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DESCRIPTION OF UNIFORM DISTRIBUTION OF REAL SEQUENCES MODULO b BY USING b-DECIMAL PART FUNCTION

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Abstract

The b-parts of real numbers were introduced and studied in [1, 2]. Those have many interesting number theoretic explanations, algebraic, and analytic properties, and the b-decimal part is as a generalization of the fractional part of real numbers. On the other hand, uniform distributed real sequences modulo 1 are defined by using their fractional parts. Considering these facts in this paper, we describe and consider uniform distribution of real sequences modulo an arbitrary fixed real number $b \neq 0$, and state its relation to the uniform distribution of real sequences modulo 1, by using b-parts.

1. Preliminaries

For a real number a denote $[a]$ as the largest integer not exceeding a, and put $(a) = a - [a]$ (the fractional part or decimal part of a). Now, let b be a nonzero constant real number. For any real number a, set

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$$[a]_b = b[\frac{a}{b}], (a)_b = b(\frac{a}{b}).$$

We call the notation $[a]_b$ *b-integer part of a* and $(a)_b$ *b-decimal part of a*. Also, $[a]_b$ and $(a)_b$ are called *b-parts of a*.

Clearly, $a = [a]_b + (a)_b$, where

$$[a]_b \in b\mathbb{Z} =< b >, (a)_b \in \mathbb{R}_b := b[0, 1) = \{bd | 0 \leq d < 1\}.$$

The b-parts have many important number theoretic explanations. For instance, $(a)_b$ is the remainder of the (generalized) division of a by b, if $b > 0$ $(\forall a, b \in \mathbb{R}$ such that $b \neq 0)$, and if b is positive integer, then $[a]_b$ is the same unique integer number of the residue class $\{[a] - b + 1, \cdots, [a]\}$ (mode b), that is, divisible by b. Also, we have $[a + \beta]_b = [a]_b + \beta, (a + \beta)_b = (a)_b$, for every $\beta \in b\mathbb{Z}$, so if m, n are integers, then

$$(ma + nc)_b = (m(a)_b + nc)_b = (ma + n(c)_b)_b = (m(a)_b + n(c)_b)_b = ((ma + nc)_b)_b.$$

Therefore, the b-decimal and b-integer part functions $(x)_b$ and $[x]_b$ are idempotent, their compositions are zero and $(x)_b$ satisfies the following functional equations

$$f(f(x) + y - f(y)) = f(x), \quad f(x + y - f(y)) = f(x).$$

More their number theoretic, algebraic, and analytic properties have been studied in [1, 2, 3, 4].

Note. Since $(a)_1 = (a)$, then to avoid any confusion between decimal and parentheses notation, sometimes we use the symbol $(a)_1$ instead of (a).

2. Uniform Distribution of Real Sequences Modulo b

Let $\omega = \{x_n\}_{n \geq 1}$ be a given sequence of real numbers. The uniform distribution of ω modulo 1 is defined as following.

Definition 2.1. The sequence ω is said to be *uniformly distributed modulo* 1 (abbreviated u.d mod 1), if for every pair α, β of real numbers with $0 \le \alpha < \beta \le 1$, we have

$$\lim_{N\to\infty} \frac{A([\alpha, \beta); N; \omega)}{N} = \alpha - \beta, \tag{2.1}$$

where $A([\alpha, \beta); N; \omega)$ is the number of terms x_n, $1 \le n \le N$, for which $(x_n)_1 \in [\alpha, \beta)$.

Thus, in simple terms, the sequence $\omega = \{x_n\}_{n\ge1}$ is u.d mod 1, if every half-open subinterval of $I = [0, 1) = \mathbb{R}_1$ eventually gets its "proper share" of fractional parts. If $\chi_{[\alpha,\beta)}$ is the characteristic function of the interval $[\alpha, \beta) \subseteq \mathbb{R}_1$, then (2.1) can be written in the form

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[\alpha,\beta)}((x_n)_1) = \int_0^1 \chi_{[\alpha,\beta)}(x)\, dx. \tag{2.2}$$

This observation, together with an important approximation technique leads to the following theorem.

Theorem 2.2. *The following statements are equivalent, for a given real sequence* $\omega = \{x_n\}_{n\ge1}$.

(a) ω *is u.d mod* 1.

(b) *For every real-valued continuous function f defined on the closed interval* $\bar{I} = \overline{\mathbb{R}_1} = [0, 1]$, *we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\, dx. \tag{2.3}$$

(c) *For every real-valued Riemann-integrable function f defined on the closed interval* [0, 1], *the Equation* (2.3) *holds.*

(d) *For every complex-valued continuous function f defined on* $\mathbb{R}$ *with period* 1, *we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,dx. \tag{2.4}$$

(e) (*Weyl criterion*)

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n} = 0, \text{ for all integers } h \neq 0. \tag{2.5}$$

Proof. See Theorem 1.1, Corollaries 1.1, 1.2, and Theorem 2.1 of [5].

□

Now, using the b-decimal part function, we describe uniform distribution of real sequences modulo an arbitrary real number $b \neq 0$, and then state its relation to uniform distribution modulo 1.

Definition 2.3. Fix real number $b \neq 0$, and consider the real sequence $\omega = \{x_n\}_{n\geq 1}$. We say the sequence x_n is uniform distributed modulo b (abbreviated u.d mod b), if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} \chi_{[\alpha,\beta)}((x_n)_b) = \frac{\beta-\alpha}{|b|}, \tag{2.6}$$

for every $\alpha, \beta \in \overline{\mathbb{R}}_b = b[0, 1]$ such that $\alpha < \beta$.

In other words, x_n is u.d mod b, if and only if the average of the sequence $\{\chi_{[\alpha,\beta)}((x_n)_b)\}_{n\geq 1}$ is equal to the average of the function $\chi_{[\alpha,\beta)}$ on $\overline{\mathbb{R}}_b$, that is,

$$\text{Avg}_{\overline{\mathbb{R}}_b}(\chi_{[\alpha,\beta)}) = \frac{1}{L(\overline{\mathbb{R}}_b)}\int_{\overline{\mathbb{R}}_b} \chi_{[\alpha,\beta)}(x)\,dx = \frac{1}{|b|}\int_{\overline{\mathbb{R}}_b} \chi_{[\alpha,\beta)}(x)\,dx, \tag{2.7}$$

where L is the Lebesgue-measure.

Lemma 2.4. *The Equation* (2.6) *holds (for a given sequence* x_n *and every* $\alpha, \beta \in \overline{\mathbb{R}}_b$ *such that* $\alpha < \beta$*), if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f((x_n)_b) = \frac{1}{|b|}\int_{\overline{\mathbb{R}}_b} f(x)\,dx, \tag{2.8}$$

for every real step function f defined on $\overline{\mathbb{R}}_b$.

Proof. Let $f = \sum_{i=0}^{k-1} d_i \chi_{[a_i, a_{i+1})}$, where $0 = a_0 < a_1 < a_2 < \cdots < a_{k-1} < a_k = b$, if $b > 0$ and $b = a_0 < a_1 < a_2 < \cdots < a_{k-1} < a_k = 0$, if $b < 0$. Then applying (2.6), we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f((x_n)_b) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{i=0}^{k-1} d_i \chi_{[a_i, a_{i+1})}((x_n)_b)$$

$$= \sum_{i=0}^{k-1} d_i \Big(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a_i, a_{i+1})}((x_n)_b) \Big)$$

$$= \sum_{i=0}^{k-1} d_i \frac{1}{|b|} \int_{\overline{\mathbb{R}}_b} \chi_{[a_i, a_{i+1})}(x)\, dx = \frac{1}{|b|} \int_{\overline{\mathbb{R}}_b} f(x)\, dx.$$

The converse of it is valid, because every characteristic function is step function. □

Theorem 2.5. *The sequence* x_n *is u.d mod b, if and only if* (2.8) *holds for every real-valued continuous* [*or Riemann-integrable*] *function f defined on* $\overline{\mathbb{R}}_b$.

Proof. Considering the above lemma, this is similar to the proof of Theorem 1.1 in [5]. □

Corollary 2.6. *The sequence* x_n *is u.d mod b, if and only if for every complex-valued continuous function f on* $\mathbb{R}$ *with period b, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \frac{1}{|b|} \int_{\overline{\mathbb{R}}_b} f(x)\, dx. \tag{2.9}$$

Proof. By applying Theorem 2.3 to the real and imaginary part of f, one shows first that (2.8) holds for complex-valued f. But, the b-periodicity condition implies $f((x_n)_b) = f(x_n - kb) = f(x_n)(k \in \mathbb{Z})$, and so we arrive at (2.8). As to the sufficiency of (2.8), in the second part of the proof of Theorem 2.3, the Equation (2.8) can be applied to the b-periodic extensions of g_1 and g_2. □

Now, we show that how u.d mod b can be returned to u.d mod 1 and vice versa.

Theorem 2.7. *Fix a real number* $b \neq 0$, *and consider a real sequence* x_n. *The following statements are equivalent*:

(a) x_n *is u.d mod b.*

(b) (*Weyl criterion*)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{\frac{2\pi i h x_n}{b}} = 0, \text{ for all integers } h \neq 0. \tag{2.10}$$

(c) *The sequence* $\frac{x_n}{b}$ *is u.d mod* 1.

Proof. Putting $f(x) = e^{\frac{2\pi i h x}{b}} (h \in \mathbb{Z} \setminus \{0\})$ in (2.9) and applying Corollary 2.4, the part (b) can be concluded from (a). Also, the part (b) implies part (c), by Theorem 2.1. Now, let (c) hold and f be an arbitrary continuous function on $\overline{\mathbb{R}}_b$. Put $g(x) = f(bx)$ for every $x \in [0, 1]$. So, g is continuous on $[0, 1]$ and Theorem 2.1 implies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g((\frac{x_n}{b})) = \int_0^1 g(x)\,dx.$$

But since $g((\frac{x_n}{b})) = f((x_n)_b)$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f((x_n)_b) = \int_0^1 f(bx)\,dx = \frac{1}{b}\int_0^b f(t)\,dt = \frac{1}{|b|}\int_{\overline{\mathbb{R}}_b} f(x)\,dx.$$

Therefore, Theorem 2.3 implies x_n is u.d mod b. □

Corollary 2.8. *The following statements* (a)-(c) *are equivalent*:

(a) *The real sequence* x_n *is u.d mod* 1.

(b) *The sequence* $y_n = bx_n$ *is u.d mod b, for every real number* $b \neq 0$.

(c) *The sequence* $y_n = cx_n$ *is u.d mod c, for some real number* $c \neq 0$.

Hence, if $b \neq 0$, *then* $\{nb\theta\}_{n\geq 1}$ *is u.d mod b, if and only if* θ *is an irrational number.*

Proof. If there exists a real number $c \neq 0$ such that the sequence $y_n = cx_n$ is u.d mod c, then $x_n = \frac{1}{c} y_n$ is u.d mod 1 (part (c) of the above theorem). So, Theorem 2.7 implies this corollary, clearly. □

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