Journal of Mathematical Sciences: Advances and Applications
Volume 7, Number 1, 2011, Pages 57-68

EXACT EXPLICIT TRAVELLING WAVE SOLUTION FOR THE GENERALIZED $(2+1)$-DIMENSIONAL BREAKING SOLITON EQUATION

SHU XIA, SHENGQIANG TANG and XINQIAN LIN

School of Mathematics and Computing Science
Guilin University of Electronic Technology
Guilin, Guangxi, 541004
P. R. China
e-mail: tangsq@guet.edu.cn

Abstract

The sine-cosine and the extended tanh method are used to construct exact solitary patterns solution and compacton solution of the generalized $(2+1)$-dimensional breaking soliton equation. The compactons solutions and solitary patterns solutions of the generalized $(2+1)$- dimensional breaking soliton equation are successfully obtained. These solutions may be important of significance for the explanation of some practical physical problems. It is shown that the sine-cosine and the extended tanh method provides a powerful mathematical tool for solving a great many nonlinear partial differential equations in mathematical physics.

2010 Mathematics Subject Classification: 35C08.

Keywords and phrases: solitary patterns solution, compacton, sine-cosine method, extended tanh method, generalized $(2+1)$- dimensional breaking soliton equation.

This research was supported by NNSF of China (10961011, 11061010).

Received December 3, 2010

1. Introduction

Recently, by using the sine-cosine method, Tascan and Bekir [8] studied the following $(2+1)$-dimensional breaking soliton equation [2]:

$$\begin{cases} u_t + \alpha u_{xxy} + 4\alpha u v_x + 4\alpha u_x v = 0, \\ u_y = v_x. \end{cases} \tag{1.1}$$

Equation (1.1) describes the $(2+1)$-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. Recently, Zhang has successfully extended the generalized auxiliary equation method to the $(2+1)$-dimensional breaking soliton equations in [12]. Some soliton-like solutions were obtained by the generalized expansion method of Riccati equation in [1]. Periodic wave solutions were obtained by the mapping method in [5]. New exact solutions were obtained by the singular manifold method in [6]. Very recently, Jacobi elliptic function solutions and their degenerate solutions are obtained by a generalized extended F-expansion method in [7].

In this paper, we consider the following generalized $(2+1)$-dimensional breaking soliton equation:

$$\begin{cases} (u^n)_t + \alpha(u^n)_{xxy} + a u v_x + b u_x v = 0, \\ u_y = v_x, \end{cases} \tag{1.2}$$

where α, a, b are arbitrary constants, n is non-zero integer. Specially, when $n = 1$, $a = b = 4\alpha$, Equation (1.2) becomes Equation (1.1).

The sine-cosine method and the tanh method will be mainly used to back up our analysis. The tanh method, developed by Malfliet [3, 4] and Wazwaz [10, 11], is a direct and effective algebraic method for handling many nonlinear equations. The sine-cosine method was proved to be powerful in handling nonlinear problems, with genuine nonlinear dispersion, where compactons and solitary patterns solutions are generated. The two methods will be described briefly, where details can be found in [3, 4, 9-11] and the references therein.

Let $u(x, y, t) = u(\xi)$, $\xi = x + y - ct$, where c is the wave speed. Then (1.2) becomes to

$$\begin{cases} -c(u^n)' + \alpha(u^n)''' + auv' + bu'v = 0, \\ u' = v', \end{cases} \tag{1.3}$$

where "$'$" is the derivative with respect to ξ. Integrating the second equation in the system and neglecting constants of integration, we find

$$u = v. \tag{1.4}$$

Substituting (1.4) into the first equation of (1.3), integrating once and setting integration constant as 0, we find

$$\frac{a+b}{2}u^2 - cu^n + \alpha(u^n)'' = 0. \tag{1.5}$$

2. Analysis of the Two Methods

The sine-cosine method, the tanh method, and the extended tanh method have been applied for a wide variety of nonlinear problems. The main features of the two methods will be reviewed briefly.

For both methods, we first use the wave variable $\xi = x + y - ct$ to carry a PDE in two independent variables

$$P(u, u_t, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots) = 0, \tag{2.1}$$

into an ODE

$$Q(u, u', u'', u''', \dots) = 0. \tag{2.2}$$

Equation (2.2) is then integrated as long as all terms contain derivatives, where integration constants are considered zeros.

2.1. The sine-cosine method

The sine-cosine method admits the use of the solution in the form

$$u(x, y, t) = \begin{cases} \lambda \cos^{\beta}(\mu\xi), & |\mu\xi| < \frac{\pi}{2}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

or in the form

$$u(x,\ y,\ t)=\begin{cases}\lambda\sin^{\beta}(\mu\xi), & |\mu\xi|<\pi,\\ 0, & \text{otherwise},\end{cases}\tag{2.4}$$

where $\lambda,\ \mu,$ and β are parameters that will be determined.

We substitute (2.3) or (2.4) into the reduced ordinary differential equation obtained above in (2.2), balance the terms of the cosine functions when (2.3) is used, or balance the terms of the sine functions when (2.4) is used, and solving the resulting system of algebraic equations by using the computerized symbolic calculations to obtain all possible values of the parameters $\lambda,\ \mu,$ and β.

2.2. The tanh method and the extended tanh method

The standard tanh method introduced in [3, 4], where the tanh is used as a new variable, since all derivatives of a tanh are represented by a tanh itself. We use a new independent variable

$$Y=\tanh(\mu\xi),\tag{2.5}$$

that leads to the change of derivatives:

$$\frac{d}{d\xi}=\mu(1-Y^2)\frac{d}{dY},$$

$$\frac{d^2}{d\xi^2}=\mu^2(1-Y^2)\left(-2Y\frac{d}{dY}+(1-Y^2)\frac{d^2}{dY^2}\right).\tag{2.6}$$

We then apply the following finite expansion:

$$u(\mu\xi)=S(Y)=\sum_{k=0}^{M}a_kY^k,\tag{2.7}$$

and

$$u(\mu\xi)=S(Y)=\sum_{k=0}^{M}a_kY^k+\sum_{k=1}^{M}b_kY^{-k},\tag{2.8}$$

where M is a positive integer that will be determined to derive a closed form analytic solution. However, if M is not an integer, a transformation formula is usually used. Substituting (2.5) and (2.6) into the simplified ODE (2.2) results in an equation in powers of Y. To determine the parameter M, we usually balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms. With M determined, we collect all coefficients of powers of Y in the resulting equation, where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters $a_k(k = 0, 1, \ldots, M)$, $b_k(k = 1, 2, \ldots, M)$, μ, and c. Having determined these parameters, knowing that M is a positive integer in most cases, and using (2.7) or (2.8), we obtain an analytic solution $u(x, t)$ in a closed form.

3. Using the Sine -Cosine Method

Substituting (2.3) into (1.5) yields

$$\frac{a+b}{2}\lambda^2 \cos^{2\beta}(\mu\xi) - c\lambda^n \cos^{n\beta}(\mu\xi)$$
$$+ \alpha\left[- n^2\mu^2\beta^2\lambda^n \cos^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta - 1)\cos^{n\beta-2}(\mu\xi)\right] = 0. \tag{3.1}$$

Equation (3.1) is satisfied only if the following system of algebraic equations holds:

$$a \neq -b,\ n\beta \neq 1,\ n\beta - 2 = 2\beta,\ -c = \alpha n^2\mu^2\beta^2,\ \frac{(a+b)\lambda^2}{2} = -\alpha n\mu^2\lambda^n\beta(n\beta - 1), \tag{3.2}$$

or

$$\lambda \neq 0,\ a + b = 0,\ n\beta - 1 = 0,\ -c = \alpha n^2\mu^2\beta^2. \tag{3.3}$$

Solving the systems (3.2) and (3.3) give

$$\beta = \frac{2}{n-2},\ \mu^2 = \frac{c(n-2)^2}{-4\alpha n^2},\ \lambda = \left[\frac{c(n+2)}{n(a+b)}\right]^{\frac{1}{n-2}},\ n \neq \pm 2,\ n\beta \neq 1,\ a \neq -b, \tag{3.4}$$

$$\lambda \neq 0,\ a + b = 0,\ \beta = \frac{1}{n},\ \mu^2 = -\frac{c}{\alpha}. \tag{3.5}$$

The results (3.4) and (3.5) can be easily obtained, if we also use the sine method (2.4). Combining (3.4) and (3.5) with (2.3) and (2.4), the following compactons solutions:

$$u_1 = \begin{cases} \pm\left[\dfrac{c(n+2)}{n(a+b)}\cos^2 \dfrac{n-2}{2n}\sqrt{\dfrac{-c}{\alpha}}(x+y-ct)\right]^{\frac{1}{n-2}}, \\ |x+y-ct| < \dfrac{n\pi}{n-2}\sqrt{\dfrac{\alpha}{-c}},\ n = \pm 2(k+1),\ k \in Z^+,\ -c\alpha > 0,\ c(a+b) > 0, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{3.6}$$

$$u_2 = \begin{cases} \left[\dfrac{c(n+2)}{n(a+b)}\cos^2 \dfrac{n-2}{2n}\sqrt{\dfrac{-c}{\alpha}}(x+y-ct)\right]^{\frac{1}{n-2}}, \\ |x+y-ct| < \dfrac{n\pi}{n-2}\sqrt{\dfrac{\alpha}{-c}},\ n = 2k+1,\ k \in Z,\ -c\alpha > 0,\ a \neq -b, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{3.7}$$

$$u_3 = \begin{cases} \lambda\cos^{\frac{1}{n}}\sqrt{\dfrac{-c}{\alpha}}(x+y-ct),\ |x+y-ct| < \dfrac{\pi}{2}\sqrt{\dfrac{\alpha}{-c}},\ -c\alpha > 0, \\ 0, \qquad \text{otherwise}, \end{cases}$$

$$\lambda \neq 0,\ a = -b, \tag{3.8}$$

and

$$u_4 = \begin{cases} \pm\left[\dfrac{c(n+2)}{n(a+b)}\sin^2 \dfrac{n-2}{2n}\sqrt{\dfrac{-c}{\alpha}}(x+y-ct)\right]^{\frac{1}{n-2}}, \\ |x+y-ct| < \dfrac{2n\pi}{n-2}\sqrt{\dfrac{\alpha}{-c}},\ n = \pm 2(k+1),\ k \in Z^+,\ -c\alpha > 0,\ c(a+b) > 0, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{3.9}$$

$$u_5 = \begin{cases} \left[\dfrac{c(n+2)}{n(a+b)} \sin^2 \dfrac{n-2}{2n} \sqrt{\dfrac{-c}{\alpha}}(x+y-ct)\right]^{\frac{1}{n-2}}, \\ |x+y-ct| < \dfrac{2n\pi}{n-2}\sqrt{\dfrac{\alpha}{-c}},\ n = 2k+1,\ k \in Z,\ -c\alpha > 0,\ a \neq -b, \\ 0, \qquad \text{otherwise,} \end{cases} \tag{3.10}$$

$$u_6 = \begin{cases} \lambda \sin^{\frac{1}{n}} \sqrt{\dfrac{-c}{\alpha}}(x+y-ct),\ |x+y-ct| < \pi\sqrt{\dfrac{\alpha}{-c}}, \\ 0, \qquad \text{otherwise.} \end{cases}$$

$$-c\alpha > 0,\ \lambda \neq 0,\ a = -b, \tag{3.11}$$

However, for $-c\alpha < 0$, we obtain the following solitary patterns and complex solutions:

$$u_7 = \pm\left[\frac{c(n+2)}{n(a+b)} \cosh^2 \frac{n-2}{2n} \sqrt{\frac{c}{\alpha}}(x+y-ct)\right]^{\frac{1}{n-2}},$$

$$n = \pm 2(k+1),\ k \in Z^+,\ c(a+b) > 0, \tag{3.12}$$

$$u_8 = \left[\frac{c(n+2)}{n(a+b)} \cosh^2 \frac{n-2}{2n} \sqrt{\frac{c}{\alpha}}(x+y-ct)\right]^{\frac{1}{n-2}},$$

$$n = 2k+1,\ k \in Z,\ a \neq -b, \tag{3.13}$$

$$u_9 = \lambda \coth^{\frac{1}{n}} \sqrt{\frac{-c}{\alpha}}(x+y-ct),\ \lambda \neq 0,\ a+b = 0, \tag{3.14}$$

and

$$u_{10} = \pm\left[-\frac{c(n+2)}{n(a+b)} \sinh^2 \frac{n-2}{2n} \sqrt{\frac{c}{\alpha}}(x+y-ct)\right]^{\frac{1}{n-2}},$$

$$n = \pm 2(k+1),\ k \in Z^+,\ c(a+b) < 0, \tag{3.15}$$

$$u_{11} = -\left[\frac{c(n+2)}{n(a+b)}\sinh^2\frac{n-2}{2n}\sqrt{\frac{c}{\alpha}}(x+y-ct)\right]^{\frac{1}{n-2}},$$

$$n = 2k+1,\ k \in Z,\ a \neq -b, \tag{3.16}$$

$$u_{12} = \lambda\left[i\sinh\sqrt{\frac{-c}{\alpha}}(x+y-ct)\right]^{\frac{1}{n}},\ \lambda \neq 0,\ a+b = 0. \tag{3.17}$$

4. Using the Extended Tanh Method

Balancing u^2 with $u^{n-1}u''$, we find

$$M = -\frac{2}{n-2}. \tag{4.1}$$

To get a closed form analytic solution, the parameter M should be an integer. A transformation formula

$$u = \phi^{-\frac{1}{n-2}}, \tag{4.2}$$

should be used to achieve our goal. This in turn transforms (1.5) to

$$\frac{a+b}{2}\phi^3 - c\phi^2 + \frac{2\alpha n(n+1)}{(n-2)^2}(\phi')^2 - \frac{n}{n-2}\phi\phi'' = 0. \tag{4.3}$$

Balancing $\phi\phi''$ and ϕ^3 gives $M = 2$. The extended tanh method allows us to use the substitution

$$\phi(\xi) = S(Y) = A_0 + A_1 Y + A_2 Y^2 + B_1 Y^{-1} + B_2 Y^{-2}. \tag{4.4}$$

Substituting (4.4) into (4.3), collecting the coefficients of each power of Y, and solve the resulting system of algebraic equations with the help of Maple to find the sets of solutions:

$$A_1 = B_1 = B_2 = 0,\ A_2 = -A_0 = -\frac{c(n+2)}{n(a+b)},\ \mu^2 = \frac{c(n-2)^2}{4\alpha n^2},\ c^2 = c^2, \tag{4.5}$$

$$A_1 = B_1 = A_2 = 0,\ B_2 = -A_0 = -\frac{c(n+2)}{n(a+b)},\ \mu^2 = \frac{c(n-2)^2}{4\alpha n^2},\ c^2 = c^2, \tag{4.6}$$

$$A_1 = B_1 = 0,\ A_0 = -2A_2 = -2B_2 = \frac{c(n+2)}{2n(a+b)},\ \mu^2 = \frac{c(n-2)^2}{16\alpha n^2},\ c^2 = c^2. \tag{4.7}$$

We obtain the solitary wave solution and the solitary patterns solutions:

$$u_{13} = \pm\left[\frac{c(n+2)}{n(a+b)}\left(1 - \tanh^2 \frac{n-2}{2n}\sqrt{\frac{c}{\alpha}}(x+y-ct)\right)\right]^{-\frac{1}{n-2}},$$

$$n = \pm 2(k+1),\ k \in Z^+,\ c\alpha > 0,\ c(a+b) > 0, \tag{4.8}$$

$$u_{14} = \left[\frac{c(n+2)}{n(a+b)}\left(1 - \tanh^2 \frac{n-2}{2n}\sqrt{\frac{c}{\alpha}}(x+y-ct)\right)\right]^{-\frac{1}{n-2}},$$

$$n = 2k+1,\ k \in Z,\ c\alpha > 0,\ a+b \neq 0, \tag{4.9}$$

$$u_{15} = \pm\left[\frac{c(n+2)}{n(a+b)}\left(1 - \coth^2 \frac{n-2}{2n}\sqrt{\frac{c}{\alpha}}(x+y-ct)\right)\right]^{-\frac{1}{n-2}},$$

$$n = \pm 2(k+1),\ k \in Z^+,\ c\alpha > 0,\ c(a+b) > 0, \tag{4.10}$$

$$u_{16} = \left[\frac{c(n+2)}{n(a+b)}\left(1 - \coth^2 \frac{n-2}{2n}\sqrt{\frac{c}{\alpha}}(x+y-ct)\right)\right]^{-\frac{1}{n-2}},$$

$$n = 2k+1,\ k \in Z,\ c\alpha > 0,\ a+b \neq 0, \tag{4.11}$$

$$u_{17} = \pm\left[\frac{c(n+2)}{2n(a+b)}\left(1 - \frac{1}{2}\tanh^2 \mu(x+y-ct) - \frac{1}{2}\coth^2 \mu(x+y-ct)\right)\right]^{-\frac{1}{n-2}},$$

$$n = \pm 2(k+1),\ k \in Z^+,\ c\alpha > 0,\ c(a+b) > 0,\ \mu = \frac{n-2}{4n}\sqrt{\frac{c}{\alpha}}, \tag{4.12}$$

$$u_{18} = \left[\frac{c(n+2)}{2n(a+b)}\left(1 - \frac{1}{2}\tanh^2 \mu(x+y-ct) - \frac{1}{2}\coth^2 \mu(x+y-ct)\right)\right]^{-\frac{1}{n-2}},$$

$$n = 2k+1,\ k \in Z,\ c\alpha > 0,\ a+b \neq 0,\ \mu = \frac{n-2}{4n}\sqrt{\frac{c}{\alpha}}. \tag{4.13}$$

However, for $c\alpha < 0$, we obtain the periodic wave and compactons solutions:

$$u_{19} = \pm\left[\frac{c(n+2)}{n(a+b)}\left(1 + \tan^2(x+y-ct)\right)\right]^{-\frac{1}{n-2}},$$

$$n = \pm 2(k+1),\ k \in Z^+,\ c(a+b) > 0, \tag{4.14}$$

$$u_{20} = \left[\frac{c(n+2)}{n(a+b)}\left(1 + \tan^2 \frac{n-2}{2n}\sqrt{\frac{-c}{\alpha}}(x+y-ct)\right)\right]^{-\frac{1}{n-2}},$$

$$n = 2k+1,\ k \in Z,\ a+b \neq 0, \tag{4.15}$$

$$u_{21} = \pm\left[\frac{c(n+2)}{n(a+b)}\left(1 + \cot^2 \frac{n-2}{2n}\sqrt{\frac{-c}{\alpha}}(x+y-ct)\right)\right]^{-\frac{1}{n-2}},$$

$$n = \pm 2(k+1),\ k \in Z^+,\ c(a+b) > 0, \tag{4.16}$$

$$u_{22} = \left[\frac{c(n+2)}{n(a+b)}\left(1 + \cot^2 \frac{n-2}{2n}\sqrt{\frac{-c}{\alpha}}(x+y-ct)\right)\right]^{-\frac{1}{n-2}},$$

$$n = 2k+1,\ k \in Z,\ a+b \neq 0, \tag{4.17}$$

$$u_{23} = \pm\left[\frac{c(n+2)}{2n(a+b)}\left(1 + \frac{1}{2}\tan^2 \mu(x+y-ct) + \frac{1}{2}\cot^2 \mu(x+y-ct)\right)\right]^{-\frac{1}{n-2}},$$

$$n = \pm 2(k+1),\ k \in Z^+,\ c(a+b) > 0,\ \mu = \frac{n-2}{4n}\sqrt{-\frac{c}{\alpha}}, \tag{4.18}$$

$$u_{24} = \left[\frac{c(n+2)}{2n(a+b)}\left(1 + \frac{1}{2}\tan^2 \mu(x+y-ct) + \frac{1}{2}\cot^2 \mu(x+y-ct)\right)\right]^{-\frac{1}{n-2}},$$

$$n = 2k+1,\ k \in Z,\ a+b \neq 0,\ \mu = \frac{n-2}{4n}\sqrt{-\frac{c}{\alpha}}. \tag{4.19}$$

5. Discussion

The sine-cosine method and the extended tanh method were used to investigate the generalized $(2+1)$-dimensional breaking soliton equation. The study revealed compactons solutions and solitary patterns solutions for some examined variants. The study emphasized the fact that, the two methods are reliable in handling nonlinear problems. The obtained results clearly demonstrate the efficiency of the two methods used in this work. Moreover, the methods are capable of greatly minimizing the size of computational work compared to other existing techniques. This emphasizes the fact that the two methods are applicable to a wide variety of nonlinear problems.

References

[1] Y. Cheng and B. Li, Symbolic computation and construction of soliton-like solutions to the $(2+1)$-dimensional breaking soliton equation, Commun. Theor. Phys. (Beijing, China) 40 (2003), 137-142.

[2] R. Hirota and Y. Ohta, Hierarchies of coupled soliton equations, I, J. Phys. Soc. Japan 60 (1991), 798-809.

[3] W. Malfliet and W. Hereman, The tanh method: I, exact solutions of nonlinear evolution and wave equations, Phys. Scripta 54 (1996), 563-568.

[4] W. Malfliet and W. Hereman, The tanh method: II, perturbation technique for conservative systems, Phys. Scripta 54 (1996), 569-575.

[5] Y. Z. Peng, New exact solutions for $(2+1)$-dimensional breaking soliton equation, Commun. Theor. Phys. (Beijing, China) 43 (2005), 205-207.

[6] Y. Z. Peng and E. V. Krishna, Two classes of new exact solutions to $(2+1)$-dimensional breaking soliton equation, Commun. Theor. Phys. (Beijing, China) 44 (2005), 807-809.

[7] Y. J. Ren, S. T. Liu and H. Q. Zhang, On a generalized extended F-expansion method, Commun. Theor. Phys. (Beijing, China) 45 (2006), 15-28.

[8] F. Tascan and A. Bekir, Analytic solutions of the $(2+1)$-dimensional nonlinear evolution equations using the sine-cosine method, Appl. Math. 215 (2009), 3134-3139.

[9] A. M. Wazwaz, A sine-cosine method for handling nonlinear wave equations, Math. Comput. Model. 40 (2004), 499-508.

[10] A. M. Wazwaz, A reliable treatment of the physical structure for the nonlinear equation $K(m, n)$, Appl. Math. Comput. 163 (2005), 1081-1095.

[11] A. M. Wazwaz, A class of nonlinear fourth order variant of a generalized Camassa-Holm equation with compact and noncompact solutions, Appl. Math. Comput. 165 (2005), 485-501.

[12] S. Zhang, New exact non-travelling wave and coefficient function solutions of the $(2+1)$- dimensional breaking soliton equations, Phys. Lett. A 368 (2007), 470-475.

■