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FINDING A MINIMAL EFFICIENT SOLUTION OF A CONVEX MULTIOBJECTIVE PROGRAM VIA A CHIP METHOD

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Abstract

In this paper, we propose a CHIP method for solving the multiobjective program. Under conditions commonly used in the literature, we get the global convergence of the CHIP method, which will lead to an implementable globally convergent algorithm.

1. Introduction

Consider the multiobjective program:

$$\min_{x \in R^n} \; f(x)$$

$$\text{s.t.} \quad g(x) \leqq 0, \tag{1.1}$$

where

$$f(x) = (f_1(x), f_2(x), \cdots, f_p(x))^T$$

and

$$g(x) = (g_1(x), g_2(x), \cdots, g_m(x))^T.$$

In this paper, $f(x)$ and $g(x)$ are three times continuously differentiable. The most famous methods for solving the multiobjective program are scalarization methods, which deal with the multiobjective program by reducing it to a single objective program with certain parameters. Furthermore, in [4], the authors considered the weight λ as variable and hence introduced the concept of the minimal weak efficient solution (see the following Definition 1.1, first introduced by Ecker et al. [2] and Armand [1] for the LMOP).

Definition 1.1. x is a minimal weak efficient solution of the multiobjective program (1.1), if (x, λ) is a solution of the problem

$$\min_{x \in R^n} \; \lambda^T f(x)$$

$$\text{s.t.} \quad g(x) \leqq 0,$$

$$\sum_{i=1}^{p} \lambda_i = 1, \quad \lambda_i \geqq 0. \tag{1.2}$$

Since (1.2) is a nonconvex programming problems, to solve (1.2), ones often solve its K-K-T system, hence, in [4], the authors develop a combined homotopy interior point method (CHIP method, for more details, see [3, 5]) to complete this work. They obtained the global convergence of that method, requiring the boundedness of the feasible set. It was also conjectured in [4] that the result may be generalized to the unbounded sets, however, till now, no such result has been given. In this paper, we relax the boundedness assumption and replace it by a coercivity condition on the gradient of $f_i(x)$, $i = 1, \cdots, p$. Under this commonly used condition in the literature, we are able to solve the K-K-T system of (1.2) on unbounded sets, besides, we also obtain the global convergence of the interior point homotopy method, which can lead to an implementable globally convergent algorithm.

2. Main Results

In this section, let $\Omega = \{x \in R^n : g_i(x) \leqq 0, i = 1, \cdots, m\}$, $\Omega^0 = \{x \in R^n : g_i(x) < 0, i = 1, \cdots, m\}$, and $\partial\Omega = \Omega \setminus \Omega^0$. In addition, denote the nonnegative and positive orthant of R^m by R^m_+ and R^m_{++}, respectively. For any $x \in \partial\Omega$, let $B(x) = \{i \in \{1, \ldots, m\} : g_i(x) = 0\}$. At last, we let

$$\Lambda^+ = \{\lambda \in R^p : \sum_{i=1}^{p} \lambda_i = 1, \quad \lambda_i \geqq 0\}, \Lambda^{++} = \{\lambda \in R^p : \sum_{i=1}^{p} \lambda_i = 1, \quad \lambda_i > 0\}.$$

In the following, we construct the following homotopy for the Karush-Kuhn-Tucker system of (1.2):

$$H(w, w^{(0)}, t) = \begin{pmatrix} (1-t)(\nabla f(x)\lambda + \nabla g(x)z) + t(x - x^{(0)}) \\ (1-t)(f(x) - \xi) - ye + t(\lambda - \lambda^{(0)}) \\ 1 - \sum_{i=1}^{p} \lambda_i \\ z \times g(x) - tz^{(0)} \times g(x^{(0)}) \\ \xi \times \lambda - t\xi^{(0)} \times \lambda^{(0)} \end{pmatrix} = 0, \tag{2.1}$$

where $w^{(0)} = (x^{(0)}, \lambda^{(0)}, y^{(0)}, z^{(0)}, \xi^{(0)}) \in \Omega^0 \times \Lambda^{++} \times \{0\} \times R^{m+p}_{++}$.

Now, we make the following assumptions:

(A_1) Ω^0 is nonempty.

(A_2) For any $x \in \partial\Omega$, the matrix $\{\nabla g_i(x) : i \in B(x)\}$ is of full column rank.

(A_3) $g_i(x)$, $i = 1, \ldots, m$ are convex.

(A_4) There exists some $\eta \in \Omega$ such that

$$\lim_{\|x\|\to\infty} (x - \eta)^T \nabla f_i(x) = +\infty, \; i = 1, \cdots, p.$$

The following Lemma 2.1 gives the existence of homotopy path.

Lemma 2.1. *Let H be defined as* (2.1), *Assumptions* (A_1)-(A_4) *hold and g_i, $i = 1, \ldots, m$ be C^3 functions. Then, for almost all $w^{(0)} = (x^{(0)}, \lambda^{(0)}, y^{(0)}, z^{(0)}, \xi^{(0)}) \in \Omega^0 \times \Lambda^{++} \times \{0\} \times R_{++}^{m+p}$, 0 is a regular value of map $H_{w^{(0)}} : \Omega \times \Lambda^{+} \times R \times R_{+}^{m+p} \times (0, 1] \to R^{n+m+2p+1}$, and $H_{w^{(0)}}^{-1}(0)$ consists of some smooth curves. Among them, a smooth curve, say $\Gamma_{w^{(0)}}$, is starting from $(w^{(0)}, 1)$.*

Proof. Similar to the proof of Lemma 3.1 in [4]. □

Next, we are devoted to proving the boundedness of the solution pathway $\Gamma_{w^{(0)}}$. Let

$$\Omega_i^{+}(\eta) = \{x \in \Omega : (x - \eta)^T \nabla f_i(x) > 0\}, \; i = 1, \ldots, p,$$

$$\Omega^{+}(\eta) = \bigcap_{i=1}^{p} \Omega_i^{+}(\eta), \quad \Omega^{-}(\eta) = \Omega \setminus \Omega^{+}(\eta).$$

Lemma 2.2. *Let H be defined as* (2.1) *and Assumptions* (A_1)-(A_4) *hold. Then, for a given $w^{(0)} = (x^{(0)}, \lambda^{(0)}, y^{(0)}, z^{(0)}, \xi^{(0)}) \in \Omega^0 \times \Lambda^{++} \times \{0\} \times R_{++}^{m+p}$, if 0 is a regular value of map $H_{w^{(0)}}$, then the projection of the smooth curve $\Gamma_{w^{(0)}}$ on the x-plane is bounded.*

Proof. If not, then there exists a sequence of points $\{(x^{(k)}, \lambda^{(k)}, y^{(k)}, z^{(k)}, \xi^{(k)}, t_k)\}_{k=1}^{\infty} \subset \Gamma_{w^{(0)}}$ such that $\|x^{(k)}\| \to \infty$ as $k \to \infty$.

By Assumption (A4), if $\|x^{(k)}\| \to \infty$, then there exists some constant σ such that

$$(x^{(k)} - \eta)^T \nabla f_i(x) \geqq \sigma > 0, \tag{2.2}$$

where $\eta \in \Omega$ is a given point.

When $x^{(k)} \in \Omega^-(\eta)$, by (2.2), then the case $\|x^{(k)}\| \to \infty$ would not happen.

Since $\Omega = \Omega^+(\eta) \cup \Omega^-(\eta)$, by the above analysis, in the following, we only need to prove that the component x of $\Gamma_{w^{(0)}}$ in $\Omega^+(\eta)$ is also bounded.

Since $g(x)$ is convex, then

$$(\eta - x^{(k)})^T \nabla g(x^{(k)}) \leqq g(\eta)^T - g(x^{(k)})^T. \tag{2.3}$$

Multiplying the first equality of (2.1) by $(x^{(k)} - \eta)^T$, we get

$$\begin{aligned}(1 - t_k)[(x^{(k)} - \eta)^T \nabla f(x^{(k)})\lambda^{(k)} + (x^{(k)} - \eta)^T \nabla g(x^{(k)})z^{(k)}]\\ = -t_k(x^{(k)} - \eta)^T (x^{(k)} - x^{(0)}).\end{aligned} \tag{2.4}$$

For any given $\eta \in \Omega$, $x^{(0)} \in \Omega^0$, and all $x \in \Omega$, we have the following inequality

$$\|x - \eta\|^2 - \|x^{(0)} - \eta\|^2 \leq 2(x - \eta)^T (x - x^{(0)}). \tag{2.5}$$

By (2.3), (2.4), (2.5), the fact that $g(\eta)^T z^{(k)} \leqq 0$, and the fourth equality of (2.1), we get

$$t_k(\|x^{(k)} - \eta\|^2 - \|x^{(0)} - \eta\|^2)$$

$$\leqq 2t_k(x^{(k)} - \eta)^T(x^{(k)} - x^{(0)})$$

$$\leqq 2(1 - t_k)[(\eta - x^{(k)})^T \sum_{i=1}^{p} \nabla f_i(x^{(k)})\lambda_i^{(k)} + (g(\eta)^T - g(x^{(k)})^T)z^{(k)}]$$

$$\leqq 2(1 - t_k)((\eta - x^{(k)})^T \sum_{i=1}^{p} \nabla f_i(x^{(k)})\lambda_i^{(k)} - g(x^{(k)})^T z^{(k)})$$

$$= 2(1 - t_k)(\eta - x^{(k)})^T \sum_{i=1}^{p} \nabla f_i(x^{(k)})\lambda_i^{(k)} - 2t_k(1 - t_k)g(x^{(0)})^T z^{(0)}. \quad (2.6)$$

By (2.6), we get

$$2(1 - t_k)(\eta - x^{(k)})^T \sum_{i=1}^{p} \nabla f_i(x^{(k)})\lambda_i^{(k)}$$

$$\geqq t_k\|x^{(k)} - \eta\|^2 - t_k\|x^{(0)} - \eta\|^2 + 2t_k(1 - t_k)g(x^{(0)})^T z^{(0)}. \quad (2.7)$$

If $\|x^{(k)}\| \to \infty,$ since $\|x^{(0)} - \eta\|^2,$ $g(x^{(0)})^T,$ and $z^{(0)}$ are constants and t_k is bounded, then there exists some k such that $\|x^{(k)} - \eta\| > M, t_k \in (0, 1],$ and the left-hand side of (2.7) is strictly smaller than 0, while the right-hand side of (2.7) is strictly larger than 0, a contradiction.

When $x^{(k)} \in \Omega^-(\eta),$ by Assumption (A_4), $\Omega^-(\eta)$ is bounded, then $\|x^{(k)}\| \to \infty$ would not happen.

Since $\Omega = \Omega^+(\eta) \cup \Omega^-(\eta),$ by the analysis above, the projection of the smooth curve $\Gamma_{w^{(0)}}$ on the x-plane in Ω is bounded. This completes the proof. □

Similar to Lemma 3.2 in [4], we have the projection of the smooth curve $\Gamma_{w^{(0)}}$ on the y-plane, z-plane, and ξ-plane are also bounded. And similar to Theorem 3.1 in [4], we have the global convergence of the homotopy method.

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