Journal of Pure and Applied Mathematics: Advances and Applications
Volume 1, Number 1, 2009, Pages 83-94

THE EXISTENCE OF COMPACT EXPONENTIAL ATTRACTING SET FOR REACTION DIFFUSION EQUATION IN H_0^1

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Abstract

Using the measure of non-compactness, we prove the existence of compact exponential attracting set for general dynamical system and give a method for proving the existence of compact exponential attracting set. As application, we prove the existence of compact exponential set for reaction diffusion equation in H_0^1.

1. Introduction

The study of the asymptotic behavior that arises in many infinite dimensional dynamical systems is one of the most important problems of modern mathematical physics. For autonomous dissipative systems, one way is to analyse the existence and structure of its global attractor [1, 2, 5]. Many authors have paid much attention to this problem for a quite

2000 Mathematics Subject Classification: 35K57, 35B40, 35B41.

Keywords and phrases: dynamical system, attractor, measure of non-compactness, reaction diffusion equation.

Submitted by Jingtang Ma

Supported by the National Natural Science Foundation of China (10771159).

Received November 23, 2008

long time and have made a lot of successful progress [3, 4, 6]. Teman [5] presents a general approach that is well suited to study equations arising in mathematic physics. In [4], the authors establish some necessary and sufficient conditions for the existence of the global attractor of an infinite dimensional dynamical system and give a new method for proving the existence of the global attractor. To verify the existence of a global attractor, one need to show that

(1) the existence of an absorbing set, and

(2) condition (C),

where condition (C) is: for any bounded set B of X and for any $\varepsilon > 0$, there exist $t(B) > 0$ and a finite dimensional subspace X_1 of X, such that $\{\|PS(t)B\|\}$ is bounded and

$$\|(I - P)S(t)x\| < \varepsilon \quad \text{for } t \geq t(B),\ x \in B,$$

where $P : X \to X_1$, is a bounded projector.

In fact, for many equations, we find not only condition (C) hold true but also condition $(\hat{C})$ [see Definition 3.5], i.e., the measure of non-compactness is exponential decay. By using the measure of non-compactness technique, we establish the existence of compact exponential attracting set for general dynamical system and include the global attractor, and give a method for proving the existence of compact exponential attracting set. As an example, we consider reaction-diffusion equation and get the existence of compact exponential attracting set in H_0^1.

Throughout this paper we will use the following notation: X is a complete metric space with the metric d, $B(X)$ is the set of all bounded subsets of X, denote by $B(x, r)$ the neighborhood $\{y \in X : d(y, x) < r\}$ of radius $r > 0$ of x, and $B(A, r)$ the neighborhood $\{y \in X : d(y, A) < r\}$ of radius $r > 0$ of A.

2. Preliminaries

Let X be a complete metric space. One parameter family of (nonlinear) mappings $S(t) : X \to X (t \geq 0)$ is called the *semigroup* provided that

(1) $S(0) = I$;

(2) $S(t + s) = S(t)S(s)$ for all $t, s \geq 0$.

The couple $(S(t), X)$ is usually referred to as a *dynamical system*, $(S(n), X)$ is called the *discrete dynamical system* generated by $(S(t), X)$.

A set $\mathcal{A} \subset X$ is called a *global attractor* to $(S(t), X)$ if (i) $\mathcal{A}$ is compact in X, (ii) $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$ and (iii) for any $B \subset X$ that is bounded, $d(S(t)B, \mathcal{A}) \to 0$ as $t \to \infty$, where $d(B, \mathcal{A}) = \sup_{b\in B} \inf_{a\in\mathcal{A}} \|b - a\|_X$.

A set B is called a *bounded absorbing set* to $(S(t), X)$, if for any bounded set $B_0 \subset X$, there exists $t_0 = t_0(B_0)$ such that $S(t)B_0 \subset B$ for all $t \geq t_0$. A set E is called *positively invariant* w.r.t. $S(t)$ if for all $t \geq 0$, $S(t)E \subset E$.

Theorem 2.1 (Teman [5])**.** *Continuous semigroup $S(t)$ has a global attractor $\mathcal{A}$ if and only if $S(t)$ has a bounded absorbing set B and for an arbitrary sequence of points $x_n \in B$, the sequence $S(t_n)x_n$ has a subsequence converging in B.*

In fact, we know that

$$\mathcal{A} = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)B}. \tag{2.1}$$

Next, we briefly review the basic concept about the Kuratowski measure of noncompact and recapitulate its basic property, which will be used to characterize the existence of compact exponential attracting set for dynamical system $(S(t), X)$.

Let X be a complete metric space and B be a bounded subset of X. The Kuratowski measure of non-compactness $\alpha(B)$ of B is defined by $\alpha(B) = \inf\{\delta > 0 \,|\, B \text{ has a finite open cover of sets of diameter} \leq \delta\}$.

It has the following properties

Lemma 2.2 (Chepyzhov and Vishik [2]). *Let* B, B_1, $B_2 \in B(X)$. *Then*

(1) $\alpha(B) = 0 \Leftrightarrow \alpha(N(B, \varepsilon)) \leq 2\varepsilon \Leftrightarrow \overline{B}$ *is compact*;

(2) $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$;

(3) $\alpha(B_1) \leq \alpha(B_2)$, *whenever* $B_1 \subset B_2$;

(4) $\alpha(B_1 \bigcup B_2) \leq \max\{\alpha(B_1), \alpha(B_2)\}$;

(5) $\alpha(B) = \alpha(\overline{B})$;

(6) *if* B *is a ball of radius* ε, *then* $\alpha(B) \leq 2\varepsilon$.

Lemma 2.3. *Let* X *be a complete metric space, and* α *be the measure of non-compactness. Assume that* F_n *is a sequence of bounded and closed subsets of* X, *satisfying*

(1) $F_n \neq \varnothing$;

(2) $F_{n+1} \subset F_n$, $n = 1, 2, \cdots$, *and*

(3) $\alpha(F_n) \to 0$ *as* $n \to \infty$.

Then $F = \bigcap_{n=1}^{\infty} F_n$ *is nonempty and compact.*

3. The Existence of Compact Exponential Attracting Set for $(S(t), X)$

Definition 3.1 (Ma et al. [4]). A semigroup $S(t)$ is called ω-*limit compact* if for every bounded B of X and for any $\varepsilon > 0$, there exists a $t_0 > 0$ such that $\alpha(\bigcup_{t \geq t_0} S(t)B) \leq \varepsilon$.

Lemma 3.2 (Ma et al. [4]). *Assume semigroup* $S(t)$ *is* ω-*limit compact, then for any sequence* $t_n \in R^+$, $t_n \to \infty$, *as* $n \to \infty$, *and any sequence* $x_n \in B$, *there exists a convergence subsequence of* $\{S(t_n)x_n\}$ *whose limit lies in* $\omega(B)$, *where* B *is bounded and* $\omega(B)$ *is* ω-*limit set of* B *defined by*

$$\omega(B) = \bigcap_{t\geq 0}\overline{\bigcup_{s\geq t} S(s)B}.$$

Next, we first consider discrete dynamical system $(S(n), X)$.

Theorem 3.3. *Assume that B is a bounded absorbing set w.r.t discrete dynamical system $S(n)$ in X. The following are equivalent*:

(1) *The dynamical system $(S(n), X)$ is exponential decay of the measure of non-compactness, i.e., there exist $k_1, k_2 > 0$ such that*

$$\alpha(\bigcup_{n\geq m} S(n)B) \leq k_1 e^{-k_2 m}.$$

(2) *There exists a compact set E such that*

(i) $S(n)E \subset E$, *and*

(ii) $d(S(n)B, E) \leq k_1 e^{-k_2 n}$, *for some $k_1, k_2 > 0$.*

Obviously, semigroup $S(n)$ is ω-limit compact if $S(n)$ is exponential decay of the measure of non-compactness.

Proof ((1) $\Rightarrow$ (2))**.** $S(n)$ is exponential decay of the measure of non-compactness w.r.t B, by Lemma 3.2 and Theorem 2.1, we get

$$\mathcal{A} = \bigcap_{m=1}^{\infty}\overline{\bigcup_{n\geq m} S(n)B}, \tag{3.1}$$

is the global attractor for $S(n)$. Using (3) of Lemma 2.2 and the assumption of (1), we find $\alpha(S(m)\mathcal{B}) \leq \alpha(\bigcup_{n\geq m} S(n)B) \leq k_1 e^{-k_2 m}$, by the definition of the measure of non-compactness. $\forall m \in \mathbb{N}$, there exist finite points $x_{m_i} \in S(m)B$, and $S(m)B \subset \bigcup B(x_{m_i}, k_1 e^{-k_2 m})$. Let $E_m = \{x_{m_i}\}$, setting

$$E_0 = \cup_{m,k\geq 1} S(m)E_k,$$

and

$$E = \mathcal{A} \cup E_0.$$

We claim that E is the compact set we are looking for.

Since $S(n)\mathcal{A} = \mathcal{A}$, $S(n)E_0 \subset E_0$, so we get $S(n)E \subset E$, in addition we find E_0 is a countable set, for arbitrary sequence of $x_n \in E$, by Lemma 3.2, there exists a convergence subsequence has an accumulation point in $\mathcal{A}$. Furthermore, $\alpha(S(m)B) \le k_1 e^{-k_2 m}$, so we get $d(S(m)B, E) \le k_1 e^{-k_2 m}$.

((2) $\Rightarrow$ (1)). Since E is compact, hence there exist $x_1, x_2, \ldots, x_l \in E$, such that $E \subset \bigcup_{i=1}^{l} B(x_i, k_1 e^{-k_2 n})$, therefore we have

$$S(n)B \subset \bigcup_{i=1}^{l} B(x_i, 2k_1 e^{-k_2 n}).$$

By (3.1) we know that $\mathcal{A} = \lim_{k\to\infty} \overline{\bigcup_{m\ge k} S(m)B}$, for $\varepsilon = k_1 e^{-k_2 n}$, there exists K, $\forall k \ge K$, we get

$$\bigcup_{m\ge k} S(m)\mathcal{B} \subset B(\mathcal{A}, k_1 e^{-k_2 n}).$$

$\mathcal{A}$ is compact, so there exist finite points $x_i' \in \mathcal{A}$ such that $\mathcal{A} \subset \bigcup B(x_i', k_1 e^{-k_2 n})$,

and $$\bigcup_{m\ge k} S(m)\mathcal{B} \subset \bigcup B(x_i', 2k_1 e^{-k_2 n}).$$

By (4) of Lemma 2.2, we find $\alpha(\bigcup_{m\ge n} S(m)B) \le 2k_1 e^{-k_2 n}$, i.e., $S(n)$ is exponential decay of the measure of non-compactness.

Theorem 3.4. *Assume B be a bounded absorbing set for $S(t)$ in X, and there exists integer number $M > 0$, for $t_0 \ge M$, $\alpha(\bigcup_{t\ge t_0} S(t)B) \le k_1 e^{-k_2 t_0}$ for some $k_1, k_2 > 0$, i.e., $S(t)$ is exponential decay of the measure of non-compactness. Then there exists a compact set E, for any bounded $B \in B(X)$, there exist η_1, η_2 and $T > 0$ such that*

$$d(S(t)B,\ E) \le \eta_1 e^{-\eta_2 t},\ \textit{for}\ t \ge T.$$

Proof. By the proof of Theorem 3.3, we know that there exists a compact set E and $k_1,\ k_2 > 0$ such that

$$d(S(n)B,\ E) \le k_1 e^{-k_2 n},\ \textit{for}\ n \ge M. \tag{3.2}$$

Since B is bounded absorbing set of $S(t)$, so we get there exists $T_1 > 0$, $\forall t \ge T_1$,

$$S(t)B \subset B.$$

Let $T = M + T_1 + 1$, $\forall t \ge T$ there exists a unique $T_0 \in [T_1,\ T_1 + 1)$ such that $t - T_0 = [t - T_0]$, where [ ] denote maximum integer number. By (3.2) we must have

$$d(S(t)B,\ E) = d(S(t - T_0)S(T_0)B,\ E) \le k_1 e^{-k_2(t-T_0)} \le k_1 e^{k_2(T_1+1)} e^{-k_2 t}.$$

Setting $\eta_1 = k_1 e^{k_2(T_1+1)}$, $\eta_2 = k_2$, we obtain

$$d(S(t)B,\ E) \le \eta_1 e^{-\eta_2 t}.$$

For any bounded set B, we take T_1 satisfies $S(t)B \subset B$ $\forall t \ge T_1$, the same conclusion is also hold true.

We present now a method to verify that the semigroup $S(t)$ is exponential decay of the measure of non-compactness.

Definition 3.5 (Condition $\hat{\mathrm{C}}$)**.** Let X be a convex Banach space, for any bounded set B of X, there exist $k_1,\ k_2$ and $T > 0$, and for any finite dimension subspace X_1 of X, such that

(i) $P_m(\bigcup_{t \ge t_0} S(t)B)$ is bounded; and

(ii) $\|(I - P_m)(\bigcup_{t \ge 2t_0} S(t)x)\| \le k_1 e^{-k_2 t_0} + k(m),\ \forall x \in B,$

for all $t_0 \ge T$, where $P_m : X \to X_1$ is a bounded projector, m is the dimension of X_1, $\|\cdot\|$ denote the norm in X and $k(s)$ is real-valued function satisfying

$$\lim_{s\to\infty} k(s) = 0.$$

Theorem 3.6. *A semigroup $S(t)$ satisfies condition $\hat{C}$ implies that the semigroup $S(t)$ is exponential decay of the measure of non-compactness.*

Proof. For any bounded set B of X, from (2) and (6) of Lemma 2.2, and by condition $\hat{\mathrm{C}}$, we get

$$\begin{aligned}\alpha(\bigcup_{t\geq t_0} S(t)B) &\leq \alpha(P_m(\bigcup_{t\geq t_0} S(t)B)) + \alpha((I-P_m)(\bigcup_{t\geq t_0} S(t)B))\\ &= \alpha((I-P_m)(\bigcup_{t\geq t_0} S(t)B))\\ &= \alpha((I-P_m)(\bigcup_{t\geq 2(t_0/2)} S(t)B))\\ &\leq k_1 e^{-\frac{k_2}{2}t_0} + k(m).\end{aligned}$$

Since $k(m)\to 0$ for $\varepsilon_0 = k_1 e^{-\frac{k_2}{2}t_0}$, there exists $M>0$, $\forall m > M$, we have

$$|k(m)| < k_1 e^{-\frac{k_2}{2}t_0}.$$

Hence, $\alpha(\bigcup_{t\geq t_0} S(t)B) \leq 2k_1 e^{-\frac{k_2}{2}t_0}$ $\forall m > M$, that is, $S(t)$ is exponential decay of the measure of non-compactness.

By Theorem 3.4 and Theorem 3.6, we have the following theorem.

Theorem 3.7. *Let X be a complete metric space and $S(t)$ be a semigroup in X. Then there is a compact set E exponential attracting every bounded set B of X if the following conditions hold true*:

(i) *$S(t)$ satisfies condition $\hat{\mathrm{C}}$, and*

(ii) *there is a bounded absorbing set $B\subset X$.*

Remark 3.8. In fact $S(t)$ satisfies condition $\hat{\mathrm{C}}$ implies that $S(t)$ is ω-limit compact, by Theorem 2.1, we get $S(t)$ has a global attractor $\mathcal{A}$, from Theorem 3.3, we know that $\mathcal{A}\subset E$.

4. Example

This section is to apply our abstract theory developed in Section 3 to obtain the existence of compact exponential attracting set for reaction diffusion equation in H_0^1.

We consider the following nonlinear reaction diffusion equation:

$$u_t - \Delta u + f(u) = g(x), \tag{4.1}$$

$$u|_{\partial\Omega} = 0, \tag{4.2}$$

$$u(x, 0) = u_0, \tag{4.3}$$

where Ω is a bounded smooth domain in $R^n, g(x) \in L^2(\Omega)$, f is a $\mathcal{C}^1$ function and there exist $p \geq 2$, $c_i > 0$, $i = 1, \ldots, 5$ such that

$$c_1|u|^p - c_2 \leq f(u)u \leq c_3|u|^p + c_4, \tag{4.4}$$

$$f_u(u) \geq -c_5, \tag{4.5}$$

for all $u \in R$.

For convenience, hereafter let $|\cdot|_p$ be the norm of $L^p(\Omega)(p \geq 1)$, and c the arbitrary positive constants, which may be different from line to line and even in the same line. We denote $H = L^2(\Omega)$ with scalar product $(\cdot)$ and norm $|\cdot|$, let $((\cdot))$ and $\|\cdot\|$ denote the scalar product and norm of $H_0^1(\Omega)$ and $((u, v)) = \int_\Omega \nabla u \nabla v dx$ for all $u, v \in H_0^1$.

For this initial boundary value problem, we know from [5], that for any initial date $u_0 \in L^2$ and any $T > 0$, there exists a unique solution $u(t) \in \mathcal{C}([0, T); H(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \cap L^p(0, T; L^p(\Omega))$.

Thanks to the existence theorem, the initial boundary value problem is equivalent to a semigroup $S(t)$ defined by

$$S(t) : H \to H. \tag{4.6}$$

Lemma 4.1 (Zhong et al. [6]). *The semigroup* $S(t)$ *has a bounded absorbing set in* $H_0^1(\Omega)$ *and* $L^{2p-2}(\Omega)$ *respectively, that is, for any bounded subset* B *in* $L^2(\Omega)$, *there exists a positive constant* T, *such that*

$$|u(t)|_{2p-2}^{2p-2} \le M \text{ for any } u_0 \in B \text{ and } t \ge T \tag{4.7}$$

and

$$\|u(t)\|^2 \le M \text{ for any } u_0 \in B \text{ and } t \ge T, \tag{4.8}$$

where M *is a positive constant independent of* B, $u(t) = S(t)u_0$.

Next we verify that $S(t)$ satisfies condition $\hat{\text{C}}$ in H_0^1.

We set $A = -\Delta$, since A^{-1} is a continuous compact operator in H, by the classical spectral theorem, there exist a sequence $\{\lambda_j\}_{j=1}^{\infty}$,

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \ldots, \quad \lambda_j \to +\infty, \text{ as } j \to \infty,$$

and a family of elements $\{e_j\}_{j=1}^{\infty}$ of $H_0^1(\Omega)$ which are orthonormal in H such that

$$Ae_j = \lambda_j e_j \quad \forall j \in N.$$

Let $H_m = span\{e_1, e_2, \ldots, e_m\}$ in H and $P_m : H \to H_m$ is a orthogonal projector. For any $u \in H$ we write

$$u = P_m u + (I - P_m)u \triangleq u_1 + u_2.$$

Lemma 4.2. *Assume* $g(x) \in L^2(\Omega)$ *and* f *satisfies* (4.4) *and* (4.5), *and let* $S(t)$ *be the semigroup associated with* (4.1)-(4.3). *Then* $S(t)$ *satisfies condition* $\hat{\text{C}}$ *in* H_0^1, *that is, for any bounded subset* B *in* $L^2(\Omega)$, *there exist* $k_1, k_2, T > 0$ *and* $k(m)$ *such that*

$$\|(I - P_m) \cup_{t \geq 2t_0} u(t)\|^2 \leq k_1 e^{k_2 t_0} + k(m), \text{ for any } u_0 \in B,$$

and

$$\lim_{m \to \infty} k(m) = 0,$$

provided that $t_0 \geq T$.

Combining Lemma 4.1 and Lemma 4.2, by using Theorem 3.7, we have the following result.

Theorem 4.3. *Assume* $g(x) \in L^2(\Omega)$ *and* f *satisfies* (4.4) *and* (4.5), *and let* $S(t)$ *be the semigroup associated with* (4.1)-(4.3). *Then* $S(t)$ *has a compact exponential attracting set* E *in* $H_0^1(\Omega)$, *i.e., for any bounded* $B \subset X$, *there exist* η_1, η_2 *and* $T > 0$ *such that*

$$d(S(t)B, E) \leq \eta_1 e^{-\eta_2 t}, \text{ for } t \geq T.$$

Proof. By Lemma 4.1, for any bounded subset B in H, there exist positive constants T and M, such that

$$|u(t)|_{2p-2}^{2p-2} \leq M, \|u(t)\| \leq M \text{ for any } u_0 \in B \text{ and } t \geq T. \tag{4.9}$$

Taking inner product of (4.1) with $-\Delta u_2$ in H, we get

$$\frac{1}{2}\frac{d}{dt}\|u_2\|^2 + |\Delta u_2|^2 \leq |f(u)||\Delta u_2| + |g(x)||\Delta u_2|, \tag{4.10}$$

since

$$|f(u)||\Delta u_2| \leq \frac{|\Delta u_2|^2}{4} + |f(u)|^2,$$

$$|g(x)||\Delta u_2| \leq \frac{|\Delta u_2|^2}{4} + |g(x)|^2,$$

(4.10) implies that

$$\frac{d}{dt}\|u_2\|^2 + |\Delta u_2|^2 \leq 2(|f(u)|^2 + |g(x)|^2).$$

Using (4.4), we find

$$|f(u)|^2 = \int_\Omega |f(u)|^2 dx \le c(|u|_{2p-2}^{2p-2} + 1),$$

thanks to (4.9) and Poincaré inequality, we get

$$\frac{d}{dt}\|u_2\|^2 + \lambda_m \|u_2\|^2 \le c, \text{ for } t \ge T,$$

applying the Gronwall's lemma, we have

$$\|u_2(t)\|^2 \le e^{-\lambda m(t-t_0)}\|u_2(t_0)\|^2 + \frac{c}{\lambda_m}$$

$$\le e^{-\lambda_1 t_0}\|u(t_0)\|^2 + \frac{c}{\lambda_m},$$

for $t \ge 2t_0$, and $t_0 > T$. Obviously, condition $\hat{C}$ hold true.

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