v) &= \int_0^1 \{(u'(t) - v'(t))[h(t, u(t), u'(t)) - h(t, v(t), v'(t))] \\ &\quad + (u(t) - v(t))[f(t, u(t), u'(t)) - f(t, v(t), v'(t))]\}dt \\ &\geq \int_0^1 \{p(t)[u'(t) - v'(t)]^2 - q(t)[u(t) - v(t)]^2\}dt \\ &\geq \delta \int_0^1 p(t)[u'(t) - v'(t)]^2 dt \geq 0, \end{aligned} \tag{2.5}$$

where $\delta > 0$ is such that

$$\frac{\delta}{1 - \delta} q(t) < \epsilon, \text{ for a.e. } t \in (0, 1).$$

Therefore, J is coercive. Letting $v = 0$, from (2.5), we have

$$(Tu, u) \geq \delta \int_0^1 p(t)[u'(t)]^2 dt + \int_0^1 h(t, 0, 0)u'(t)dt + \int_0^1 f(t, 0, 0)u(t)dt.$$

Noticing that

$$\pi^2 \int_0^1 u^2(t)dt \leq \int_0^1 [u'(t)]^2 dt, \text{ for } u \in H_0^1(0, 1), \tag{2.6}$$

we have

$$(Tu, u) \geq C_1 \int_0^1 [u'(t)]^2 dt - C_2 \{ \int_0^1 [u'(t)]^2 dt \}^{\frac{1}{2}}, \tag{2.7}$$

for some positive constant C_1 and C_2 (independent of u). Therefore, T is coercive. From Lemma 2, we have $T(X) = X$, and hence, (1.1)-(1.2) has at least one solution.

Suppose that both x_1 and x_2 are solutions of (1.1)-(1.2); in view of (2.5) and (2.6), we have for some positive constant ϵ that

$$0 = (Tx_1 - Tx_2, x_1 - x_2) \geq \delta \int_0^1 p(t)[(x_1(t) - x_2(t))']^2 dt \geq \epsilon \int_0^1 (x_1(t) - x_2(t))^2 dt,$$

it follows that $x_1 = x_2$.

Proof of Theorem 2. We will use Lemma 3. Define $T : H_0^1(0, 1) \to H_0^1(0, 1)$, $u \to Tu$ by

$$(Tu, v) = \int_0^1 a(t, u(t))u'(t)v'(t)dt + \int_0^1 f(t, u(t))v(t)dt, \text{ for every } v \in H_0^1(0, 1).$$

From assumption (2) of the theorem, we have

$$\begin{aligned}(Tu, u) &\geq \int_0^1 p(t)[u'(t)]^2 dt - \int_0^1 q(t)u^2(t)dt \\ &\quad + \int_{|u(t)|<r} q(t)u^2(t)dt + \int_{|u(t)|<r} f(t, u(t))u(t)dt \\ &\geq C_1 \int_0^1 [u'(t)]^2 dt - C_3,\end{aligned}$$

where C_1 is as in (2.7) and C_3 is independent of u. And hence, T is coercive. Let $x_n \rightharpoonup x$ in $H_0^1(0, 1)$, from the compact imbedding from

$H_0^1(0, 1)$ to $C[0, 1]$, we have $x_n \to x$ in $C[0, 1]$ and therefore, for every $u \in H_0^1(0, 1)$, we have

$$(Tx_n, u) = \int_0^1 a(t, x_n(t))x_n'(t)u'(t)dt + \int_0^1 f(t, x_n(t))u(t)dt,$$

that is, $Tx_n \rightharpoonup Tx$ in $H_0^1(0, 1)$. Hence, condition (1) of Lemma 3 is valid. Since f is continuous with respect to x, it is easily seen that condition (2) of Lemma 3 is satisfied. The proof is complete.

3. Applications

In this section, we discuss the problem

$$x'' + f(t, x) = 0, \tag{3.1}$$

$$x(0) = 0 = x(1).$$

From Theorems 1 and 2, we have

Corollary 1. *Assume that* $f : [0, 1] \times \mathbf{R} \to \mathbf{R}$ *satisfies the Caratheodory conditions such that*

$$f(t, x)/x \le q(t), \text{ for every } |x| \ge r \text{ and a.e. } t \in (0, 1), \tag{3.2}$$

where $q \in L^\infty(0, 1)$ *with* $q < q_0 \in H_0(p)$. *Then the problem* (3.1)-(3.2) *has at least one solution. If condition* (3.2) *is replaced by*

$$[f(t, x_1) - f(t, x_2)](x_1 - x_2) \le q(t)(x_1 - x_2)^2, \text{ for every } x_1, x_2 \in \mathbf{R},$$

$$\text{a.e. } t \in (0, 1),$$

then the problem (3.1)-(3.2) *has a unique solution.*

Lemma 4. *Assume that there exist two constants* $A > 0$ *and* $B > \pi^2$, *and* $\beta \in L^\infty(0, 1)$ *such that*

$$-A \le \beta(t) \le B, \int_0^1 \beta(t)dt \le (A + B)\alpha_1 - A,$$

where $\alpha = \alpha_1$ *is the minimal real root of equation*

$$\sqrt{A}cth(\sqrt{A}\,\frac{1-\alpha}{2} = \sqrt{B}\tan(\sqrt{B}\,\frac{\alpha}{2}).$$

Then, there exists a function $q_0 \in H_0(1)$ *such that* $\beta < q_0$.

Proof. From [7, Corollary 1], the problem

$$x'' + q(t)x = 0,$$

$$x(0) = 0 = x(1)$$

has only the trivial solution for every $q \in L^{\infty}(0, 1)$ such that $-A \le q \le \beta$. In view of [3, Theorem 3], the proof is complete.

From Corollary 1 and Lemma 4, we have

Corollary 2. *Assume that* $f : [0,1]\times\mathbf{R} \to \mathbf{R}$ *satisfies the Caratheodory conditions, and* β *satisfies the conditions in Lemma* 4 *such that,*

$$(f(t, x_1) - f(t, x_2))(x_1 - x_2) \le \beta(t)(x_1 - x_2)^2,$$

for every $x_1, x_2 \in \mathbf{R}$, a.e. $t \in (0, 1)$.

Then the problem (3.1)-(3.2) *has a unique solution.*

Remark. This corollary is a generalization of [7, Theorem 1], where the authors assume that $f, f_x : [0, 1]\times\mathbf{R} \to \mathbf{R}$ are continuous, and

$$-A \le f_x(t, x) \le \beta(t), \text{ for every } (t, x) \in [0, 1]\times\mathbf{R}.$$

4. An Improvement of a Result by Bobisud

In this section, by making use of Lemma 2 as in the proof of Theorem 2, we give a generalization of some parts of [1, Theorem 1].

Theorem 3. *Assume that* $k, f : [0, 1]\times\mathbf{R} \to \mathbf{R}$ *are Caratheodory functions such that,*

(1) $f(t, x)/x \ge -\delta_1$ *for* $|x| \ge r$, a.e. $t \in (0, 1)$, *where* δ_1 *is a positive constant and* $r > |\alpha| + |\beta|$.

(2) $k(t, x) \geq \delta > \delta_1$ *for* $x \in [-m, m]$, a.e. $t \in (0, 1)$, *where* δ *is a constant and* $m = r\delta / (\delta - \delta_1)$. *Then the problem*

$$(k(t, x)x')' = f(t, x), \tag{4.1}$$

$$x(0) = \alpha, \; x(1) = \beta \tag{4.2}$$

has at least one solution.

Proof. Set

$$\begin{aligned} \bar{k}(t, x) &= k(t, x), \; -m \leq x \leq m, \text{ a.e. } t \in (0, 1) \\ &= k(t, -m), \; x \leq -m, \text{ a.e. } t \in (0, 1) \\ &= k(t, m), \; x \geq m, \text{ a.e. } t \in (0, 1), \end{aligned}$$

we will show that any solution $x = x(t)$ of

$$(\bar{k}(t, x)x')' = f(t, x), \tag{4.3}$$

$$x(0) = \alpha, \; x(1) = \beta, \tag{4.4}$$

satisfies

$$|x(t)| \leq m \text{ for } t \in [0, 1].$$

In fact, suppose that $x = x(t)$ has a maximum exceeding m at $t_0 \in (0, 1)$. It is easily seen that $(\bar{k}(t, x(t))x'(t))|_{t=t_0} = 0$, and as far to the right as $x(t) \leq r$, we have that

$$\bar{k}(t, x(t))x'(t) \geq -\delta_1 x(t_0).$$

From this, we get that

$$x'(t) \geq -(\delta_1 / \delta)x(t_0).$$

Integrating to the nearest point to the right at which $x = r$, we have that

$$x(t_0) \leq \delta r / (\delta - \delta) = m,$$

a contradiction. Therefore, we only need to show that the problem (4.2)-(4.3) has at least one solution. Set $x = y + \alpha + (\beta - \alpha)t$. Then x is a solution of (4.2)-(4.3), if and only if y satisfies

$$(\bar{k}(t,\ y + \alpha + (\beta - \alpha)t)(y' + \beta - \alpha))' = f(t,\ y + \alpha + (\beta - \alpha)t),$$

$$y(0) = 0 = y(1).$$

Define $T : H_0^1(0, 1) \to H_0^1(0, 1),\ y \to Ty$ by

$$(Ty,\ v) = \int_0^1 \bar{k}(t,\ y(t) + \alpha + (\beta - \alpha)t)(y'(t) + (\beta - \alpha))v'(t)dt$$

$$+ \int_0^1 f(t,\ y(t) + \alpha + (\beta - \alpha)t)v(t)dt.$$

Just as the proof of Theorem 2, we can show that all of the assumptions of Lemma 3 are valid. Therefore, the proof is complete.

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