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INVERSE EIGENVALUE PROCEDURES FOR DAMAGE DETECTION IN BARS AND BEAMS

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Abstract

This work is related to the damage detection problem in bars and beams and consists of finding the location and severity of damage. It is shown that the discrete models based on the geometric parameters of the fixed-fixed bar, fixed-free bar and the beams can be reconstructed from one eigen-pair, that is, one eigenvalue and its respective eigenvector. This reconstruction permits development of inverse eigenvalue procedures for damage detection in bar and beams. Also, numerical algorithms are given.

1. Introduction

If a structure is damaged, its vibratory behavior will change. The vibratory behavior of a structure may be characterized by its natural frequencies and corresponding principal mode shapes. Therefore, the damage detection (or called the identification of damage) in a vibrating

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structure from changes in vibratory behavior is an inverse problem [3]. Since such damage detection has potentially important practical value, it is appropriate for it to be included in any treatment of inverse problems. It is essentially an application of inverse techniques and must be combined with numerical methods.

The inverse problems in structural vibration are related to the determination or estimation of the physics properties of a vibratory system (density or mass, conductivity, elastic constants, crack lengths, etc.) from a known dynamic behavior (natural frequencies, forced response, stresses, etc.). An important class of these problems are detection problems, which consist of finding the amount and localization of changes in a structure.

In particular, in this work we are interested in the damage detection problem, which appears increasingly interesting in areas such as civil, mechanical, and aerospace engineering [5, 6, 7]. The damage detection problem consists of finding the location and severity of damage in a structure. In general terms, damage leads to a loss of local stiffness in a structure. It is necessary to compare two different states of the system, the first one reserved to represent the initial state, named state undamaged, and the second one, the state after the occurrence of damage. To detect the stiffness changes in the structure, we propose to apply an inverse eigenvalue method for the reconstruction of the damaged systems from one eigenpair.

The outline of this paper is as follows. In the next section, we define the structural damage involving geometric parameter of bars. For this, we consider the discrete models of the fixed-fixed and fixed-free bar respectively. The inverse eigenvalue procedures to detect damage in bars, which can be carried out from one eigenpair, will be developed. In section 3, an inverse eigenvalue procedure for damage detection in discrete model of beams is developed. Finally, we make some concluding remarks in the last section.

2. Damage Detection in Bars

Consider the governing equation for the free axial vibration $v(x)$ of a bar

$$\frac{d}{dx}\left[EA(x)\frac{dv}{dx}\right] + \lambda\rho A(x)v(x) = 0, \tag{1}$$

where $A(x) > 0$ and $\rho > 0$ are the cross-sectional area and density of the bar, respectively, and λ is the square of the natural frequency [8].

2.1. Damage detection for fixed-fixed bars using one eigenpair

The discrete form of Equation (1) is given by

$$(K - \lambda M)u = 0, \tag{2}$$

where K and M are stiffness and mass matrives, respectively, u is the mode shape vector, and λ is the square of the natural frequency [2, 4, 9, 13].

The stiffness and mass matrices can be written in term of the bar's cross-sectional area and given material properties. When the bar is divided into N finite elements, there will be N unknowns $A_i (i = 1, 2, \cdots, N)$ denoting the cross-section areas of elements. There will be $N + 1$ nodes numbered $0, 1, \cdots, N$ with zeroth and N^{th} nodes specified to be zero for the fixed-fixed boundary condition. Therefore, the mode shape vector u in Equation (1) is of size $(N - 1) \times 1$ after the boundary conditions are applied. The stiffness and mass matrices for fixed-fixed condition are given by

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & -k_3 & k_3 + k_4 & \ddots & \\ & & \ddots & \ddots & -k_{N-1} \\ & & & -k_{N-1} & k_{N-1} + k_N \end{bmatrix}_{(N-1)\times(N-1)} \tag{3a}$$

$$M = \begin{bmatrix} 2m_1 + 2m_2 & m_2 & & & \\ m_2 & 2m_2 + 2m_3 & m_3 & & \\ & m_3 & 2m_3 + 2m_4 & \ddots & \\ & & \ddots & \ddots & m_{N-1} \\ & & & m_{N-1} & 2m_{N-1} + 2m_N \end{bmatrix}_{(N-1)\times(N-1)}, \tag{3b}$$

where $m_i = \rho A_i l / 6$ and $k_i = EA_i / l$ in which E is Young's Modulus, l is the length of each element, and ρ is the density.

Since the A_i's are unknowns now, Equation (2) can be rewritten so that the displacements u_i appear in a matrix and the A_i's in the unknown vector as

$$PA = \begin{bmatrix} P_{1,1} & P_{1,2} & 0 & 0 & 0 & \cdots & 0 \\ 0 & P_{2,2} & P_{2,3} & 0 & 0 & \cdots & 0 \\ 0 & 0 & P_{3,3} & P_{3,4} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & P_{N-2,N-2} & P_{N-2,N-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & P_{N-1,N-1} & P_{N-1,N} \end{bmatrix}_{(N-1)\times N} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_N \end{bmatrix}_{N\times 1} = 0, \tag{4}$$

where $P_{1,1} = cu_1$, $P_{1,2} = cu_1 - du_2$, $P_{i,i} = cu_i - dui$, $P_{i,i+1} = cu_i - du_{i+1}$, for $i = 2, 3, \cdots, N-2$, $P_{N-1,N-1} = cu_{N-1} - du_{N-2}$, $P_{N-1,N} = cu_{N-1}$, with $c = E/l - \lambda\rho l/3$, $d = E/l + \lambda\rho l/6$.

We note that, without loss of generality, if we assume that A_N is unity, each row of Equation (4) can be solved to obtain the next elemental area of cross-section as

$$\frac{A_2}{A_1} = \frac{cu_1}{du_2 - cu_1}, \frac{A_{i+1}}{A_i} = \frac{cu_i - du_{i-1}}{du_{i+1} - cu_i}, \frac{A_N}{A_{N-1}} = \frac{cu_{N-1}}{du_{N-2} - cu_{N-1}}, i = 2, 3, \cdots, N-2. \tag{5}$$

In another word, knowing the eigenpair (λ, u) and the last area A_N we can determine recursively the remaining areas by (5).

Hence, the first procedure to detect damage in fixed-fixed bar can be summarized as follows.

Algorithm 1. damage detection algorithm for fixed-fixed bar

- Input: The areas $(\overline{A}_i)_1^{n-1}$ of the initial rod (undamaged), one eigenpair (λ, u) and the N^{th} area A_N of the damaged rod, and the tolerance $\varepsilon > 0$.

- Output: The areas $(A_i)_1^N$ of the damaged rod, and therefore, the amount and localization of the damage denoted by $d_i = |\overline{A}_i - A_i|$, $i = 1, 2, \cdots, p$, with $1 \leq p \leq N$.

Compute A_i, $i = 1, 2, \cdots, N-1$, using (5)

For $i = 1, 2, \cdots, N - 1.$

Compute d_i

If $(|d_i| \leq \varepsilon * \overline{A}_i)$

'the i^{th} area is not damaged'

End if

2.2. Damage detection for fixed-free bars using one eigenpair

The discretized formulation of the fixed-free case is similar to that of the fixed-fixed case with a small difference. First, the stiffness and mass matrices are given by

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{N-1} & k_{N-1} + k_N & k_N \\ & & & -k_N & k_N \end{bmatrix}_{N \times N} \tag{6a}$$

$$M = \begin{bmatrix} 2m_1 + 2m_2 & m_2 & & & \\ m_2 & 2m_2 + 2m_3 & m_3 & & \\ & \ddots & \ddots & \ddots & \\ & & m_{N-1} & 2m_{N-1} + 2m_N & m_N \\ & & & m_N & 2m_N \end{bmatrix}_{N \times N} \tag{6b}$$

for N elements with cross-section areas $A_i (i = 1, 2, \cdots, N)$ and $(N + 1)$ nodes numbered $0, 1, \cdots, N$ with zeroth node specified to be zero. The size of the mode shape vector u is now $N \times 1$. Therefore, in the rearranged form that is similar to Equation (4), there are N unknown A_i and N equations. That is

$$P_{N \times N} A_{N \times 1} = 0. \tag{7}$$

The last row of the above equation can be used to solve for λ :

$$\lambda = \frac{6E(-u_{N-1} + u_N)}{\rho L^2 (u_{N-1} + 2u_N)}. \tag{8}$$

This means that the natural frequency is automatically determined when a mode shape is prescribed for a fixed-free bar. Using the remained $(N-1)$ rows of Equation (7), the following recurrence relationship can be obtained:

$$A_i = \frac{(\lambda\rho l^2/6E)(2u_i + u_{i+1}) - (u_i - u_{i+1})}{(u_i - u_{i-1}) - (\lambda\rho l^2/6E)(2u_{i-1} + u_i)} A_{i+1}, \; i = N-1, N-2, \cdots, 2, 1. \qquad (9)$$

As before, without loss of generality, A_N can be assumed to be unity to successively determine the remaining A_i using (9). In another word, knowing the eigenpair (λ, u) and the last area A_N we can determine recursively the remaining areas by (9).

Hence, the procedure to detect damage in fixed-free bar can be summarized as follows.

Algorithm 2. damage detection algorithm for fixed-free bar

• Input: The areas $(\overline{A}_i)_1^{N-1}$ of the initial rod (undamaged), one eigenpair (λ, u) and the N^{th} area A_N of the damaged rod, and the tolerance $\varepsilon > 0$.

• Output: The areas $(A_i)_1^N$ of the damaged rod, and therefore, the amount and localization of the damage denoted by $d_i = |\overline{A}_i - A_i|$, $i = 1, 2, \cdots, p$, with $1 \leq p \leq N$.

Compute A_i, $i = 1, 2, \cdots, n-1$, using (9)

For $i = 1, 2, \cdots, N-1$

Compute d_i

If $(|d_i| \leq \varepsilon * \overline{A}_i)$

'the i^{th} area is not damaged'

End if

3. Damage Detection in Beams

As in the case of bars, in beams too arbitrarily specified mode shapes do not result in physically meaningful geometries. The condition for a valid mode shape depend on the boundary conditions of the beam. The discussion in this paper is limited to the case of a cantilever beam for which the governing differential equation for free vibration is given by

$$\frac{d^2}{dx^2}\left(EI\frac{d^2u}{dx^2}\right) - \lambda\rho Au = 0,\ \lambda = \omega^2, \tag{10}$$

where $I(x)$ and $A(x)$ are the second moment of inertia and area of cross-section along the longitudinal axis of the beam respectively; E, L and ρ are Young's modulus, length, and density of the beam; $u(x)$ is the mode shape corresponding to a frequency ω. Discretization of this equation using the finite difference method gives rise to the matrix eigenvalue problem [1, 8, 10, 11, 12]

$$Ku = \lambda Mu, \tag{11}$$

where the mass matrix M is diagonal matrix with diagonal entries $m_i = \rho A_i l$, and the stiffness matrix $K = EL^{-1}E\overline{K}E^{\mathsf{T}}L^{-1}E^{\mathsf{T}}$ is the pentadiagonal as shown below:

$$K = \begin{bmatrix} a_1 & -b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 \\ -b_1 & a_2 & -b_2 & c_2 & 0 & \cdots & 0 & 0 \\ c_2 & -b_2 & a_3 & -b_2 & c_3 & \cdots & 0 & 0 \\ 0 & c_2 & -b_3 & a_4 & -b_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a_{N-1} & -b_{N-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & -b_{N-1} & a_N \end{bmatrix} \tag{12a}$$

with coefficients

$$a_i = (k_i + 4k_{i+1} + k_{i+2})/l^2,\ b_i = 2(k_{i+1} + k_{i+2})/l^2,$$

$$c_i = k_{i+2}/l^2,\ i = 1, 2, \cdots, N-2$$

$$a_{N-1} = (k_{N-1} + 4k_n)/l^2,\ b_{N-1} = 2k_N/l^2,\ a_N = k_N/l^2$$

and $\overline{K} = diag(k_i)$ with $k_i = Eh^2 A_i / l^3$; L is identity matrix multiplied by l, and

$$E = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{12b}$$

The above discretization assumed finite difference grid spacing of length l, where A_i is the area of cross-section across the ith grid spacing. A rectangular cross-section was assumed with thickness h while the width is assumed to vary along the axis. The transverse displacement in the mode shape is given by $u = (u_1, u_2, \cdots, u_N)$. The mass and stiffness matrices together with the mode shape vector u can be rewritten so that the modal displacements u_i appear in the matrices and A_i as the unknown vector:

$$CA = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} & 0 & 0 & \cdots & 0 \\ 0 & C_{2,2} & C_{2,3} & C_{2,4} & 0 & \cdots & 0 \\ 0 & 0 & C_{3,3} & C_{3,4} & C_{3,5} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 & C_{N-2,N-2} & C_{N-2,N-1} & C_{n-2,n} \\ 0 & 0 & \cdots & 0 & 0 & C_{N-1,N-1} & C_{N-1,N} \\ 0 & 0 & \cdots & 0 & 0 & 0 & C_{N,N} \end{bmatrix}_{N\times N} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ \vdots \\ A_N \end{bmatrix}_{N\times 1} = 0, \tag{13}$$

where

$$C_{1,1} = \frac{Eh^2}{12l^3} u_1 - \lambda\rho l u_1,\ C_{i,i+1} = \frac{Eh^2}{6l^3}(u_{i-1} - 2u_i + u_{i+1}),$$

$$C_{i,i+2} = \frac{Eh^2}{12l^3}(u_i - 2u_{i+1} + u_{i+2}),$$

$$C_{i,i} = \frac{Eh^2}{12l^3}(u_{i-2} - 2u_{i-1} - u_i) - \lambda\rho l u_i,\ C_{N,N} = \frac{Eh^2}{12l^3}(u_{N-2}$$

$$-2u_{N-1} - u_N) - \lambda\rho l u_N. \tag{14}$$

The eigenvalue λ can be determined by solving the last row of the above equation:

$$\lambda = \frac{Eh^2}{12\rho l^4}\frac{(u_{N-2} - 2u_{N-1} + u_N)}{u_N}. \tag{15}$$

Thus, as in the case of the bars, the frequency cannot be arbitrarily chosen when a mode shape is specified. The remaining rows of Equations (13) can be solved sequentially starting from the second row from bottom and moving up to get

$$A_{N-1} = -\frac{C_{N-1,N}}{C_{N-1,N-1}}A_N,\ A_i = -\frac{(C_{i,i+1}A_{i+1} + C_{i,i+2}A_{i+2})}{C_{i,i}},$$

$$i = N-2,\ N-3, \cdots, 2, 1. \tag{16}$$

As before, without loss of generality, A_N can be assumed to be unity to successively determine the remaining A_i using (16). In another word, knowing the eigenpair (λ, u) and the last area A_N we can determine recursively the remaining areas by (16).

Hence, the procedure to detect damage in beams can be summarized as follows.

Algorithm 3. damage detection algorithm for beams

- Input: The areas $(\overline{A}_i)_1^{N-1}$ of the initial beam (undamaged), one eigenpair (λ, u) and the N^{th} area A_N of the damaged beam, and the tolerance $\varepsilon > 0$.

- Output: The areas $(A_i)_1^N$ of the damaged beam, and therefore, the amount and localization of the damage denoted by $d_i = |\overline{A}_i - A_i|$, $i = 1, 2, \cdots, p$, with $1 \leq p \leq N$.

Compute A_i, $i = 1, 2, \cdots, N-1$, using (16)

For $i = 1, 2, \cdots, N-1$

Compute d_i

If $(|d_i| \leq \varepsilon * \overline{A}_i)$

'the ith area is not damaged'

End if

Conclusion and Remarks

In this paper, we have proposed the inverse eigenvalue procedures for damage detection in bars and beams. So this problem is more attractive, valuable and interesting both in theory and in practice, especially in fields of applied structural mechanics.

Just like the other damage detection problems, there remain many problems to be studied. For example, how to analyze the sensitivity of the inverse procedure for a large order model?

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