Journal of Mathematical Sciences: Advances and Applications
Volume 4, Number 2, 2010, Pages 227-244

GENERALIZED MATRICES FOR LIPSCHITZ FUNCTIONS

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Abstract

It is proved that matrices $p''_{xq}(\cdot,\cdot)$ of the second mixed derivatives of the support function $p(\cdot,\cdot)$ for an introduced set-valued mapping $D_\alpha f(\cdot)$ for Lipschitz function $f(\cdot)$ exist almost everywhere in $R^n \times R^n$. Matrices $p''_{xq}(\cdot,\cdot)$ are similar to matrices of second derivatives for smooth functions. The Clarke subdifferential of the set-valued mapping $D_\alpha f(\cdot)$, α- generalized and their continuous extension α, δ- generalized matrices for $f(\cdot)$ were defined. Under some assumption, a continuous extension of the Clarke, subdifferential and the form of $p''_{xq}(\cdot,\cdot)$, when $f(\cdot)$ is twice continuous differentiable were found. An optimization method is proposed for using these matrices in nonsmooth optimization.

1. α- Subdifferential for Lipschitz Functions

In [3], [4], qualities of a set-valued mapping $D_\alpha f(\cdot)$, $\alpha > 0$, for a Lipschitz function $f(\cdot) : R^n \to R$ were defined and studied. For a convex

2010 Mathematics Subject Classification: Lipschitz Functions.

Keywords and phrases: Lipschitz functions, Lipschitz set-valued mappings, support function, generalized matrices for Lipschitz functions, Clarke subdifferential for set-valued mappings, Newton's optimization process.

Submitted by Zeyad Abdel Aziz Mah'd Al Zhour.

Received April 1, 2009; Revised June 7, 2009

function $f(\cdot): R^n \to R$, this mapping has the analogous qualities similar to qualities of the ε-subdifferential mapping [5]. We will call the set $D_\alpha f(x)$ α-subdifferential of the function $f(\cdot)$ at the point x.

According to the definition [4],

$$D_\alpha f(x) = \overline{co}\{v \in R^n \mid v = \alpha^{-1}\int_0^\alpha f(r(x, \tau, g))\, d\tau$$

$$\forall r(x, ., g) \in \eta(x), \forall g \in S_1^{n-1}(0) = \{w \in R^n \mid \|w\| = 1\}\},$$

where $\eta(x)$ is a set of the curves $r(x, \tau, g) = x + \alpha g + o(\alpha)$, and a curve $o(\cdot)$ satisfies the following conditions: $\|o(\alpha)\| / \alpha \to 0$ when $\alpha \to +0$, and a norm of the derivative $o'(\cdot)$ bounded by some constant $c > 0$ for some $\alpha_0 > 0$, i.e.,

$$\max_{\tau \in [o, \alpha_0]} \|o'(\tau)\| \le c.$$

The following theorem was proved [4].

Theorem 1.1. *If $f(\cdot)$ is a Lipschitz function, then the set-valued mapping $D_\alpha f(\cdot)$ is a Lipschitz set-valued mapping with a Lipschitz constant L_α.*

In this paper, the Clarke subdifferential of the set-valued mapping $D_\alpha f(\cdot)$ is introduced. The important problem for optimization is to find a continuous extension of the Clarke subdifferential. Under some assumption such extension is found.

The Clarke subdifferential of $D_\alpha f(\cdot)$ is defined using $p''_{xq}(\cdot, \cdot)$, where $p(\cdot, \cdot)$ is the support function of images of the set-valued mapping $D_\alpha f(\cdot)$. Below we will find the form of $p''_{xq}(\cdot, \cdot)$, when $f(\cdot)$ is twice continuous differentiable. The problem is to find a form of $p''_{xq}(\cdot, \cdot)$ in common case, when $f(\cdot)$ is any Lipschitz function.

2. The Clarke Subdifferential for $D_\alpha f(\cdot)$

Let us define $p_{D\alpha}(\cdot,\cdot) : R^n \times R^n \to R$

$$p_{D\alpha}(\cdot,\cdot) = \max_{v \in D_\alpha f(x)}(v, q).$$

This is called the support function of the set-valued mapping $D_\alpha f(\cdot)$. Interesting qualities of the support function $p_{D\alpha}(\cdot,\cdot)$ will follow from Theorem 1.1. It can be proved that the function $p_{D\alpha}(\cdot,\cdot)$ is Lipschitz in the variables (x, q), i.e.,

$$| p_{D\alpha}(x_1, q_1) - p_{D\alpha}(x_2, q_2) | \le L_2 \| z_1 - z_2 \|$$

in any compact set $D \times B_m^1(0) \subset R^n \times R^m$, where $z_1 = (x_1, q_1)$, $z_2 = (x_2, q_2)$,

$$L_1(x) = \max_{v \in D_\alpha f(x)} \|v\|, \; L_2 = \max_{x \in D}(L_\alpha, L_1(x)),$$

and $B_1^m(0) = \{w \in R^m \mid \|w\| \le 1\}$ is a unit ball in R^m.

The following theorem was proved in [1].

Theorem 2.1. *The support function $p_{D\alpha}(\cdot,\cdot)$ has the second mixed derivatives $p''_{xq}(\cdot,\cdot)$ almost everywhere (a.e.) in $R^n \times R^n$ for every Lipschitz function $f(\cdot)$.*

The Theorem 2.1 is true for any Lipschitz set-valued mapping $G(\cdot) : R^n \to R^n$. If $G(\cdot) = f(\cdot)$, where $f(\cdot) : R^n \to R$ is any Lipschitz function, then $p''_{xq}(x, q) = \nabla f(x)$ for any $q \in S_1^{n-1}(0)$, $x \in N(f)$, where $N(f)$ is a set where $f(\cdot)$ is differentiable.

We will denote a set, where $p''_{xq}(\cdot,\cdot)$ exists for the set-valued mapping $D_\alpha f(\cdot)$ by Λ_α.

We can define the Clarke subdifferential for any Lipschitz set-valued mappings like the Clarke subdifferential for a Lipschitz function.

Consider sets defined for any Lipschitz set-valued mapping $G(\cdot) : R^n \to R^m$ with convex compact images and the support function $p(\cdot, \cdot)$ having the second mixed derivative $p''_{xq}(\cdot, \cdot)$ on a set $N(G)$, that is, a dense set in $R^n \times R^m$ of full measure

$$M(x, q) := co\,\{A \in R^{n\times m} \mid A = \lim_{k\to\infty} p''_{xq}(\xi_k, q_k),$$

$$(\xi_k, q_k) \to_k (x, q), (\xi_k, q_k) \in N(G)\},$$

$$M(x) := co \bigcup\nolimits_{q\in S_1^{m-1}(0)} M(x, q),$$

where $M(x)$ is a convex compact subset in $R^{n\times m}$ [1].

Consider the following function

$$\psi(x, g, q) = \overline{\lim}_{\alpha\to+0, u\to 0, \Delta q\to 0} \frac{p(x + u + \alpha g, q + \Delta q) - p(x + u, q + \Delta q)}{\alpha}.$$

Theorem 2.2 [1]**.** *The equality*

$$\psi(x, g, q) = \max\nolimits_{A\in M(x)} (Ag, q)$$

is true for any $g \in R^n$ *and* $q \in R^m$.

The set $M(x)$ is similar to the Clarke subdifferential of Lipschitz functions. All qualities of the Clarke subdifferential are true for the set-valued mapping $M(\cdot)$ (for instance, the upper semicontinuity).

Let us introduce a set for any $x \in R^n$ and $\alpha > 0$

$$D_{2,\alpha} f(x) = co\,\{A[n \times n] \mid \exists\{\alpha_k\},\ \alpha_k \to +0,\ \exists g,\ q \in S_1^{n-1}(0),$$

$$\exists (r_1(x_m, .., g), r_2(., q)) \in \eta_2(x_m),$$

$$x_m \to_m x,\ A = \lim_{\alpha_k\to 0, m\to\infty} \alpha_k^{-1} \int_0^{\alpha_k} p''_{xq}(r_1(x_m, \tau, g), r_2(\tau, q))\, d\tau\},$$

where $\eta_2(x_m)$ is a set of all pairs of curves $(r_1(x_m, .., g), r_2(., q))$ for all $g, q \in R^n$:

$$r_1(x_m, \alpha, g) = x_m + \alpha g + o_{1,m}(\alpha),\ r_2(\alpha, q) = q + O_{2,m}(\alpha).$$

The vector-functions $o_{1,m}(\cdot)$, $O_{2,m}(\cdot)$, $i = 1, 2$, $m = 1, 2, \ldots$, satisfy the following conditions

$$\lim_{\alpha \to +0} o_{1m}(\alpha) / \alpha = 0, \quad \max_{\tau \in [0, \alpha_0]} \| o'_{i_m}(\tau) \| \le c,$$

$$\lim_{\alpha \to +0} O_{2m}(\alpha) = 0, \quad \max_{\tau \in [0, \alpha_0]} \| O'_{2_m}(\tau) \| \le c,$$

for some $c > 0$ and $\alpha_0 > 0$. Moreover, the matrices $p''_{xq}(\cdot, \cdot)$ exist almost everywhere (a.e.) along pairs of the curves $(r_{1,m}(\cdot), r_{2,m}(\cdot))$, $i = 1, 2$, for $\tau \in [0, \alpha_0]$.

Lemma 2.1. *The set $D_{2,\alpha} f(x_0)$ is a convex compact set.*

Proof. Since $\| p''_{xq}(r_1(\cdot), r_2(\cdot)) \| \le L_\alpha$ [1], where L_α is a Lipschitz constant of the set valued mapping $D_\alpha f(\cdot)$, the set $D_{2,\alpha} f(x_0)$ is bounded.

Let $(r_1(x_m, ., g_k), r_2(., q_k)) \in \eta_2(x_m)$ for some sequences $g_k, q_k \in S_1^{n-1}(0)$, $k = 1, 2, \ldots$, for that $g_k \to g \in S_1^{n-1}(0)$, $q_k \to q \in S_1^{n-1}(0)$, $k_m \to_m +\infty$, and

$$A_{k_m} = \lim_{m \to \infty} \alpha_{k_m}^{-1} \int_0^{\alpha_{k_m}} p''_{xq}(r_1(x_m, \tau, g_k), r_2(\tau, q_k)) d\tau \to_k A_0$$

for $\alpha_{k_m} \to_m +0$.

We will construct a pair of curves $(r_1(x_m, \tau, g), r_2(\tau, q)) \in \eta_2(x_m)$ that for each k and m coincide with the curves $r_1(x_m, ., g_k)$, $r_2(., q_k)$ for $\alpha \in [0, \alpha_0]$. Replacing g_k in g and q_k in q, we change the curves $r_1(x_m, \cdot, g_k)$, $r_2(\cdot, q_k)$ by vectors $\alpha_{k_m}(g_k - g) = o(\alpha_{k_m})$ and $q_k - q = O(\alpha_{k_m})$, and get the curves $r_1(x_m, ., g)$, $r_2(., q)$ from the set $\eta_2(x_m)$. It is not difficult to prove that by replacing the vectors g_k and q_k in vectors g and q correspondently the parameter α_{k_m} will replace in $\overline{\alpha}_{k_m}$. The correlation

$$\frac{\overline{\alpha}_{k_m}}{\alpha_{k_m}} \to 1,$$

when $m \to \infty$ is true. Therefore, we can put the parameter α_{k_m}, instead of $\overline{\alpha}_{k_m}$. From here, we have

$$A_0 = \lim_{k \to +\infty,\, m \to +\infty} \alpha_{k_m}^{-1} \int_0^{\alpha_{k_m}} p''_{xq}(r_1(x_m, \tau, g), r_2(\tau, q))\, d\tau.$$

Consequently, $A_0 \in D_{2,\alpha} f(x_0)$. Lemma 2.1 is proved. □

We will prove that $D_{2,\alpha} f(x)$ is also the Clarke subdifferential for the set-valued mapping $D_\alpha f(\cdot)$.

Theorem 2.3. *The sets $M(x)$ and $D_{2,\alpha} f(x)$ coincide with each other.*

Proof. First, we will prove that

$$M(x) \subset D_{2,\alpha} f(x). \tag{1}$$

Let $A \in M(x)$ and

$$\psi(x, g, q) = \sup \lim_{\alpha \to +0,\, u \to 0,\, \Delta q \to 0} \frac{p(x + u + \alpha g, q + \Delta q) - p(x + u, q + \Delta q)}{\alpha}$$

$$= \lim_{k \to \infty} (1 / \alpha_k) \int_0^{\alpha_k} (p''_{xq}(\xi_k(\tau), \eta_k(\tau)) \xi'_k(\tau), \eta_k(\tau))\, d\tau + o(\alpha_k) / \alpha_k$$

$$= \lim_{k \to \infty} (A_k g, q + \Delta g_k),$$

where $\xi_k(\tau) = x + u_k + \tau g + o_1(\tau)$, $\eta_k(\tau) = q + \Delta q_k + o_2(\alpha)$, $A_k = p''_{xq}$ $(\xi_k(\tau), \eta_k(\tau))$, and $u_k, \Delta q_k \to 0$ when $k \to \infty$.

According to the supposition, the limit A_0 of the matrices A_k, when $k \to \infty$ exists at the point, where the upper limit is achieved and A_0 belongs to the set

$$Q(x, g, q) = \{A[n \times n] \in M(x) \,|\, (Ag, q) = \max_{B \in M(x)} (Bg, q)\}.$$

From this, the inclusion (1) follows.

We will prove the reverse inclusion. From the upper semicontinuity of the set-valued mapping $M(\cdot)$, it follows that for any $\varepsilon > 0$ and $g, q \in S_1^{n-1}(0)$, there is $\delta > 0$ that

$$p''_{xq}(x + h, q) \in M(x) + B_\varepsilon^{n^2}(0), \tag{2}$$

if $\|h\| \le \delta$, where $B_\varepsilon^{n^2}(0)$ is a ball of radius ε in R^{n^2} and with the center in 0.

Let be for some sequences $\{\alpha_k\}, \{h_m\}, \{q_m\}$

$$A = \lim_{k\to\infty}(1/\alpha_k)\int_0^{\alpha_k}(p''_{xq}(\xi_k(\tau), \eta_k(\tau))\xi'_k(\tau), \eta_k(\tau))\,d\tau,$$

where $\xi_k(\tau), \eta_k(\tau)$ are the same as above. It is easy to check that from (2), the inequality

$$\rho_H(A, M(x)) \le \varepsilon$$

follows, ρ_H is the Hausdorff metric. Since ε is any positive number,

$$A \in M(x),$$

i.e., the inclusion

$$D_{2,\alpha}f(x) \subset M(x) \tag{3}$$

is true. From (1) and (3), we will get the statement of the theorem. □

Since the set-valued mapping $M(\cdot)$ is upper semicontinuous, as it follows from Theorem 2.3, the set-valued mapping $D_{2,\alpha}f(\cdot)$ is upper semicontinuous as well.

3. Generalized Matrices

Any element from the set $D_{2,\alpha}f(x_0)$, we will call an α-generalized matrix of the function $f(\cdot)$ at the point x_0.

Let us define for any fixed $\alpha, \delta > 0$ a set

$$D_{2,\alpha,\delta}f(x_0) = \overline{co}\{A[n\times n] \mid \exists g,\ q \in S_1^{n-1}(0),\ \exists(r_1(x_0,\ .\ ,\ g),\ r_2(.\ ,\ q)) \in \eta_2(x_0)$$

$$A = \delta^{-1}\int_0^{\delta} p''_{xq}(r_1(x_0,\ \tau,\ g), r_2(\tau,\ q))\,d\tau\},$$

where the integration is taken along a pair of curves $(r_1(x_0,\ .\ ,\ g),\ r_2(.\ ,\ q)) \in \eta_2(x_0)$. The elements of the set $D_{2,\alpha,\delta}f(x_0)$, we will call α, δ- generalized matrices.

We will not prove the upper, lower semicontinuities and continuity of the set-valued mapping $D_{2,\alpha,\delta}f(\cdot)$, as soon as, these qualities follow from Theorem 3.1.

Theorem 3.1. *The set-valued mapping $D_{2,\alpha,\delta}f(\cdot)$ is a Lipschitz set-valued mapping.*

Proof. It is well known that

$$\rho_H(D_{2,\alpha,\delta}f(x_1),\ D_{2,\alpha,\delta}f(x_2)) = \max_{g\in S_1^{n^2-1}(0)} \mid p_{D_{2,\alpha,\delta}}(x_1,\ g) - p_{D_{2,\alpha,\delta}}(x_2,\ g)\mid,$$

where

$$p_{D_{2,\alpha,\delta}}(x,\ g) = \max_{v\in D_{2,\alpha,\delta}f(x)}(v,\ g)$$

is the support function of the set-valued mapping $D_{2,\alpha,\delta}f(\cdot)$. Without loss of generality, we will consider that

$$\max_{g\in S_1^{n^2-1}(0)} \mid p_{D_{2,\alpha,\delta}}(x_1,\ g) - p_{D_{2,\alpha,\delta}}(x_2,\ g)\mid = p_{D_{2,\alpha,\delta}}(x_1,\ \overline{g}) - p_{D_{2,\alpha,\delta}}(x_2,\ \overline{g}),$$

for some $\overline{g} \in S_1^{n^2-1}(0)$.

Let

$$A_i \in \arg\max_{A\in D_{2,\alpha,\delta}f(x_i)}(A,\ \overline{g}),$$

where $(\cdot,\cdot)$ is the scalar production of elements of R^{n^2}.

We will take curves $(r_1(x_1,\ .\ ,\ \hat{g}),\ r_2(.\ ,\ \hat{q})) \in \eta_2(x_1)$, $\overline{g} = \overline{g}(\hat{g},\ \hat{q})$, for that

$$A_1 = \delta^{-1}\int_0^{\delta} p''_{xq}(r_1(x_1, \tau, \hat{g}), r_2(\tau, \hat{q}))\, d\tau.$$

Let us consider also a matrix

$$\overline{A}_2 = \delta^{-1}\int_0^{\delta} p''_{xq}(r_1(x_2, \tau, \hat{g}), r_2(\tau, \hat{q}))\, d\tau,$$

where $(r_1(x_2, \tau, \hat{g}), r_2(\tau, \hat{q}))$ is any pair of curves of the set $\eta_2(x_2)$. Since

$$(A_2, \bar{g}) \geq (\overline{A}_2, \bar{g}),$$

then

$$p_{D_{2,\alpha,\delta}}(x_1, \bar{g}) - p_{D_{2,\alpha,\delta}}(x_2, \bar{g}) \leq (A_1, \bar{g}) - (\overline{A}_2, \bar{g}) \leq \|A_1 - \overline{A}_2\|. \quad (4)$$

Here $\|\cdot\|$ is the norm that corresponds to the scalar production $(\cdot, \cdot)$ introduced earlier.

By analogy with the proof of the theorem about Lipschitz quality of the set-valued mapping $D_\alpha f(\cdot)$ [4] and taking into account that [1].

$$\|p''_{xq}(\cdot, \cdot)\| \leq L_\alpha,$$

where L_α is a Lipschitz constant for the set-valued mapping $D_\alpha f(\cdot)$, we can prove that the curves $(r_1(x_2, ., g), r(., q)$ can be chosen such that

$$\|A_1 - \overline{A}_2\| \leq \delta^{-1} L_\alpha c_4 \|x_1 - x_2\|, \quad (5)$$

where c_4 is some constant. Then from (4) and (5), the statement of the theorem follows. The theorem is proved. □

Let us construct the set-valued mapping $\overline{D}_{2,\alpha,\delta} f(\cdot) : R^n \to R^n$, for any $\delta > 0$

$$\overline{D}_{2,\alpha,\delta} f(x) = \overline{co} \bigcup_{\eta \in [0,\delta]} D_{2,\alpha,\eta} f(x).$$

We define $D_{2,\alpha,0} f(x)$ as a convex hull of all limit vectors $v_k \in D_{2,\alpha,\eta_k} f(x)$ for any sequence $\{\eta_k\}$ going to 0.

Theorem 3.2. *The set-valued mapping* $\overline{D}_{2,\alpha,\delta}f(\cdot)$ *is continuous at the point x in the Hausdorff metric, if* $D_{2,\alpha,0}f(x) = D_{2,\alpha}f(x)$.

Proof. Consider the set-valued mapping $B_{\delta_1}(\cdot) : R^n \to R^n$ for any small $\delta_1 > 0$,

$$B_{\delta_1}(\cdot) = co \bigcup\nolimits_{\eta\in[\delta_1,\delta]} D_{2,\alpha,\eta}f(\cdot).$$

It is apparent that

$$B_{\delta_1}(x) \subset \overline{D}_{2,\alpha,\delta}f(x),$$

and

$$\lim\nolimits_{\delta_1\to+0} \rho_H(B_{\delta_1}(x),\ \overline{D}_{2,\alpha,\delta}f(x)) = 0,$$

where ρ_H is the Hausdorff metric.

As long as, the set-valued mapping $D_{2,\alpha,\eta}f(\cdot)$ is continuous for all $\eta \in [\delta_1, \delta]$, then $B_{\delta_1}(\cdot)$ is also continuous.

It will follows from the upper semicontinuity of the set-valued mapping $D_{2,\alpha}f(\cdot)$ that for any $\varepsilon > 0$, a number $\beta = \beta(\varepsilon) > 0$ exists that a vector

$$\gamma^{-1}\int_0^{\gamma} p''_{xq}(r_1(x_k,\ \tau, g),\ r_2(\tau,\ q))\,d\tau,$$

belongs to the set $D_{2,\alpha}f(x) + B_{\varepsilon}^{n^2}(0)$, if $\|x_k - x\| < \beta$ and $0 < \gamma < \beta$.

The set $\overline{D}_{2,\alpha,\delta}f(x_k)$ can be represented as the convex hull of the sets $B_{\delta_1}(x_k)$ and $\overline{D}_{2,\alpha,\delta}f(x_k)$. It is not difficult to prove the inequality

$$\rho_H(\overline{D}_{2,\alpha,\delta}f(x_k),\ \overline{D}_{2,\alpha,\delta}f(x)) \le \rho_H(B_{\delta_1}(x),\ B_{\delta_1}(x_k))$$

$$+ \rho_H(\overline{D}_{2,\alpha,\delta}f(x) \setminus B_{\delta_1}(x),\ \overline{D}_{2,\alpha,\delta}f(x_k) \setminus B_{\delta_1}(x_k)).$$

Since all terms of this inequality from the right side can be made any small for small β and δ_1, it will follows the statement of the theorem. □

To construct the sets $D_{2,\alpha}f(\cdot)$, $D_{2,\alpha,\delta}f(\cdot)$, and $\overline{D}_{2,\alpha,\delta}f(\cdot)$, it is necessary to write the matrix $p''_{xq}(\cdot,\cdot)$ in a clear form. Now, we give an expression for the matrix $p''_{xq}(\cdot,\cdot)$, when the function $f(\cdot,\cdot)$ is twice continuously differentiable, i.e., $f(\cdot) \in C^2$. Further, in the following articles, we will consider common case.

Let $v(x_0, q)$ be a vector for that

$$v(x_0, \overline{q}) = \arg\max_{v \in D_\alpha f(x_0)}(v, \overline{q}), \|\overline{q}\| = 1,$$

and $(x_0, \overline{q}) \in \Lambda_\alpha$. There are several cases of location of the vector $v(x_0, \overline{q})$.

(a) The vector $v(x_0, \overline{q})$ has a support cone $K(v(x_0, \overline{q}))$ with nonempty interior, i.e., $\operatorname{int} K(v(x_0, \overline{q})) \neq \emptyset$. Since, $(x_0, \overline{q}) \in \Lambda_\alpha$, $\overline{q} \in \operatorname{int} K(v(x_0, \overline{q}))$. Let

$$v(x_0, \overline{q}) = \alpha^{-1}\int_0^\alpha \nabla f(r(x_0, \tau, g))\, d\tau, \tag{6}$$

for some $r(x_0, ., g) \in \eta(x_0)$. It is obvious, when we move to a point $x_0 + \Delta x$, then the vector $v(x_0, \overline{q})$ moves to the vector $v(x_0 + \Delta x, \overline{q})$ and $\overline{q}$ belongs to a cone $K(v(x_0 + \Delta x, \overline{q}))$ for small $\|\Delta x\|$.

Consequently, we can take at the point $x_0 + \Delta x$, a curve $r(x_0 + \Delta x, \tau, g) = x_0 + \Delta x + \tau(g + o(\Delta x, \tau)) + o(\tau)$ to obtain a vector $v(x_0 + \Delta x, \overline{q})$. It is easy to calculate the derivative of the vector $v(x_0, \overline{q})$ as a function of the variable x_0. Indeed, we can cover the curve $r(x_0, \tau, g)$ by balls with centers on the curve $r(x_0, \tau, g)$, where the function $f(\cdot)$ is twice differentiable and choose a finite number of such balls. So, we will have an approximation for the integral

$$\alpha^{-1}\int_0^{\alpha}\nabla f(r(x_0+\Delta x,\,\tau,g))\,d\tau.$$

Taking limit on number of the balls and expanding the integrable function in powers of Δx, we obtain

$$v(x_0+\Delta x,\,\overline{q}) = \alpha^{-1}\int_0^{\alpha}\nabla f(r(x_0+\Delta x,\,\tau,g))\,d\tau$$

$$= \alpha^{-1}\int_0^{\alpha}(\nabla f(r(x_0,\,\tau,\,g)) + \nabla^2 f(r(x_0,\,\tau,g))\Delta x + o(\Delta x))\,d\tau$$

$$= v(x_0,\,\overline{q}) + \alpha^{-1}(\int_0^{\alpha}\nabla^2 f(r(x_0,\,\tau,g))\,d\tau)\Delta x + o(\Delta x). \qquad (7)$$

The linear part of the expression (7) is

$$\alpha^{-1}\int_0^{\alpha}\nabla^2 f(r(x_0,\,\tau,g))\,d\tau.$$

Consequently,

$$p''_{xq}(x_0,\,\overline{q}) = \nabla v(x_0,\,\overline{q}) = \alpha^{-1}\int_0^{\alpha}\nabla^2 f(r(x_0,\,\tau,g))\,d\tau. \qquad (8)$$

Since $(x_0,\,\overline{q}) \in \Lambda_\alpha$, the value of $\nabla v(x_0,\,\overline{q})$ does not depend on choice of the curve $r(x_0,\,.\,,\,g)$ for that Equation (6) is true.

(b) Consider the case when $K(v(x_0,\,\overline{q})) = \{\overline{q}\}$. Let us define a function $\varphi(\cdot,\cdot) : R^n \times R^n \to R$

$$\varphi(x_0,\,w) = \sup_{o(\alpha):x_0+\alpha w+o(\alpha)\in\eta(x_0)}(f(x_0+\alpha w+o(\alpha)) - f(x_0))/\alpha,$$

for $\|w\| = 1$, and

$$\varphi(x_0,\,\lambda w) = \lambda\varphi(x_0,\,w),\ \lambda \geq 0.$$

We can assume that the function $\varphi(x_0,\,.)$ is convex in a neighborhood of the point $\overline{q}$. Since $(x_0,\,\overline{q}) \in \Lambda_\alpha$, we can also assume that there is an one to one relation between the boundary vectors near the vector $v(x_0,\,\overline{q})$ and the vectors $q \in S_1^{n-1}(O)$ near the vector $\overline{q}$.

The case when the function $\varphi(x_0, .)$ is concave in a neighborhood of the point $\overline{q}$ can be considered similarly.

Let equality (6) be true for the vector $v(x_0, \overline{q})$ and the curve $r(x_0, \alpha, g)$. To obtain a point $v(x_0 + \Delta x, \overline{q})$, we must choose in this case a curve $r(x_0 + \Delta x, \tau, g) = x_0 + \Delta x + \tau(g + o(\Delta x, \tau)) + o(\Delta x)$ for some $o(\Delta x, \tau)$, i.e.,

$$v(x_0 + \Delta x, \overline{q}) = \alpha^{-1} \int_0^{\alpha} \nabla f(r(x_0 + \Delta x, \tau, g))\, d\tau + o(\Delta x), \tag{9}$$

where $o(\Delta x) / \Delta x \to_{\Delta x \to 0} 0$ and

$$\lim_{\tau \to 0} \frac{o(\Delta x, \tau)}{\tau} = 0,$$

uniformly on small Δx.

The vector function $o(\Delta x)$ appears because we take instead of the vector g, a vector $g + o(\Delta x, \tau)$. For case, when the function $f(\cdot)$ is twice differentiable, from (9), where $r(x_0 + \Delta x, \tau, g) = x_0 + \Delta x + \tau g$, we obtain also formula (8). It will be independent of choice of the curve $r(x_0, ., g)$, if (6) is true, as long as, we took the point $(x_0, \overline{q})$ from the set Λ_α.

Remark 3.1. It is not easy to calculate the matrix $p''_{xq}(\cdot, \cdot)$ because we have to know a curve $r(\cdot)$, on which the support function $p(\cdot, \cdot)$ reaches its value.

Remark 3.2. If a curve $r(x_0, ., g)$ does not exist for the vector $v(x_0, \overline{q}) \in D_\alpha f(x_0)$, $(x_0, \overline{q}) \in \Lambda_\alpha$ that

$$v(x_0, \overline{q}) = \alpha^{-1} \int_0^{\alpha} \nabla f(r(x_0, \tau, g))\, d\tau,$$

then there is a sequence $\{v_k\}$, $v_k(x_0) \in D_\alpha f(x_0)$, $v_k \to v$, and a sequence $\{r_k(\cdot) \in \eta_\alpha\}$ that

$$v_k(x_0) = \alpha^{-1}\int_0^{\alpha} \nabla f(r_k(x_0, \tau, g))\, d\tau.$$

We will take such differentiable vector-function $v_k(x_0)$ that the following relations hold for some sequence $\{\alpha_k\}$ and any vector $g_1 \in S_1^{n-1}(0)$:

1. $|(v_k(x_0 + \alpha_k g_1), \overline{q}) - (v_k(x_0), \overline{q}) - p_{D_\alpha}(x_0 + \alpha_k g_1, \overline{q}) + p_{D_\alpha}(x_0, \overline{q})| \le \varepsilon_k$,

2. $v_k(x_0 + \alpha_k g_1) = v_k(x_0) + \alpha_k \nabla v_k(x_0) g_1 + o_k(\alpha_k)$,

where $\varepsilon_k / \alpha_k \to 0$ and $o_k(\alpha_k)/\alpha_k \to 0$ as $k \to \infty$. We can always satisfy with two conditions. To satisfy with the second condition, it is sufficiently to choose such sequence $\{v_k(x_0)\}$ for that the limit

$$\lim_{k\to\infty} \nabla v_k(x_0),$$

exists that we can do easily because $\|\nabla v_k(x_0)\|$ are bounded for all k. For such sequence $\{v_k\}$, we will have

$$\nabla v(x_0, \overline{q}) = \lim_{k\to\infty} \nabla v_k(x_0).$$

Consider the following example

Example 3.1. Let us define the function $f(\cdot) : R^n \to R$

$$f(x) = c^T x + \frac{1}{2} x^T A x,$$

for that

$$\nabla f(x) = c^T + Ax.$$

If the matrix A is positive, then $f(\cdot)$ is convex.

Calculate $p''_{xq}(\cdot, \cdot)$, if A is a positive symmetric matrix. The set $D_\alpha f(0)$ is equal to (under suggestion that $o(\cdot) \equiv 0$)

$$D_\alpha f(0) = co\{v(g) \mid v(g) = c^T + \frac{\alpha}{2} Ag \; \forall g \in S_1^{n-1}(0)\}.$$

It is apparent that

$$v(0, q) = \arg \max_{w \in D_\alpha f(0)} (w, q) = c^T + \frac{\alpha}{2} Ag,$$

where $q = \dfrac{Ag}{\|Ag\|}$. Move to a point Δx. The set $D_\alpha f(\Delta x)$ is equal to

$$D_\alpha f(\Delta x) = co\{w(g) \in R^n \mid w(g) = c^T + A\Delta x + \frac{\alpha}{2} Ag \; \forall g \in S_1^{n-1}(0)\}.$$

It is obviously that

$$v(\Delta x, q) = \arg \max_{w \in D_\alpha f(\Delta x)} (w, q) = c^T + A\Delta x + \frac{\alpha}{2} Ag,$$

where q is the same as above. From here, $p''_{xq}(0, g) = \nabla v(0, g) = A$.

With the help of the matrix $p''_{xq}(\cdot, \cdot)$, we can build an optimization process for the function $f(\cdot)$. Let us consider the following optimization problem

$$f(\cdot) \to \inf_{x \in R^n}.$$

We will use an idea from [2]. Firstly, we will consider instead of the function $f(\cdot)$, a function $\bar{f}(\cdot) : R^n \to R$

$$\bar{f}(x) = f(x) + \overline{L}\|x - x_k\|,$$

where $\overline{L} = \max(L, L_\alpha)$, if we stay at a point x_k. Secondly, we will construct the set $D_\alpha \bar{f}(x_k)$:

$$D_\alpha \bar{f}(x_k) = \overline{co}\{v \in R^n \mid v = \alpha^{-1} \int_0^\alpha \nabla \bar{f}(r(x_k, \tau, g))\, d\tau\}.$$

Thirdly, we must find a decent direction g_k of the function $f(\cdot)$ at the point x_k, if the necessary optimal condition for α-optimality at the point x_k

$$\overline{L} B_1^n(0) \subset D_\alpha \bar{f}(x_k)$$

is not true. In this case, we must find the vectors

$$v_k = \arg_v \max_{w \in \overline{L} B_1^n(0)} \min_{v \in D_\alpha \bar{f}(x_k)} \|v - w\|, \tag{10}$$

and

$$w_k = \arg_w \max_{w \in \overline{L} B_1^n(0)} \min_{v \in D_\alpha \bar{f}(x_k)} \|v - w\|, \tag{11}$$

and then put

$$q_k = \frac{w_k - v_k}{\|w_k - v_k\|}.$$

If the second mixed derivative $A(x_k, q_k) = p''_{xq}(x_k, q_k)$ of the support function of the set-valued mapping $D_\alpha \bar{f}(\cdot)$ exists together with its inverse matrix (it can be always reached, if we add to our function some quadratic function like $M\|x - x_k\|^2$, where $M > 2\overline{L}$), then we can use this matrix to find the next approximate point like in Newton's method

$$x_{k+1} = x_k - \beta_k A(x_k, q_k)^{-1}(v_k - w_k),$$

where β_k is a step satisfying the condition

$$f(x_{k+1}) < f(x_k) - \beta_k (L_A / 2)\|\Delta_k\|,$$

where

$$\Delta_k = A(x_k, q_k)^{-1}(v_k - w_k),$$

and L_A is the lower bound of $\|A(x_k, q_k)\|$.

If the vector q_k is not unique, we will take any vector q_k for that the following expression is true

$$q_k = \arg_q \min \|A(x_k, q)^{-1}(v - w)\|,$$

where $q = q(v, w)$ for $v \in D_\alpha \bar{f}(x_k)$, $w \in \overline{L} B_1^n(0)$, satisfying to (10) and (11), and

$$q(v, w) = \frac{w - v}{\|w - v\|}.$$

In such a way, we can construct processes like Newton's process for smooth functions to find an α-optimal point. If the second derivative $p''_{xq}(x_k, q_k)$ does not exist, then we can choose in any neighborhood of the point (x_k, q_k) and another point $(\bar{x}_k, \bar{q}_k)$, where $p_{xq}(\bar{x}_k, \bar{q}_k)$ exists. We can do it because the set-valued mapping $D_\alpha \bar{f}(\cdot)$ is continuous.

The difficulty of this problem is that in common case, the matrix $A(x_k, q_k)$ is not continuous in x_k and q_k. Continuity of matrix of second derivatives of a target function is very important in Newton's method. Therefore, we must construct some continuous approximation for the matrix $A(x_k, q_k)$ that we have to look for it among the matrices $D_{2,\alpha,\delta} f(x)$.

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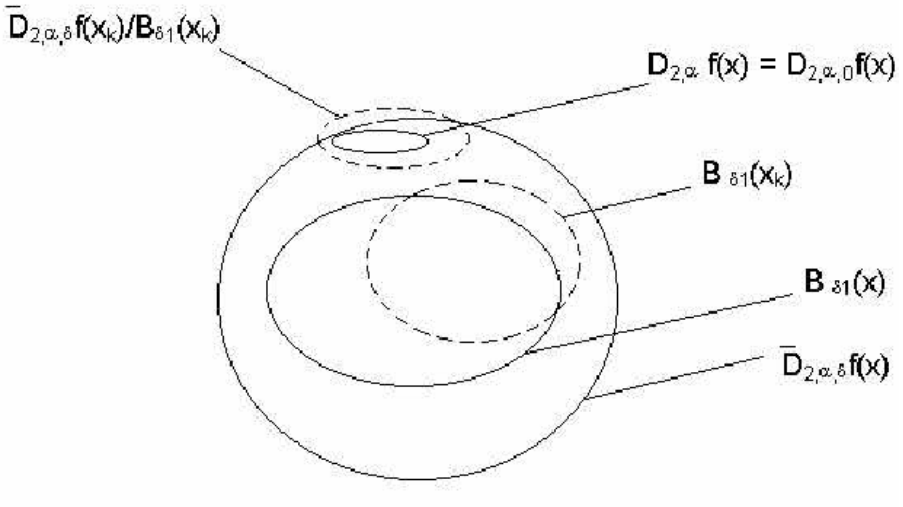
$\bar{D}_{2,\alpha,\delta} f(x_k)/B_{\delta 1}(x_k)$
$D_{2,\alpha} f(x) = D_{2,\alpha,0} f(x)$
$B_{\delta 1}(x_k)$
$B_{\delta 1}(x)$
$\bar{D}_{2,\alpha,\delta} f(x)$

Figure 1.