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THE EXISTENCE OF LEAST ENERGY SOLUTION FOR A SINGULARLY PERTURBED EQUATION

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Abstract

In this paper, we study the existence of least energy solution of nonlinear singularly perturbed equation

$$\begin{cases} -\varepsilon^2 \Delta u + u = f(u), & \text{in } \Omega, \\ u > 0, \text{ in } \Omega, \ u = 0, \text{ on } \partial\Omega, \end{cases}$$

under a super-quadratic condition, which is weaker than the Ambrosetti-Rabinwitz condition. We also give asymptotic expansion for critical value of corresponding functional.

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1. Introduction

In this paper, we consider a semi-linear equation of the form

$$\begin{cases} -\varepsilon^2 \Delta u + u = f(u), & \text{in } \Omega, \\ u > 0, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, Ω is a domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $\varepsilon > 0$ is a small parameter, and $f : \mathbb{R} \to \mathbb{R}$ is a C^1 mapping.

First of all, we recall some definitions. $H^1(\Omega) := \{u \in L^2(\Omega);\ \partial^\alpha u \in L^2(\Omega),\ |\alpha| \le 1\}$. $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm

$$\| u \| = \left(\int_\Omega (|\nabla u|^2 + u^2) \right)^{\frac{1}{2}}.$$

In recent years, the singularly perturbed elliptic equations have received considerable attention, due to their significant applications in many fields, such as in quantum mechanics, optical materials, chemical reactor theory, and the biological population. For background of this equation, we refer to the introduction in [6].

In [11], Del Pino et al. found the solutions of (1.1), and proved these solutions concentrate at prescribed finite set of local maxima of the distance function $d(x, y)(x \in \Omega,\ y \in \partial\Omega)$. In [20], Wei and Weth showed that for ε small, (1.1) has a nodal solution with k positive local maximum points and k negative local minimum points, this solution has at least $k+1$ nodal domains, and the locations of the maximum and minimum points are related to the mean curvature on $\partial\Omega$. Byeon and Park in [7] studied (1.1) on a connected compact smooth Riemannian manifold, they proved there exists a mountain pass solution u_ε of (1.1), which exhibits a spike layer.

For the equation

$$-\varepsilon^2 \Delta u + u = |u|^{p-2} u, \tag{1.2}$$

where $1 < p < \frac{N+2}{N-2}$. Floer and Weinstein first constructed a standing wave of (1.2) in one dimension space [12], and they based on the Lyapunov-Schmidt reduction. Oh extended their result to higher dimensions in [17, 18]. Ni and Wei established the existence of a "spike-layer" solution, and determined the location of the peak as well as the profile of the spike in [16]. In [4, 5], Benci and Cerami studied the multiplicity solutions of (1.2) by using of category and Morse theory. In [19], Wei obtained necessary conditions for the existence of two-peaked solutions of (1.2). The works by Ambrosetti et al. [3], extended some work of Ambrosetti et al. [2], which proved solutions concentrating on spheres.

Among various assumptions forced on f, the Ambrosetti-Rabinowitz growth condition (A-R condition for short) is most frequent appeared in super-linear problems.

A-R condition. There exists $\theta > 2$, such that for $t > 0$,

$$0 < \theta F(t) \le tf(t),$$

where and in the following, $F(t) = \int_0^t f(s)ds$.

It implies that there is a $C > 0$ such that $F(u) \ge C|u|^\theta$. This condition plays an important role in establishing the mountain pass geometry, as well as in obtaining the boundness of (PS) sequence.

Instead of the A-R condition, Liu and Wang [15] first posed a much natural super-quadratic conditions:

(f_1) $\lim_{t\to\infty} \frac{F(t)}{t^2} = \infty$,

to get the bounds of minimizing sequence on Nehari manifold, and under coercive condition of $V(x)$, they proved the existence of three solutions of equation $-\Delta u + V(x)u = f(x,u)(u \in H^1(\mathbb{R}^N))$, one positive, one negative, and one sign-changing. The results in [14] by Li et al. are natural generalization of that in [15] to noncompact cases, the two cases do not have compact embedding.

Moreover, the solvability of (1.1) depends on the growth rate of $f(t)$ at infinity, which is always compared with the function $|t|^{2^*-1}$, where $2^* = \frac{2N}{N-2}$, if $N \geq 3$. In the current paper, we consider the following assumption:

(f_2) $f(t) = O(t^{q-1})$, as $t \to +\infty$, where $2 < q < 2^*$, if $N \geq 3$, and $2 < q < +\infty$, if $N = 2$.

In this paper, we try to find a least energy solution of (1.1), under a weaker assumption on the nonlinearity. We also give asymptotic expansion for critical value of corresponding functional. The proof of main result relies on variational arguments, together with techniques in [14, 15] and a penalization type method.

2. Main Result

The energy functional associated to Equation (1.1) is

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega (\varepsilon^2|\nabla u|^2 + u^2) - \int_\Omega F(u), \quad u \in H_0^1(\Omega).$$

We denote $H_0^1(\Omega)$ by X. Assume domain Ω satisfies assumption (H), that is:

(H) There is a smooth bounded domain Λ, such that $0 \in \Lambda$, $\overline{\Lambda} \subset \Omega$, and $\text{int}(\Lambda) \neq \emptyset$.

And the function f in Equation (1.1) satisfies the following conditions in addition to (f_1) and (f_2):

(f_3) $f(t) \equiv 0$ for $t \leq 0$, $f(t) = o(t)$ as $t \to 0$.

(f_4) $t \to \frac{f(t)}{t}$ is strictly increasing for small $t > 0$.

To state the mainly result Theorem 2.1, we need following preparations:

Consider the problem in the whole space

$$\begin{cases} -\Delta w + w = f(w), \quad w > 0 \quad \text{in } \mathbb{R}^N, \\ w(0) = \max\limits_{z\in\mathbb{R}^N} w(z), \quad w(z) \to 0, \ |z| \to \infty. \end{cases} \tag{2.1}$$

The corresponding functional is

$$I(w) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) - \int_{\mathbb{R}^N} F(w). \tag{2.2}$$

From [16, p.733], it is known that any solution to (2.1) needs to be spherically symmetric about the origin, and strictly decreasing in $r = |z|$. And from [16], we see that a solution w to (2.1) is said to be *nondegenerate*, if the linearized operator $\Delta - 1 + f'(w)$ has a bounded inverse on the space $\{u \in L^2(\mathbb{R}^N) \mid u(z) = u(|z|)\}$. We assume

(f_5) Equation (2.1) has a unique solution w and it is nondegenerate.

Define $\Omega_\varepsilon = \{y \in \mathbb{R}^N \mid x_\varepsilon + \varepsilon y \in \Omega\}$, and define w_{Ω_ε} to be the unique solution of the following linear equation:

$$\begin{cases} -\Delta u + u = f(w), \quad u > 0, \quad \text{in } \Omega_\varepsilon, \\ u = 0, \quad \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2.3}$$

Now we state our main result.

Theorem 2.1. *Assume* $(f_1) \sim (f_5)$ *and* (H), *for* ε *sufficiently small, Equation* (1.1) *has a least energy solution* u_ε, *and asymptotic expansion for critical value of corresponding functional is*

$$c_\varepsilon = \varepsilon^N \{I(w) + (\frac{1}{2}\int_{\mathbb{R}^N} f(w)\tilde{v}) \cdot h(x) + o(h(x))\}, \tag{2.4}$$

where $h(x) = \varphi_\varepsilon(\dfrac{x - x_\varepsilon}{\varepsilon})$, $\varphi_\varepsilon = w - w_{\Omega_\varepsilon}$, x_ε *is the peak point of* u_ε, $\tilde{v}$ *satisfies*

$$-\Delta\tilde{v} = \tilde{v}, \ \tilde{v} > 0, \ \tilde{v}(0) = 1, \quad in \ \mathbb{R}^N. \tag{2.5}$$

Remark 2.2. The function $f(t) = t^p (t \geq 0, 1 < p < 2^* - 1)$ satisfies the assumptions $(f_1) \sim (f_5)$. About (f_5), the uniqueness of the solution of (2.1), one can see [8, 13].

3. Preliminaries

We modify the function f outside the set Λ as that done in [9]. Define

$$g(\cdot, t) = \chi_\Lambda(\cdot) f(t) + (1 - \chi_\Lambda(\cdot))\widetilde{f}(t), \tag{3.1}$$

where χ_Λ denotes the characteristic function on Λ, and

$$\widetilde{f}(t) = \begin{cases} f(t), & \text{if } t \leq a, \\ f(a) + f'(a)(t - a), & \text{if } t > a, \end{cases} \tag{3.2}$$

where $a > 0$ such that $f'(a) < 1$, and for any $t \geq a$, $f(t) \geq f(a) + f'(a)(t - a)$. Now we define a new equation

$$\begin{cases} -\varepsilon^2 \Delta u + u = g(x, u), \text{ in } \Omega, \\ u > 0, \text{ in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

and the corresponding functional is

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega (\varepsilon^2 |\nabla u|^2 + u^2) - \int_\Omega G(x, u),$$

where $G(x, t) = \int_0^t g(x, s)ds$. It is easy to check that $(f_1) \sim (f_4)$ implies g defined above is a Caratheodory function, and the following conditions are satisfied:

(g_1) (Λ is defined in the assumption (H)):

(i) $\lim\limits_{t\to\infty} \dfrac{G(z, t)}{t^2} = \infty$, for all $z \in \Lambda$, $t > 0$,

(ii) $0 \leq 2G(z, t) \leq g(z, t)t \leq f'(a)t^2$, for all $t > a$, $z \notin \Lambda$.

(g_2) There is a $2 < q < 2^*$ such that, $g(z, t) = O(t^{q-1})$, as $t \to \infty$, uniformly in $z \in \Omega$.

(g_3) $g(z, t) \equiv 0$, for $t \le 0$; $g(z, t) = o(t)$, as $t \to 0$, $z \in \Omega$.

(g_4) $t \to \dfrac{g(z, t)}{t}$ is strictly increasing for small $t > 0$, $z \in \Omega$.

Remark 3.1. If $f(t)$ is replaced by t^p and modify $f(t)$ as (3.1), then $g(z, t)$ satisfies conditions $(g_1) \sim (g_4)$.

Theorem 3.2. *Under the conditions* $(g_1) \sim (g_4)$ *and* (H), *problem* (3.3) *has a least energy solution* u_ε, *such that*

$$J_\varepsilon(u_\varepsilon) = \max_{t>0} J_\varepsilon(tu_\varepsilon) = \inf_{v\in X} \max_{t>0} J_\varepsilon(tv) > 0.$$

Proof. By (g_3), for given $\sigma > 0$, there exists $\delta > 0$ such that $|G(x, u)| \le \dfrac{\sigma}{2}|u|^2$ for $|u| \le \delta$. By (g_2), there exists $R > 0$ such that $|G(x, u)| \le M|u|^q$ for $|u| \ge R$. So $|G(x, u)| \le \dfrac{\sigma}{2}|u|^2 + M|u|^q$, then

$$\int_\Lambda G(x, u) \le \frac{\sigma}{2}\int_\Lambda |u|^2 + M\int_\Lambda |u|^q \le C_0\left(\frac{\sigma}{2} + M\|u\|^{q-2}\right)\|u\|^2.$$

Choosing $\|u\| \le \left(\dfrac{\sigma}{2M}\right)^{\frac{1}{q-2}}$, we have $\int_\Lambda G(x, u) \le C_0\sigma\|u\|^2$. Thus,

$$J_\varepsilon(u) \ge \frac{1}{2}\varepsilon^2\|u\|^2 + \frac{1-\varepsilon^2}{2}\int_\Omega u^2 - C_0\sigma\|u\|^2 - \int_{\Omega\setminus\Lambda}\frac{1}{2}f'(a)u^2$$

$$\ge \left(\frac{\varepsilon^2}{2} - C_0\sigma\right)\|u\|^2 + \left(\frac{1-\varepsilon^2}{2} - \frac{1}{2}f'(a)\right)\int_\Omega u^2$$

$$\ge \left(\frac{\varepsilon^2}{2} - C_0\sigma\right)\|u\|^2 + C_0\left(\frac{1-\varepsilon^2}{2} - \frac{1}{2}f'(a)\right)\|u\|^2.$$

For ε small, choosing $\sigma < \dfrac{\varepsilon^2}{2}\left(\dfrac{1}{C_0} - 1\right) + \dfrac{1}{2}(1 - f'(a))$ and taking $\|u\| = \left(\dfrac{\sigma}{2M}\right)^{\frac{1}{q-2}}$, we have $J_\varepsilon(u) \ge \alpha > 0$. Thus $J_\varepsilon(u)$ has a strict local minimum at 0. Note that for any $u \ne 0$, $J_\varepsilon(tu) \to -\infty$ as $t \to \infty$. Hence,

$$c = \inf_{v \in X \setminus \{0\}} \max_{t>0} J_\varepsilon(tu) > 0$$

is well defined. Let $\{u_n\}$ be a minimizing sequence of c such that $J_\varepsilon(u_n) = \max_{t>0} J_\varepsilon(tu_n) \to c$ as $n \to \infty$. In fact, $\{u_n\}$ is bounded. If not, define $v_n = u_n / \|u_n\|$, passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in X. Then $v_n \to v$ in $L^p_{loc}(\Omega)$, $p \in [2, 2^*)$.

If $v(x) \equiv 0$ in Ω, we choose $s > \dfrac{\sqrt{2c}}{\varepsilon}$, then

$$J_\varepsilon(u_n) \geq J_\varepsilon(sv_n) = \frac{1}{2} s^2\varepsilon^2 + \int_\Omega (1-\varepsilon^2)s^2 v_n^2 - \int_\Omega G(x, sv_n).$$

The left hand side of this inequality tends to c, but the right hand side converges to $\dfrac{\varepsilon^2 s^2}{2} > c$, so $v(x) \not\equiv 0$ in Ω. Then there is a set $A \subset \Omega$, $\mathrm{mes}(A) > 0$, such that $v(x) \neq 0$ in A. From $u_n = \|u_n\| \cdot v_n$, we have $|u_n| \to \infty$ on A as $n \to \infty$. Moreover,

$$\frac{c + o(1)}{\|u_n\|^2} = \frac{\varepsilon^2}{2} - \int_\Omega (1-\varepsilon^2)\frac{u_n^2}{\|u_n\|^2} - \int_\Omega \frac{G(x, u_n)}{\|u_n\|^2},$$

then

$$\frac{\varepsilon^2}{2} \geq \int_\Omega \frac{G(x, u_n)}{\|u_n\|^2} = \int_\Omega \frac{G(x, u_n)}{u_n^2} v_n^2.$$

From Fatou's lemma, for any fixed $\varepsilon > 0$,

$$\frac{\varepsilon^2}{2} \geq \liminf_{n\to\infty} \int_A \frac{G(x, u_n)}{u_n^2} v_n^2 \geq \int_A \liminf_{n\to\infty} \frac{G(x, u_n)}{u_n^2} \cdot \liminf_{n\to\infty} v_n^2.$$

By (g_1), the right side tends to $+\infty$, which gives a contradiction. Thus $\{u_n\}$ is bounded. We may assume $u_n \rightharpoonup u$, then by [1, Lemma 2.1],

$$\int_\Omega g(x, u_n)u_n - (1-\varepsilon^2)\int_\Omega u_n^2 \to \int_\Omega g(x, u)u - (1-\varepsilon^2)\int_\Omega u^2, \text{ as } n \to \infty,$$

and $u \neq 0$. Since $u > 0$, by (g_4), there exists $s \geq 0$, $su \in \mathcal{N}(\mathcal{N} = \{u \in X \setminus \{0\} : J'_\varepsilon(u)u = 0\})$ such that $J_\varepsilon(su) = \max_{t>0} J_\varepsilon(tu)$ (see [15, Theorem 2.3]). It follows that

$$J_\varepsilon(su) \leq \liminf_{n\to\infty} J_\varepsilon(su_n) \leq \liminf_{n\to\infty} J_\varepsilon(u_n) = c.$$

Hence su is a minimizer of J_ε on $\mathcal{N}$. Define $h(t) = J_\varepsilon(tu)$, we prove $h(t)$ has a unique critical point for $t > 0$. Take derivative,

$$h'(t) = \langle J'_\varepsilon(tu), u\rangle = \int_\Omega (t\varepsilon^2|\nabla u|^2 + tu^2) - \int_\Omega ug(x, tu).$$

Let $h'(t) = 0$. We have

$$h''(t) = \int_\Omega (\varepsilon^2|\nabla u|^2 + u^2) - \int_\Omega u^2 g'(x, tu)$$

$$= \int_\Omega \frac{u}{t} g(x, tu) - \int_\Omega u^2 g'(x, tu).$$

It follows from (g_4) that

$$h''(t) = \frac{1}{t^2}\int_\Omega tug(x, tu) - \int_\Omega (tu)^2 g'(x, tu) < 0.$$

And $h'(t) = 0$ implies for some $t_0 > 0$,

$$\int_\Omega \varepsilon^2|\nabla u|^2 + u^2 = t_0^{-1}\int_\Omega ug(x, t_0u) = \int_\Omega u^2 \frac{g(x, t_0u)}{t_0u}.$$

For small $t > 0$, $h(t) > 0$, $h(t) \to -\infty$ as $t \to +\infty$. Again by (g_4), there exists a unique $t_0 > 0$ such that $h'(t_0) = 0$. It can be easily shown that su is a critical point of J_ε. □

Theorem 3.3. *Let u_ε be the solution obtained by Theorem* 3.2, *then*

$$\lim_{\varepsilon\to 0} \max_{x\in\partial\Lambda} u_\varepsilon(x) = 0.$$

Proof. We argue by contradiction. If $\varepsilon_n \downarrow 0$ and $x_n \in \overline{\Lambda}$ such that $u_{\varepsilon_n}(z_n) \geq b$ for some constant $b > 0$. Let

$$v_n(x) = u_{\varepsilon_n}(x_n + \varepsilon_n x).$$

Then $v_n(x)$ satisfies the equation

$$\begin{cases} -\Delta v_n + v_n = g(x_n + \varepsilon_n x, v_n), & \text{in } \Omega_n, \\ v_n = 0, & \text{on } \partial\Omega_n, \end{cases} \tag{3.4}$$

where $\Omega_n = \varepsilon_n^{-1}\{\Omega - x_n\}$. We can get $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Without loss of generality, we may assume that $v_n \rightharpoonup v$. From Lemma 2.3 in [10], we can further assume that $\{v_n\}$ converges to $v \in H^1(\mathbb{R}^N)$ and $\{\chi_\Lambda(z_n + \varepsilon_n z)\}$ converges weakly in L^p to some function $\chi \equiv 1$. Therefore, v satisfies the equation

$$-\Delta v + v = \overline{g}(x, v), \quad x \in \mathbb{R}^N, \tag{3.5}$$

where $\overline{g}(z, s) = \chi(z)f(s) + (1 - \chi(z))\widetilde{f}(s)$. Define $\overline{J} : H^1(\mathbb{R}^N) \to \mathbb{R}$ as

$$\overline{J}(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2) - \int_{\mathbb{R}^N}\overline{G}(x, u),$$

where $\overline{G}(z, s) = \int_0^s \overline{g}(z, \tau)d\tau$. Then v is a critical point of $\overline{J}$. We also set

$$J_n(u) = \frac{1}{2}\int_{\Omega_n}(|\nabla u|^2 + u^2) - \int_{\Omega_n}\overline{G}(x_n + \varepsilon_n x, u),$$

then $J_n(v_n) = \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n})$. Furthermore, we claim that

$$\liminf_{n\to\infty} J_n(v_n) \geq \overline{J}(v). \tag{3.6}$$

Let $h_n(x) = \frac{1}{2}(|\nabla v_n|^2 + v_n^2) - \overline{G}(x_n + \varepsilon_n x, v_n)$. Choose $R > 0$ sufficiently large, it follows from $v_n \to v$ in $H^1(\mathbb{R}^N)$ that

$$\lim_{n\to\infty}\int_{B_R} h_n(x) = \frac{1}{2}\int_{B_R}(|\nabla v|^2 + v^2) - \int_{B_R}\overline{G}(x, v).$$

Since $v \in H^1(\mathbb{R}^N)$, for $\delta > 0$ and $R > 0$ sufficiently large, it holds that

$$\lim_{n\to\infty} \int_{B_R} h_n \geq \overline{J}(v) - \delta. \tag{3.7}$$

Choose a C_0^∞ cut-off function η_R such that $\eta_R \equiv 0$ on B_{R-1}, $\eta_R \equiv 1$ on $\mathbb{R}^N \setminus B_R$, $0 \leq \eta_R \leq 1$, and $|\nabla \eta_R| \leq C$. Let $\hat{v}_n = \eta_R v_n \in H^1(\Omega_n)$. Then

$$\begin{aligned} J_n'(v_n)\hat{v}_n &= \int_{\Omega_n} \nabla v_n \nabla \hat{v}_n + v_n \hat{v}_n - \int_{\Omega_n} \overline{g}(x_n + \varepsilon_n x, v_n)\hat{v}_n \\ &= \int_{B_R \setminus B_{R-1}} \nabla v_n \nabla(\eta_R v_n) + \eta_R v_n^2 \\ &\quad - \int_{B_R \setminus B_{R-1}} \overline{g}(x_n + \varepsilon_n x, v_n)\eta_R v_n + \int_{\Omega_n \setminus B_R} 2h_n \\ &\quad + \int_{\Omega_n \setminus B_R} 2\overline{G}(x_n + \varepsilon_n x, v_n) - \overline{g}(x_n + \varepsilon_n x, v_n)v_n. \end{aligned}$$

So, for $R > 0$ sufficiently large,

$$\lim_{n\to\infty} \left| \int_{B_R \setminus B_{R-1}} \nabla v_n \nabla(\eta_R v_n) + \eta_R v_n^2 - \int_{B_R \setminus B_{R-1}} \overline{g}(x_n + \varepsilon_n x, v_n)\eta_R v_n \right| \leq \delta.$$

It follows from $J_n'(v_n) = 0$ and $2\overline{G}(x_n + \varepsilon_n x, v_n) - \overline{g}(x_n + \varepsilon_n x, v_n)v_n \leq 0$ on $\Omega_n \setminus B_R$ that

$$\begin{aligned} \int_{\Omega_n \setminus B_R} 2h_n &= -\int_{\Omega_n \setminus B_R} (2\overline{G}(x_n + \varepsilon_n x, v_n) - \overline{g}(x_n + \varepsilon_n x, v_n)v_n) \\ &\quad - \Big(\int_{B_R \setminus B_{R-1}} \nabla v_n \nabla(\eta_R v_n) + \eta_R v_n^2 \\ &\quad - \int_{B_R \setminus B_{R-1}} \overline{g}(x_n + \varepsilon_n x, v_n)\,\eta_R v_n\Big) \\ &\geq -\int_{\Omega_n \setminus B_R} (2\overline{G}(x_n + \varepsilon_n x, v_n) - \overline{g}(x_n + \varepsilon_n x, v_n)v_n) - \delta \\ &\geq -\delta. \end{aligned} \tag{3.8}$$

From (3.7) and (3.8), (3.6) follows.

On the other hand, since v is a critical point of $\overline{J}$, $\overline{g}(x, v)$ satisfies (g_4), a similar proof to that of Theorem 3.2 gives that $\overline{J}(v) = \max_{\tau \geq 0} \overline{J}(\tau v)$. Since $f(s) \geq \widetilde{f}(s)$ for all s,

$$\overline{J}(v) \geq \inf_{u \in H^1(\Omega), u \neq 0} \sup_{\tau > 0} I(\tau u), \tag{3.9}$$

where $I(u)$ is defined in (2.2). This contradicts with (3.6). □

In order to prove Theorem 2.1, we need the following lemmas. Set $\widetilde{v}_\varepsilon = \frac{\varphi_\varepsilon}{h(x_\varepsilon)}$ ($h(x)$ is defined in Theorem 2.1), we have $\widetilde{v}_\varepsilon$ satisfies the equation $\Delta \widetilde{v}_\varepsilon = \widetilde{v}_\varepsilon$ in Ω_ε, $\widetilde{v}_\varepsilon(0) = 1$. And define $v_\varepsilon(y) = u_\varepsilon(x_\varepsilon + \varepsilon y)$, $\phi_\varepsilon(y) = \frac{v_\varepsilon(y) - w_{\Omega_\varepsilon}(y)}{h(x_\varepsilon)}$.

Lemma 3.4 [16, Proposition 3.4(ii)]**.** *For* $0 < \delta < 1$, *there is a constant* C, *such that for* $y \in \Omega_\varepsilon$,

$$v_\varepsilon(y) \leq C \cdot e^{-(1-\delta)|y|}.$$

Lemma 3.5 [16, Proposition 6.2]**.** (i) *For* $s > n$, *we have* $\|e^{-\mu|y|}\phi_\varepsilon\| \leq C(s)$, $1 - \sigma < \mu < 1$.

(ii) *For every sequence* $\varepsilon_k \to 0$, *there is a subsequence* $\varepsilon_{k_l} \to 0$ *such that* $\widetilde{v}_{\varepsilon_{k_l}} \to \widetilde{v}$, *where* $\widetilde{v}$ *is a positive solution of* (2.5).

(iii) *For every sequence* $\varepsilon_k \to 0$, *there is a subsequence* ε_{k_l} *and a solution* $\widetilde{v}$ *of* (2.5), *such that*

$$\|e^{-\mu|y|}(\phi_{\varepsilon_{k_l}} - \phi_0)\|_{L^\infty(B_{(1-\frac{1}{4}\sigma)})} \to 0, \quad \text{as } \varepsilon_{k_l} \to 0,$$

where ϕ_0 *is a solution of*

$$\Delta\phi_0 - \phi_0 + f'(w)(\phi_0 - \widetilde{v}) = 0, \text{ in } \mathbb{R}^N, \tag{3.10}$$

and w *is the unique solution of* (2.1).

Remark 3.6. The proof of Lemmas 3.4 and 3.5 needs the assumption (f_5) (see [16]).

4. Proof of Theorem 2.1

Proof of Theorem 2.1. By Theorem 3.3, there exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$,

$$u_\varepsilon(x) < a, \quad x \in \partial\Lambda,$$

where a is defined in (3.2). We claim that $u_\varepsilon(x) \le a$ for all $x \in \Omega \setminus \Lambda$. Choose $(u_\varepsilon - a)_+ = \max\{u_\varepsilon - a, 0\}$ as a test function in Equation (3.3). Integrating by parts, we have

$$\int_{\Omega\setminus\Lambda} \varepsilon^2 |\nabla(u_\varepsilon - a)_+|^2 + (1 - \frac{g(x, u_\varepsilon(x))}{u_\varepsilon(x)})(u_\varepsilon - a)_+^2 = 0. \tag{4.1}$$

From the definition of g, $1 - \frac{g(x, u_\varepsilon(x))}{u_\varepsilon(x)} > 0$ in $\Omega \setminus \Lambda$, hence all terms in (4.1) are zero. So, we get $u_\varepsilon(x) \le a$ for all $x \in \Omega \setminus \Lambda$, and then the solution u_ε of (3.3) is also a solution to (1.1).

Next, we give improved asymptotic expansion for c_ε upon [16, Theorem 2.3]. Define $c_\varepsilon = I_\varepsilon(u_\varepsilon)$,

$$\begin{aligned} c_\varepsilon &= \frac{1}{2}\int_\Omega (\varepsilon^2|\nabla u_\varepsilon|^2 + u_\varepsilon^2) - \int_\Omega F(u_\varepsilon) = \frac{1}{2}\int_\Omega u_\varepsilon f(u_\varepsilon) - \int_\Omega F(u_\varepsilon) \\ &= \varepsilon^n \{\frac{1}{2}\int_{\Omega_\varepsilon} v_\varepsilon f(v_\varepsilon) - \int_{\Omega_\varepsilon} F(v_\varepsilon)\}. \end{aligned} \tag{4.2}$$

Lemma 3.5 (i) and the Sobolev imbedding theorem imply that $\phi_\varepsilon \le Ce^{\mu|y|}$ for ε sufficiently small, where the constant C is independent of ε. By the mean-value theorem,

$$v_\varepsilon f(v_\varepsilon) = wf(w) + h(x_\varepsilon)(f(w_1) + w_1 f'(w_1))(\phi_\varepsilon - \widetilde{v}_\varepsilon), \tag{4.3}$$

where w and w_{Ω_ε} are defined in Section 2, w_1 lies between w and v_ε. Taking $\delta = \frac{\sigma}{20}$ in Lemma 3.4, we have

$$\begin{aligned}\int_{\Omega_\varepsilon} (f(w_1) + w_1 f'(w_1))(\phi_\varepsilon - \widetilde{v}_\varepsilon) &\le \int_{\Omega_\varepsilon} |(f(w_1) + w_1 f'(w_1))(\phi_\varepsilon - \widetilde{v}_\varepsilon)| \\ &\le C\int_{\Omega_\varepsilon} w_1^{1+\sigma} |\widetilde{v}_\varepsilon - \phi_\varepsilon| \\ &\le C\int_{\Omega_\varepsilon} e^{-(1-\delta)(1+\sigma)|y|}(e^{(1+\frac{1}{4}\sigma)|y|} + e^{\mu|y|}) \\ &\le C.\end{aligned}$$

By Lemma 3.5 (ii) and (iii), we have as $\varepsilon_{k_l} \to 0$ that

$$\int_{\Omega_{\varepsilon_{k_l}}} (f(w_1) + w_1 f'(w_1))(\phi_{\varepsilon_{k_l}} - \widetilde{v}_{\varepsilon_{k_l}}) \to \int_{\mathbb{R}^N} (f(w) + wf'(w))(\phi_0 - \widetilde{v}). \tag{4.4}$$

Thus

$$\int_{\mathbb{R}^N \setminus \Omega_\varepsilon} w|f(w)| \le C \cdot e^{-\frac{(2+\sigma_0)}{\varepsilon}} \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} e^{-(\sigma-\sigma_0)|y|} = o(h(x_\varepsilon)), \text{ for } \sigma_0 < \sigma.$$

Observe that ϕ_0 satisfies (3.10),

$$\begin{aligned}\int_{\mathbb{R}^N} wf'(w)\widetilde{v} &= \int_{\mathbb{R}^N} (\Delta\phi_0 - \phi_0 + f'(w)\phi_0)w \\ &= \int_{\mathbb{R}^N} (\Delta w - w)\phi_0 + f'(w)\phi_0 w \\ &= \int_{\mathbb{R}^N} (wf'(w) - f(w))\phi_0. \end{aligned} \tag{4.5}$$

Again by Lemma 3.5 and (4.3) ~ (4.5), then

$$\frac{1}{2}\int_{\Omega_\varepsilon} v_\varepsilon f(v_\varepsilon)$$

$$= \frac{1}{2}\int_{\Omega_{\varepsilon_{k_l}}} \{wf(w) - \varphi_{\varepsilon_{k_l}}\left(\frac{x - x_{\varepsilon_{k_l}}}{\varepsilon_{k_l}}\right)(f(w_1) + w_1 f'(w_1))(\tilde{v}_{\varepsilon_{k_l}} - \phi_{\varepsilon_{k_l}})\}$$

$$= \frac{1}{2}\int_{\mathbb{R}^N} wf(w) - \varphi_{\varepsilon_{k_l}}\left(\frac{x - x_{\varepsilon_{k_l}}}{\varepsilon_{k_l}}\right)\int_{\mathbb{R}^N} \frac{1}{2}(f(w) + wf'(w))(\tilde{v} - \phi_0)$$

$$+ o(\varphi_{\varepsilon_{k_l}}\left(\frac{x - x_{\varepsilon_{k_l}}}{\varepsilon_{k_l}}\right))$$

$$= \frac{1}{2}\int_{\mathbb{R}^N} wf(w) - \varphi_{\varepsilon_{k_l}}\left(\frac{x - x_{\varepsilon_{k_l}}}{\varepsilon_{k_l}}\right)\int_{\mathbb{R}^N}\left(\frac{1}{2}f(w)(\tilde{v} - \phi_0) + \frac{1}{2}wf'(w)\tilde{v}\right.$$

$$\left. - \frac{1}{2}wf'(w)\phi_0\right) + o(\varphi_{\varepsilon_{k_l}}\left(\frac{x - x_{\varepsilon_{k_l}}}{\varepsilon_{k_l}}\right))$$

$$= \frac{1}{2}\int_{\mathbb{R}^N} wf(w) - \varphi_{\varepsilon_{k_l}}\left(\frac{x - x_{\varepsilon_{k_l}}}{\varepsilon_{k_l}}\right)\int_{\mathbb{R}^N}\left(\frac{1}{2}f(w)\tilde{v} - f(w)\phi_0\right)$$

$$+ o(\varphi_{\varepsilon_{k_l}}\left(\frac{x - x_{\varepsilon_{k_l}}}{\varepsilon_{k_l}}\right)). \tag{4.6}$$

We estimate $\int_{\Omega_\varepsilon} F(v_\varepsilon)$ in (4.2) by a similar way as (4.6),

$$\int_{\Omega_\varepsilon} F(v_\varepsilon) = \int_{\mathbb{R}^N} F(w) + \varphi_{\varepsilon_{k_l}}\left(\frac{x - x_{\varepsilon_{k_l}}}{\varepsilon_{k_l}}\right)\int_{\mathbb{R}^N} f(w)(\phi_0 - \tilde{v}) + o(\varphi_{\varepsilon_{k_l}}\left(\frac{x - x_{\varepsilon_{k_l}}}{\varepsilon_{k_l}}\right)). \tag{4.7}$$

From (4.6), (4.7), and $\int_{\mathbb{R}^N} f(w)\tilde{v} > 0$, (2.4) holds for $\varepsilon = \varepsilon_{k_l}$. Since the sequence ε_k is arbitrary, it is easy to see that (2.4) holds for ε sufficiently small. The proof of Theorem 2.1 is complete.

□

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