Journal of Mathematical Sciences: Advances and Applications
Volume 4, Number 2, 2010, Pages 265-277

REGULAR STURM-LIOUVILLE OPERATORS WITH TRANSMISSION CONDITIONS AT FINITE INTERIOR DISCONTINUOUS POINTS

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Abstract

In this paper, we investigate a class of Sturm-Liouville problems with transmission conditions at finite interior discontinuous points. We define the maximal and minimal operators associated with conditions at the interior points in an appropriate Hilbert space. By construction of the maximal and minimal operators, we establish necessary and sufficient conditions, which guarantee self-adjointness of the ordinary differential operators with coupled boundary conditions and real interface conditions at the interior points.

Introduction

The Sturm-Liouville theory is very important for solving problems in mathematics physics. In this paper, we study regular Sturm-Liouville problems, which have discontinuities at finite interior points. This research is motivated by the theory of heat and mass transfer, varied assortment of physical transfer problems [2], and electrostatic potential

2010 Mathematics Subject Classification: 34B24.

Keywords and phrases: Sturm-Liouville operators, discontinuous points, transmission conditions, self-adjointness.

This work was supported by the National Nature Science Foundation of China (10861008).

Received October 23, 2009

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problems of a point charge [3]. To deal with interior discontinuities, some conditions are imposed on the discontinuous points and such conditions involve left and right limits of solutions and their quasi-derivatives at the discontinuous points, which are often called *transmission conditions* or *interface conditions*. Following Mukhtarov and Yakubov in [4], we use an appropriate Hilbert space H with an inner product, which depends on the transmission conditions, and give necessary and sufficient conditions for self-adjointness of SLP's with real transmission conditions in H. We define the maximal and minimal operators L_{MT}, L_{0T} in Section 1, which depends on the transmission conditions at the interior points of the interval (a, b). It is the key point for solving the problem.

We prove that the minimal operator L_{0T} is symmetric and L_{0T}, L_{MT} are an adjoint pair. Thus, the characterizations of self-adjointness of the problem is equivalent to considering all self-adjoint extensions of L_{0T}. Note that our real transmission conditions are not required to be self-adjoint, i.e., the coefficient matrices of transmission conditions are arbitrary real coupling matrices, whose determinant is positive. This contrasts with the usual theory, which requires the coupling matrix to have determinant one.

1. The Maximal and Minimal Operators Associated with the Operator T

Consider the regular symmetric differential equation

$$ly := -(p(x)y')' + q(x)y = \lambda y \quad \text{on } I,\ \lambda \in \mathbb{C}, \tag{1.1}$$

with

$$I = (a, c_1) \cup (c_1, c_2) \cup \cdots \cup (c_m, b),\ -\infty \le a < b \le \infty,\ c_i \in (a, b),\ i = 1, \cdots, m,$$

$$p(x)^{-1},\ q(x) \in L^1(I, (R));$$

boundary conditions

$$AY(a) + BY(b) = 0, \quad Y = \begin{pmatrix} y \\ py' \end{pmatrix}; \tag{1.2}$$

and interface conditions

$$l_i^1 := \gamma_{i1}y(c_i-) + \gamma_{i2}y^{[1]}(c_i-) + \delta_{i1}y(c_i+) + \delta_{i2}y^{[1]}(c_i+) = 0,\ i = 1, \cdots, m, \tag{1.3}$$

$$l_i^2 := \gamma_{i3}y(c_i-) + \gamma_{i4}y^{[1]}(c_i-) + \delta_{i3}y(c_i+) + \delta_{i4}y^{[1]}(c_i+) = 0,\ i = 1, \cdots, m, \tag{1.4}$$

where $y^{[1]}$ is the 1-th quasi-derivatives of y; $A = (a_{ij})$, $B = (b_{ij})$ $(i, j = 1, 2)$ are complex matrices such that the 2×4 matrix $(A|B)$ has full rank; and the coefficients $\gamma_{ij}, \delta_{ij} (i = 1, \cdots, m, j = 1, \cdots, 4)$ are real numbers satisfying $\gamma_i = \begin{vmatrix} \gamma_{i1} & \gamma_{i2} \\ \gamma_{i3} & \gamma_{i4} \end{vmatrix} > 0$, $\delta_i = \begin{vmatrix} \delta_{i1} & \delta_{i2} \\ \delta_{i3} & \delta_{i4} \end{vmatrix} > 0$.

Following Mukhtarov and Yakubov [4], in this paper, we define the similar inner product in H as follows:

$$(f, g) = \int_a^{c_1} f_1\overline{g}_1 dx + \frac{\delta_1}{\gamma_1}\int_{c_1}^{c_2} f_2\overline{g}_2 dx + \frac{\delta_1\delta_2}{\gamma_1\gamma_2}\int_{c_2}^{c_3} f_3\overline{g}_3 dx + \cdots + \frac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m}\int_{c_m}^{b} f_{m+1}\overline{g}_{m+1} dx,$$

where $f, g \in L^2(I)$, $f_1(x) = f(x)|_{(a, c_1)}$, $f_i(x) = f(x)|_{(c_{i-1}, c_i)} (i = 2, \cdots, m)$, $f_{m+1}(x) = f(x)|_{(c_m, b)}$. Then H is a Hilbert space with this inner product.

Definition 1.1. The maximal operator L_M generated by the differential expression ly in H is defined as

$$\mathcal{D}(L_M) = \{y \in H \mid y_1, y_1^{[1]} \in AC_{loc}(a, c_1),\ y_i, y_i^{[1]} \in AC_{loc}(c_{i-1}, c_i)\ (i = 2, \cdots, m),\ y_{m+1}, y_{m+1}^{[1]} \in AC_{loc}(c_m, b), \text{ and } ly \in H\},$$

$$L_M = ly, \quad y \in (D)(L_M).$$

Definition 1.2. The minimal operator L_0 generated by ly in H is defined as

$$\mathcal{D}(L_0) = \{y \in \mathcal{D}(L_M) | y(a) = y^{[1]}(a) = y(c_1-) = y^{[1]}(c_1-) = y(c_1+)$$

$$= y^{[1]}(c_1+) \cdots = y(c_m-) = y^{[1]}(c_m-) = y(c_m+) = y^{[1]}(c_m+) = 0\},$$

$$L_0 y = ly, \quad y \in \mathcal{D}(L_0).$$

Remark. Here, $y(c_i-) = y_i(c_i) = \lim_{x \to c_i-} y_i(x) (i = 1, \cdots, m)$. By Theorem 1.5.2 in [7], these limits exist and are finite for any maximal domain function generated by ly restricted to $(a, c_1)(i = 1)$, and $(c_{i-1}, c_i)(i = 2, \cdots, m)$. Similarly, $y(c_i+) = y_{i+1}(c_i)$, and $y^{[1]}(c_i+) = y_{i+1}^{[1]}(c_i)(i = 1, \cdots, m)$ defined by right limits exist and are finite.

Definition 1.3. We define the operator T generated by ly in H as follows:

$$\mathcal{D}(T) = \{y \in \mathcal{D}_M | AY(a) + BY(b) = 0, l_i^1 = l_i^2 = 0, i = 1, \cdots, m\},$$

$$Ty = ly, \quad \text{for } y \in \mathcal{D}(T).$$

In order to investigate the question of self-adjointness for all SLP (1.1)-(1.4), we introduce the new maximal and minimal operators associated with T.

Definition 1.4. The maximal operator L_{MT} associated with T is defined as follows:

$$\mathcal{D}(L_{MT}) = \{y \in \mathcal{D}(L_M) | l_i^1 y = l_i^2 y = 0\},$$

$$L_{MT} y = ly, \quad \text{for } y \in \mathcal{D}(L_{MT}).$$

Definition 1.5. The minimal operator L_{0T} associated with T is defined by

$$\mathcal{D}(L_{0T}) = \{y \in \mathcal{D}(L_{MT}) | y(a) = y^{[1]}(a) = y(b) = y^{[1]}(b) = 0\},$$

$$L_{0T} y = ly, \quad \text{for } y \in \mathcal{D}(L_{0T}).$$

Obviously, we have $L_0 \subset L_{0T} \subset T \subset L_{MT} \subset L_M$.

Lemma 1.1. *The minimal operator* L_0 *is a closed and dense-defined symmetric differential operator and* $L_0^* = L_M$, $L_M^* = L_0$.

By property of the adjoint operator, we have

$$L_0 \subset L_{MT}^* \subset T^* \subset L_{0T}^* \subset L_M. \tag{1.5}$$

For every $u, v \in \mathcal{D}(L_{MT})$, we have

$$(lu, v) - (u, lv) = \frac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m} W(u, \bar{v}; b) - W(u, \bar{v}; a), \tag{1.6}$$

where $W(u, \bar{v}; x) = u(x)\bar{v}^{[1]}(x) - u^{[1]}(x)\bar{v}(x)$. Hence, for every $u \in \mathcal{D}(L_{0T})$ and $v \in \mathcal{D}(L_{MT})$, we have

$$(L_{0T}u, v) = (u, L_{MT}v). \tag{1.7}$$

It is clear that $L_{MT} \subset L_{0T}^*$ and L_{0T} is a dense-defined symmetric operator.

Theorem 1.1. *For any complex numbers* $\alpha_1, \beta_1, \alpha_2, \beta_2$, *there exists* $u \in \mathcal{D}(L_{MT})$ *such that* $u(a) = \alpha_1$, $u^{[1]}(a) = \beta_1$, $u(b) = \alpha_2$, $u^{[1]}(b) = \beta_2$.

Proof. Let $y_{m+1}(x)$ and $z_{m+1}(x)$ be linearly independent solutions of the equation $-(p(x)u')' + q(x)u = 0\,(x \in (c_m, b))$ with the initial-value

$$y_{m+1}(b) = 1,\ y_{m+1}^{[1]}(b) = 0, \text{ and } z_{m+1}(b) = 0,\ z_{m+1}^{[1]}b = 1,$$

respectively.

Let $y_i(x)\,(i = m, \cdots, 1)$ and $z_i(x)\,(i = m, \cdots, 1)$ be linearly independent solutions of the equation $-(p(x)u')' + q(x)u = 0$ defined on the interval $(c_{i-1}, c_i)\,(i = m, \cdots, 2)$, $(a, c_1)\,(i = 1)$, satisfying

$$y_i(c_i) = \frac{1}{\gamma_i}[(\gamma_{i2}\delta_{i3} - \gamma_{i4}\delta_{i1})y_{i+1}(c_i) + (\gamma_{i2}\delta_{i4} - \gamma_{i4}\delta_{i2})y_{i+1}^{[1]}(c_i)],$$

$$y_i^{[1]}(c_i) = \frac{1}{\gamma_i}[(\gamma_{i3}\delta_{i1} - \gamma_{i1}\delta_{i3})y_{i+1}(c_i) + (\gamma_{i3}\delta_{i2} - \gamma_{i1}\delta_{i4})y_{i+1}^{[1]}(c_i)],$$

and

$$z_i(c_i) = \frac{1}{\gamma_i}[(\gamma_{i2}\delta_{i3} - \gamma_{i4}\delta_{i1})z_{i+1}(c_i) + (\gamma_{i2}\delta_{i4} - \gamma_{i4}\delta_{i2})z_{i+1}^{[1]}(c_i)],$$

$$z_i^{[1]}(c_i) = \frac{1}{\gamma_i}[(\gamma_{i3}\delta_{i1} - \gamma_{i1}\delta_{i3})z_{i+1}(c_i) + (\gamma_{i3}\delta_{i2} - \gamma_{i1}\delta_{i4})z_{i+1}^{[1]}(c_i)].$$

It is easy to verify that $y_i(x)$ and $z_i(x)$ are linearly independent.

Put

$$y(x) = \begin{cases} y_1(x), & x \in (a, c_1), \\ y_2(x), & x \in (c_1, c_2), \\ \cdots & \\ y_{m+1}(x), & x \in (c_m, b). \end{cases} \qquad z(x) = \begin{cases} z_1(x), & x \in (a, c_1), \\ z_2(x), & x \in (c_1, c_2), \\ \cdots & \\ z_{m+1}(x), & x \in (c_m, b). \end{cases}$$

We see that $y(x)$ and $z(x)$ are in $\mathcal{D}(L_{MT})$, and they are linearly independent solutions of $lu = 0$.

Let

$$f(x) = \begin{cases} \tau_1 y_1(x) + \tau_2 z_1(x), & x \in (a, c_1), \\ \tau_1 y_2(x) + \tau_2 z_2(x), & x \in (c_1, c_2), \\ \cdots & \\ \tau_1 y_{m+1}(x) + \tau_2 z_{m+1}(x), & x \in (c_m, b), \end{cases}$$

where τ_1, τ_2 satisfy the following conditions:

$$(f, y) = \tau_1(y, y) + \tau_2(z, y) = -\frac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m}\beta_2,$$

$$(f, z) = \tau_1(y, z) + \tau_2(z, z) = \frac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m}\alpha_2. \tag{1.8}$$

Since, $y(x)$ and $z(x)$ are linearly independent, the Gram determinant $\begin{vmatrix} (y, y) & (z, y) \\ (y, z) & (z, z) \end{vmatrix} \neq 0$. Hence, c_1 and c_2 are uniquely determined.

Let $w(x) \in \mathcal{D}(L_{MT})$ be the solution of the initial-value problem

$$\begin{cases} lw = f,\ x \in I, \\ w(a) = 0, \\ w^{[1]}(a) = 0. \end{cases}$$

In term of (1.8), we can obtain that $w(x)$ satisfies $w(a) = 0$, $w^{[1]}(a) = 0$, $w(b) = \alpha_2$, and $w^{[1]}(b) = \beta_2$. In the same way, we have $v(x) \in \mathcal{D}(L_{MT})$ such that $v(a) = \alpha_1$, $v^{[1]}(a) = \beta_1$, $v(b) = 0$, and $v^{[1]}(b) = 0$. Put $u = w + v$. Then $u(x)$ is the required function. ■

Theorem 1.2. *Let $f(x) \in H$. Then, the equation $ly = f$ has a solution $\psi(x) \in \mathcal{D}(L_{0T})$, if and only if f is orthogonal to all solutions of $ly = 0$, which are in $\mathcal{D}(L_{MT})$.*

Proof. Necessity. For $f \in H$, where $f = \begin{cases} f_1, & x \in (a, c_1) \\ \cdots & \\ f_{m+1}, & x \in (c_m, b) \end{cases}$, if $ly = f$ has a solution $\psi(x) \in \mathcal{D}(L_{0T})$, then for every $g \in \mathcal{D}(L_{MT})$ such that $lg = 0$, we have

$$(f, g) = (l\psi, g) = (l_{0T}\psi, g) = (\psi, L_{0T}^{*}g) = (\psi, L_{MT}g) = 0.$$

Sufficiency. Consider the initial-value problem

$$\begin{cases} -(py')' + qy = f_1, & x \in (a, c_1), \\ y(a) = 0, \\ y^{[1]}(a) = 0. \end{cases}$$

By the theory of existence and uniqueness of solutions, the problem has a unique solution $\psi_1(x)$. Similarly, let $\psi_{i+1}(x) (i = 1, \cdots, m)$ be the unique solution of the following initial-value problem;

$$\begin{cases} -(py')' + qy = f_{i+1}, \text{ if } i = 1, \cdots, m-1, \ x \in (c_i, c_{i+1}) \text{ and if } i = m, \ x \in (c_m, b), \\ y(c_i) = \dfrac{1}{\delta_i}[(\gamma_{i3}\delta_{i2} - \gamma_{i1}\delta_{i4})\psi_i(c_i) + (\gamma_{i4}\delta_{i2} - \gamma_{i2}\delta_{i4})\psi_i^{[1]}(c_i)], \\ y^{[1]}(c_i) = \dfrac{1}{\delta_i}[(\gamma_{i1}\delta_{i3} - \gamma_{i3}\delta_{i1})\psi_i(c_i) + (\gamma_{i2}\delta_{i3} - \gamma_{i4}\delta_{i1})\psi_i^{[1]}(c_i)]. \end{cases}$$

Then, $\psi(x) = \begin{cases} \psi_1(x), & x \in (a, c_1), \\ \psi_2(x), & x \in (c_1, c_2), \\ \cdots \\ \psi_{m+1}(x), & x \in (c_m, b), \end{cases}$ is a solution of $ly = f$ satisfying the transmission conditions (1.3) and (1.4), and $\psi(a) = \psi^{[1]}(a) = 0$. Obviously, $\psi(x) \in \mathcal{D}(L_{MT})$. Let $g \in \mathcal{D}(L_{MT})$, such that $lg = 0$. Then

$$(L_{MT}\psi, g) - (\psi, L_{MT}g) = \frac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m}(\psi(b)\overline{g}^{[1]}(b) - \psi^{[1]}(b)\overline{g}(b)).$$

Note that $(L_{MT}\psi, g) - (\psi, L_{MT}g) = (L_{MT}\psi, g) = (f, g) = 0$. Therefore, for each $g \in \mathcal{D}(L_{MT})$ such that $lg = 0$, we have $\psi(b)\overline{g}^{[1]}(b) - \psi^{[1]}(b)\overline{g}(b) = 0$. By Theorem 1.1, we may choose $g \in \mathcal{D}(L_{MT})$ such that $g(b) = 0$, but $g^{[1]}(b) \neq 0$. So $\psi(b) = 0$. Similarly, we get $\psi^{[1]}(b) = 0$. ■

By above some preliminary theorems, we have the following theorem:

Theorem 1.3. *The maximal and minimal operators L_{0T}, L_{MT} associated with T are an adjoint pair in the sense that $L_{0T}^* = L_{MT}$, and $L_{MT}^* = L_{0T}$.*

Proof. (1) By (1.7), $L_{MT} \subset L_{0T}^*$. In the following, we prove that $L_{0T}^* \subset L_{MT}$. Let $g, g^* \in H$, and for every $f \in \mathcal{D}(L_{0T})$, $(lf, g) = (f, g^*)$. Denote $\psi(x) \in \mathcal{D}(L_{MT})$ be a solution of $ly = g^*$. Then for every $f \in \mathcal{D}(L_{0T})$, from (1.7), we have

$$(lf, g) = (f, g^*) = (f, l\psi) = (lf, \psi).$$

Therefore, $(lf, g - \psi) = 0$. Since, $f \in \mathcal{D}(L_{0T})$ is arbitrary, the above equation shows that $g - \psi \in \mathcal{R}_{L_{0T}}^{\perp}$. By Theorem 1.2, we obtain $y = g - \psi \in \mathcal{N}(L_{MT})$. So $g - \psi = y \in \mathcal{D}(L_{MT})$, i.e., $g = \psi + y \in \mathcal{D}(L_{MT})$, and $lg = l\psi + ly = l\psi = g^*$. It implies that $L_{0T}^* \subset L_{MT}$. Hence, $L_{0T}^* = L_{MT}$.

(2) $L_{MT}^* \subset L_{0T}^* = L_{MT}$ by $L_{0T} \subset L_{MT}$. Therefore, L_{MT}^* is a differential operator generated by ly, and $\mathcal{D}(L_{MT}^*) \subset \mathcal{D}(L_{MT})$. For any $u, v \in \mathcal{D}(L_{MT})$, by (1.6), we know that $v \in \mathcal{D}(L_{MT}^*)$, if and only if for any $u \in \mathcal{D}(L_{MT})$,

$$\frac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m}(u(b)\overline{v}^{[1]}(b) - u^{[1]}(b)\overline{v}(b)) - (u(a)\overline{v}^{[1]}(a) - u^{[1]}(a)\overline{v}(a)) = 0. \quad (1.9)$$

By Theorem 1.1, we may choose $u(b)$, $u^{[1]}(b)$, $u(a)$, and $u^{[1]}(a)$ independently. We assert that for all $u \in \mathcal{D}(L_{MT})$, (1.9) holds, if and only if $v(b) = v^{[1]}(b) = v(a) = v^{[1]}(a) = 0$, i.e., $v \in \mathcal{D}L_{0T}$. So, $\mathcal{D}(L_{MT}^*) = \mathcal{D}(L_{0T})$. ■

Theorem 1.4. *The adjoint operator T^* of T satisfies $L_{0T} \subset T^* \subset L_{MT}$.*

Proof. This follows from (1.5) and Theorem 1.3. ■

2. The Necessary and Sufficient Conditions for Self-Adjointness

Theorem 2.1. *Let T^* denote the adjoint operator of T. Then $v \in \mathcal{D}(T^*)$, if and only if $v \in \mathcal{D}(L_{MT})$ and*

$$\frac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m} W(u, \overline{v}; b) - W(u, \overline{v}; a) = 0,$$

for every $u \in \mathcal{D}(T)$.

Proof. Let $v \in \mathcal{D}T^*$. By Theorem 1.4, $v \in \mathcal{D}(L_{MT})$. For every $u \in \mathcal{D}(T)$, we have $(Tu, v) - (u, T^*v) = 0$. Hence, $\frac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m} W(u, \overline{v}; b) - W(u, \overline{v}; a) = 0$. For the converse, if $v \in \mathcal{D}(L_{MT})$ and $\frac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m} W(u, \overline{v}; b) - W(u, \overline{v}; a) = 0$, for each $u \in \mathcal{D}(T)$, then $(lu, v) - (u, lv) = 0$, i.e., $v \in \mathcal{D}(T^*)$. ■

Theorem 2.2. *The operator T is self-adjoint, if and only if $\mathcal{D}$ satisfies*

(1) $\mathcal{D}(T) \subset \mathcal{D}(L_{MT})$;

(2) *For any* $u, v \in \mathcal{D}(T)$, $\frac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m} W(u, \overline{v}; b) - w(u, \overline{v}; a) = 0$;

(3) *If* $v \in \mathcal{D}(L_{MT})$ *and the equality* $\frac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m} W(u, \overline{v}; b) - w(u, \overline{v}; a) = 0$, *holds for any* $u \in \mathcal{D}(T)$, *then* $v \in \mathcal{D}(T)$.

Proof. The proof is similar to Theorem 2.1. We omit the details. ■

Below, we will give the directly analytic description of the self-adjoint boundary conditions. For terminology, such as boundary forms, mutually adjoint boundary conditions, etc., see [5]. We use the notation $*$ to denote the complex conjugate transpose of matrix.

By Theorems 2.1 and 2.2, we obtain the next conclusion, which specifies the conditions on the adjoint boundary conditions of operator T.

Theorem 2.3. *Let* $y \in \mathcal{D}(L_{MT})$, *and* $V(y) = SY(a) + BY(b)$ *be two-dimensional boundary forms. Then, the boundary conditions* $U(y) = AY(a) + BY(b) = 0$ *of* T *and* $V(y) = 0$ *are mutually adjoint, if and only if*

$$\frac{1}{\gamma_1 \cdots \gamma_m} AJ^{-1}S^* = \frac{1}{\delta_1 \cdots \delta_m} BJ^{-1}F^*, \; J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proof. Set $\widetilde{C}(y) = \begin{pmatrix} Y(a) \\ Y(b) \end{pmatrix}$, then $U(y) = AY(a) + BY(b) = (A|B)\widetilde{C}(y)$, where $\operatorname{rank}(A|B) = 2$. Let A_1, B_1 be 2×2 matrices, such that $N = \begin{pmatrix} A & B \\ A_1 & B_1 \end{pmatrix}$ is a nonsingular matrix. Then $U_c(y) = A_1Y(a) + B_1Y(b) = (A_1|B_1)\widetilde{C}(y)$ are boundary forms, which are complementary to the original boundary forms $U(y)$. Thus, $\widetilde{U}(y) = \begin{pmatrix} U(y) \\ U_c(y) \end{pmatrix} = N\widetilde{C}(y)$. Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $Z = \begin{pmatrix} z \\ z^{[1]} \end{pmatrix}$, then $W(y, \bar{z}; x) = \langle JY(x), Z(x) \rangle = Z^*(x)JY(x)$, where $\langle ., . \rangle$ denotes the inner product of complex Euclidean space. Put $\widetilde{J} = \begin{pmatrix} -J & 0 \\ 0 & \dfrac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m} J \end{pmatrix}$. Then

$$\frac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m} W(y, \bar{z}; b) - W(y, \bar{z}; a) = \langle \widetilde{J}\widetilde{C}(y), \widetilde{C}(z) \rangle = \langle \widetilde{U}(y)(\widetilde{J}N^{-1})^*, \widetilde{C}(z) \rangle.$$

Obviously, $\widetilde{J}N^{-1}$ is nonsingular. Set

$$\widetilde{V}(z) = \begin{pmatrix} V_c(z) \\ V(z) \end{pmatrix} = (\widetilde{J}N^{-1})^*\widetilde{C}(z),$$

where $V(z) = SZ(a) + FZ(b)$, $V_c(z) = \widetilde{S}Z(a) + \widetilde{F}Z(b)$. Then, we have

$$\widetilde{V}(z) = \begin{pmatrix} \widetilde{S} & \widetilde{F} \\ S & F \end{pmatrix} \widetilde{C}(z), \quad (\widetilde{J}N^{-1})^* = \begin{pmatrix} \widetilde{S} & \widetilde{F} \\ S & F \end{pmatrix}.$$

Hence,

$$\widetilde{J} = \begin{pmatrix} -J & 0 \\ 0 & \dfrac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m} J \end{pmatrix} = \begin{pmatrix} \widetilde{S}^* & S^* \\ \widetilde{F}^* & F^* \end{pmatrix} N = \begin{pmatrix} \widetilde{S}^* & S^* \\ \widetilde{F}^* & F^* \end{pmatrix} \begin{pmatrix} A & B \\ A_1 & B_1 \end{pmatrix},$$

we have

$$E = \begin{pmatrix} -J^{-1}\widetilde{S}^* & -J^{-1}S^* \\ \dfrac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m} J^{-1}\widetilde{F}^* & \dfrac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m} J^{-1}F^* \end{pmatrix} \begin{pmatrix} A & B \\ A_1 & B_1 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} A & B \\ A_1 & B_1 \end{pmatrix} \begin{pmatrix} -J^{-1}\widetilde{S}^* & -J^{-1}S^* \\ \dfrac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m} J^{-1}\widetilde{F}^* & \dfrac{\delta_1 \cdots \delta_m}{\gamma_1 \cdots \gamma_m} J^{-1}F^* \end{pmatrix} = E.$$

We obtain $\dfrac{1}{\gamma_1 \cdots \gamma_m} AJ^{-1}S^* = \dfrac{1}{\delta_1 \cdots \delta_m} BJ^{-1}F^*$.

Conversely, assume that there exist 2×2 matrices S_1 and F_1 such that

$$\operatorname{rank}(S_1|F_1) = 2, \text{ and } \frac{1}{\gamma_1 \cdots \gamma_m} AJ^{-1}S^* = \frac{1}{\delta_1 \cdots \delta_m} BJ^{-1}F^*, \qquad (2.10)$$

by $\operatorname{rank}(A|B) = 2$, let $\mathcal{F}$ denote the space generated by two linearly independent row vectors of $(A|B)$. From (2.10), we have

$$(A|B) \begin{pmatrix} -J^{-1}S^* \\ \dfrac{\gamma_1 \cdots \gamma_m}{\delta_1 \cdots \delta_m} J^{-1}F_1{}^* \end{pmatrix} = 0.$$

This shows that all column vectors of the matrix $R_1 = \begin{pmatrix} -J^{-1}S^* \\ \dfrac{\gamma_1 \cdots \gamma_m}{\delta_1 \cdots \delta_m} J^{-1}F_1^* \end{pmatrix}$ belong to $\mathcal{F}^{\perp}$. However,

$$R_1 = \begin{pmatrix} -J^{-1} & 0 \\ 0 & \dfrac{\gamma_1 \cdots \gamma_m}{\delta_1 \cdots \delta_m} J^{-1} \end{pmatrix} \begin{pmatrix} S_1^* \\ F_1^* \end{pmatrix} = \widetilde{F}^{-1}(S_1|F_1)^*.$$

It is obvious that $\operatorname{rank}(R_1) = 2$, by $\operatorname{rank}(S_1|F_1) = 2$. So, two linearly independent column vectors of R_1 form of a base of $\mathcal{F}^{\perp}$. Similarly, we have $\operatorname{rank}(R) = \operatorname{rank}\begin{pmatrix} -J^{-1}S^* \\ \dfrac{\gamma_1 \cdots \gamma_m}{\delta_1 \cdots \delta_m} J^{-1}F_1^* \end{pmatrix} = 2$, and its two column vectors consist of a base of the space $\mathcal{F}^{\perp}$. Therefore, there exists 2×2

nonsingular matrix P such that $R_1 = RP^*$, i.e., $\widetilde{J}^{-1}(S|F)^* = \widetilde{J}^{-1}P^*$. So, $(S_1|F_1) = P(S|F)$. We have $S_1 = PS$ and $F_1 = PF$. Hence, $V_1(z) = S_1Z(a) + F_1Z(b) = PV(z)$, and $V_1(z) = 0$ are the adjoint boundary conditions of $U(y) = 0$. ■

Lemma 2.1. *The differential operator T is self-adjoint, if and only if* $U(y) = AY(a) + BY(b) = 0$ *are self-adjoint boundary conditions.* ■

The next theorem is our main result in this section; it gives the sufficient and necessary condition on the boundary conditions for self-adjointness of the operator T.

Theorem 2.4. *The operator T is self-adjoint, if and only if*

$$\frac{1}{\gamma_1 \cdots \gamma_m} AJ^{-1}A^* = \frac{1}{\delta_1 \cdots \delta_m} BJ^{-1}B^*, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proof. From Lemma 2.1 and Theorem 2.3, we can obtain the direct conclusion. ■

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