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ON INVERTIBILITY OF MULTIPLICATIVE COMBINATION OF TWO IDEMPOTENTS

XIYAN YAO and GUIBAO GAO

Department of Applied Mathematics
Yuncheng University
Shanxi 044000
P. R. China
e-mail: yaoxiyan63@163.com

Abstract

Let P_1 and P_2 be two idempotents on a Hilbert space. In this note, using the technique of block-operator matrices of bounded linear operators on a Hilbert space, we prove that the invertibility of multiplicative combination of two idempotents $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_1 P_2$ is independent of the choice of λ_i, $i = 1, 2, 3$, if $\lambda_1 \lambda_2 \lambda_3 \neq 0$.

1. Introduction

In recent years, a number of researchers has considered questions concerning the idempotents and linear combination of idempotents [2-6, 11]. Particularly, some researchers pay much attention to the study of linear combination of idempotents [2, 6]. For example, if P_1 and P_2 be two idempotents in the finite-dimensional space $\mathbb{C}^n$, Baksalary and Baksalary [2] have proved that the non-singularity of $P_1 + P_2$ is equivalent to the non-singularity of any linear combination $c_1 P_1 + c_2 P_2$,

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where $c_1 + c_2 \neq 0$. In this note, we will study the invertibility of multiplicative combination of two idempotents on an infinite-dimensional Hilbert space, by using the technique of block-operator matrices of bounded linear operators on a Hilbert space. A similar theorem involving the right multiplicative combination of two idempotents.

We begin with some terminology, notation, and lemmas. Let $\mathcal{H}$ be a Hilbert space and let all bounded linear operators on $\mathcal{H}$ be denoted by $\mathcal{B}(H)$. An operator $P \in \mathcal{B}(H)$ is said to be *idempotents*, if $P^2 = P$. If P is an idempotent operator and $T \in \mathcal{B}(H)$ is an invertible operator, then $T^{-1}PT$ is still an idempotent, since $(T^{-1}PT)^2 = T^{-1}PTT^{-1}PT = T^{-1}P^2$ $T = T^{-1}PT$. An operator $P \in \mathcal{B}(H)$ is said to be *orthogonal projection*, if $P^2 = P = P^*$, where P^* is the adjoint of P. Moreover, for an idempotent P, there exists an invertible operator $T \in \mathcal{B}(H)$ such that $T^{-1}PT$ is an orthogonal projection. In fact, if P is an idempotent, then P can be written in the form of

$$P = \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix},$$

with respect to the space decomposition $H = \mathcal{R}(P) \oplus \mathcal{R}(P)^{\perp}$. In this case, we have

$$\begin{pmatrix} I & P_1 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & -P_1 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where $\begin{pmatrix} I & -P_1 \\ 0 & I \end{pmatrix}$ is invertible, and $\begin{pmatrix} I & -P_1 \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & P_1 \\ 0 & I \end{pmatrix}$. Let $A \in \mathcal{B}(H)$, the $\mathcal{R}(A)$ denotes the range of the operator. An operator $A \in \mathcal{B}(H)$ is said to be *positive*, if $(Ax, x) \geq 0$ for $x \in \mathcal{H}$, if A is positive, then $A^{\frac{1}{2}}$ denotes the positive square root of A. An operator A is said to be a *contraction* (*strict contraction*), if $\| A \| \leq 1 (\| A \| < 1)$.

To prove the main results, we give some lemmas which are well known, so the proofs are omitted.

Lemma 1.1. *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ *be a bounded linear operator on* $\mathcal{H} \oplus \mathcal{K}$. *Then A is a positive, if and only if* $A_{11} > 0$, $A_{22} > 0$, $A_{12} = A_{21}^*$, *and there exists a contraction operator D from* $\mathcal{K}$ *into* $\mathcal{H}$ *such that*

$$A = \begin{pmatrix} A_{11} & A_{11}^{\frac{1}{2}} D A_{22}^{\frac{1}{2}} \\ A_{22}^{\frac{1}{2}} D^* A_{11}^{\frac{1}{2}} & A_{22} \end{pmatrix}.$$

Lemma 1.2. *Let* $\widetilde{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{B}(H \oplus K)$ *and* $D \in \mathcal{B}(K)$ *be invertible. Then* $\widetilde{A}$ *is invertible, if and only if* $A - BD^{-1}C$ *is invertible.*

Lemma 1.3. *Let* $A \in \mathcal{B}(H)$ *be a positive operator. Then, the following statements hold*:

(1) $\mathcal{R}(A) \subseteq \mathcal{R}(A^{\frac{1}{2}})$ *and* $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A^{\frac{1}{2}})}$, *where* $\overline{K}$ *denotes the closure of K*;

(2) $\mathcal{R}(A)$ *is closed, if and only if* $\mathcal{R}(A) = \mathcal{R}(A^{\frac{1}{2}})$;

(3) $\mathcal{R}(A) = \mathcal{H}$, *if and only if A is invertible.*

2. Main Results

In this note, the main result is the following theorem.

Definition 2.1. The multiplicative combination associated with two idempotents of a Hilbert space $\mathcal{H}$ is defined by $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_1 P_2$, for $\lambda_i \in \mathbb{C}$, $i = 1, 2, 3$.

In fact, multiplicative combination can be considered an operator valued function from $\mathbb{C}^3$ into $\mathcal{H}$.

Theorem 2.2. *Let* P_1 *and* P_2 *in* $\mathcal{H}$ *be two idempotents. If* $\lambda_1\lambda_2\lambda_3 \neq 0$, *then the invertibility of* $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_1 P_2$ *is independent of the choice of* $\lambda_i \in \mathbb{C}$, $i = 1, 2, 3$.

Proof. Let P_1 and P_2 be the idempotent. By the discussion of above, since $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_1 P_2$ is invertible, if and only if

$$\lambda_1 T_1^{-1} P_1 T_1 + \lambda_2 T_2^{-1} P_2 T_2 + \lambda_3 T_1^{-1} P_1 T_1 T_2^{-1} P_2 T_2$$

is invertible, to discuss the invertible of $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_1 P_2$, without loss of generality, we can assure that one of P_1 and P_2 is an orthogonal projection. For example, we assume that P_2 is an orthogonal projection, of course, P_2 is a positive operator. In this case, by Lemma 1.1, P_1 and P_2 have the following operator matrix forms:

$$P_1 = \begin{pmatrix} I & P_1^{1} \\ 0 & 0 \end{pmatrix}, \text{ and } P_2 = \begin{pmatrix} P_{11} & P_{11}^{\frac{1}{2}} D P_{22}^{\frac{1}{2}} \\ P_{22}^{\frac{1}{2}} D^* P_{11}^{\frac{1}{2}} & P_{22} \end{pmatrix},$$

with respect to the space decomposition $H = \mathcal{R}(P_1) \oplus \mathcal{R}(P_1)^{\perp}$, where P_{11} and P_{22} are positive operators on $\mathcal{R}(P_1)$ and $\mathcal{R}(P_1)^{\perp}$, respectively, and D is a contraction operator from $\mathcal{R}(P_1)^{\perp}$ into $\mathcal{R}(P_1)$. Now, $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_1 P_2$ has the following operator matrix forms:

$$\begin{aligned}
&\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_1 P_2 \\
&= \lambda_1 \begin{pmatrix} I & P_1^{1} \\ 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} P_{11} & P_{11}^{\frac{1}{2}} D P_{22}^{\frac{1}{2}} \\ P_{22}^{\frac{1}{2}} D^* P_{11}^{\frac{1}{2}} & P_{22} \end{pmatrix} \\
&\quad + \lambda_3 \begin{pmatrix} P_{11} + P_1^{1} P_{22}^{\frac{1}{2}} D^* P_{11}^{\frac{1}{2}} & P_{11}^{\frac{1}{2}} D P_{22}^{\frac{1}{2}} + P_1^{1} P_{22} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1 I + \lambda_2 P_{11} & \lambda_1 P_1^{1} + \lambda_2 P_{11}^{\frac{1}{2}} D P_{22}^{\frac{1}{2}} \\ + \lambda_3 (P_{11} + P_1^{1} P_{22}^{\frac{1}{2}} D^* P_{11}^{\frac{1}{2}}) & + \lambda_3 (P_{11}^{\frac{1}{2}} D P_{22}^{\frac{1}{2}} + P_1^{1} P_{22}) \\ \lambda_2 P_{22}^{\frac{1}{2}} D^* P_{11}^{\frac{1}{2}} & \lambda_2 P_{22} \end{pmatrix},
\end{aligned}$$

with respect to the space decomposition $H = \mathcal{R}(P_1) \oplus \mathcal{R}(P_1)^{\perp}$. Since $\lambda_1\lambda_2\lambda_3 \neq 0$. If $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_1 P_2$ is invertible, that is, the operator matrix

$$\begin{pmatrix} \lambda_1 I + \lambda_2 P_{11} & \lambda_1 P_1^1 + \lambda_2 P_{11}^{\frac{1}{2}} D P_{22}^{\frac{1}{2}} \\ + \lambda_3 (P_{11} + P_1^1 P_{22}^{\frac{1}{2}} D^* P_{11}^{\frac{1}{2}}) & + \lambda_3 (P_{11}^{\frac{1}{2}} D P_{22}^{\frac{1}{2}} + P_1^1 P_{22}) \\ \lambda_2 P_{22}^{\frac{1}{2}} D^* P_{11}^{\frac{1}{2}} & \lambda_2 P_{22} \end{pmatrix}$$

is invertible, then $\mathcal{R}(\lambda_2 P_{22}^{\frac{1}{2}} D^* P_{11}^{\frac{1}{2}}, \lambda_2 P_{22}) = \mathcal{R}(P_1)^{\perp}$. By Lemma 1.3, $\mathcal{R}(P_{22}) \subseteq \mathcal{R}(P_{22})^{\frac{1}{2}}$ and $\mathcal{R}(\lambda_2 P_{22}^{\frac{1}{2}} D^* P_{11}^{\frac{1}{2}}, \lambda_2 P_{22}) \subseteq \mathcal{R}(P_{22})^{\frac{1}{2}} \subseteq \mathcal{R}(P_1)^{\perp}$, then $\mathcal{R}(P_{22})^{\frac{1}{2}} = \mathcal{R}(P_1)^{\perp}$. By Lemma 1.3 again, we have $\mathcal{R}(P_{22}) = \mathcal{R}(P_1)^{\perp}$. That shows that P_{22} is invertible. In this case, By Lemma 1.2, $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_1 P_2$ is invertible, if and only if

$$\begin{aligned} &\lambda_1 I + \lambda_2 P_{11} + \lambda_3 (P_{11} + P_1^1 P_{22}^{\frac{1}{2}} D^* P_{11}^{\frac{1}{2}}) \\ &- [\lambda_1 P_1^1 + \lambda_2 P_{11}^{\frac{1}{2}} D P_{22}^{\frac{1}{2}} + \lambda_3 (P_{11}^{\frac{1}{2}} D P_{22}^{\frac{1}{2}} + P_1^1 P_{22})] \lambda_2^{-1} P_{22}^{-1} \lambda_2 P_{22}^{\frac{1}{2}} D^* P_{11}^{\frac{1}{2}} \\ = \lambda_1 &(I - P_1^1 P_{22}^{-\frac{1}{2}} D^* P_{11}^{\frac{1}{2}}) \end{aligned}$$

is invertible. This shows that the invertibility of $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_1 P_2$ is only dependent on the invertible of the operator $I - P_1^1 P_{22}^{-\frac{1}{2}} D^* P_{11}^{\frac{1}{2}}$, if $\lambda_1\lambda_2\lambda_3 \neq 0$.

In other words, the invertibility of $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_1 P_2$ is independent of the choice of $\lambda_i \in \mathbb{C}$, $i = 1, 2, 3$, if $\lambda_1\lambda_2\lambda_3 \neq 0$. □

Definition 2.3. The right multiplicative combination associated with the idempotents P_1 and P_2 by $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_2 P_1$ for $\lambda_i \in \mathbb{C}$, $i = 1, 2, 3$.

Proposition 2.4. *Let P_1 and P_2 in $\mathcal{B}(H)$ be two idempotents. If $\lambda_1\lambda_2\lambda_3 \neq 0$, then the invertibility of right multiplicative combination $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_2 P_1$ is independent of the choice of $\lambda_i \in \mathbb{C}$, $i = 1, 2, 3$.*

Proof. Let P_1 and P_2 in $\mathcal{B}(H)$ be two idempotents. To discuss the invertible of $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_2 P_1$, we can assure that P_1 is an orthogonal projection, of course, P_1 is a positive operator. In this case, P_1 and P_2 have the following operator matrix forms:

$$P_1 = \begin{pmatrix} Q_{11} & Q_{11}^{\frac{1}{2}} D Q_{22}^{\frac{1}{2}} \\ Q_{22}^{\frac{1}{2}} D^* Q_{11}^{\frac{1}{2}} & Q_{22} \end{pmatrix}, \text{ and } P_2 = \begin{pmatrix} I & Q_1^1 \\ 0 & 0 \end{pmatrix},$$

with respect to the space decomposition $H = \mathcal{R}(P_2) \oplus \mathcal{R}(P_2)^{\perp}$, where Q_{11} and Q_{22} are positive operators on $\mathcal{R}(P_2)$ and $\mathcal{R}(P_2)^{\perp}$, respectively. D is a contraction operator from $\mathcal{R}(P_2)^{\perp}$ into $\mathcal{R}(P_2)$. Now, $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_2 P_1$ has the following operator matrix forms:

$$\begin{aligned} &\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_2 P_1 \\ &= \lambda_1 \begin{pmatrix} Q_{11} & Q_{11}^{\frac{1}{2}} D Q_{22}^{\frac{1}{2}} \\ Q_{22}^{\frac{1}{2}} D^* Q_{11}^{\frac{1}{2}} & Q_{22} \end{pmatrix} + \lambda_2 \begin{pmatrix} I & Q_1^1 \\ 0 & 0 \end{pmatrix} \\ &\quad + \lambda_3 \begin{pmatrix} Q_{11} + Q_1^1 Q_{22}^{\frac{1}{2}} D^* Q_{11}^{\frac{1}{2}} & Q_{11}^{\frac{1}{2}} D Q_{22}^{\frac{1}{2}} + Q_1^1 Q_{22} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_2 I + \lambda_1 Q_{11} & \lambda_2 Q_1^1 + \lambda_1 Q_{11}^{\frac{1}{2}} D Q_{22}^{\frac{1}{2}} \\ + \lambda_3 (Q_{11} + Q_1^1 P_{22}^{\frac{1}{2}} D^* Q_{11}^{\frac{1}{2}}) & + \lambda_3 (Q_{11}^{\frac{1}{2}} D Q_{22}^{\frac{1}{2}} + Q_1^1 Q_{22}) \\ \lambda_1 Q_{22}^{\frac{1}{2}} D^* Q_{11}^{\frac{1}{2}} & \lambda_1 Q_{22} \end{pmatrix}, \end{aligned}$$

with respect to the space decomposition $H = \mathcal{R}(P_2) \oplus \mathcal{R}(P_2)^{\perp}$, then the proof is similar with Theorem 2.2, so omitted. □

Corollary 2.5. *Let* P_1 *and* P_2 *in* $\mathcal{B}(H)$ *be two orthogonal projections. If* $\lambda_1\lambda_2\lambda_3 \neq 0$, *then the invertibility of* $\lambda_1P_1 + \lambda_2P_2 + \lambda_3P_1P_2$ *and* $\lambda_1P_1 + \lambda_2P_2 + \lambda_3P_2P_1$ *are independent of the choice of* $\lambda_i \in \mathbb{C}$, $i = 1, 2, 3$.

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