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BIFURCATIONS OF TRAVELLING WAVE SOLUTIONS FOR A CLASS OF GENERALIZED NONLINEAR EVOLUTION EQUATIONS*

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Abstract

By using the bifurcation theory of dynamical systems to a class of generalized nonlinear evolution equations, the existence of solitary wave solutions, kink and anti-kink wave solutions, periodic cusp wave solutions and uncountably infinite many smooth and non-smooth periodic wave solutions is obtained. Under different parametric conditions, various sufficient conditions to guarantee the existence of the above solutions are given. Some exact explicit parametric representations of the above travelling wave solutions are determined.

1. Introduction

In this paper, we would like to seek the travelling wave solutions of a class of generalized nonlinear evolution equations, namely

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$$(u^n)_{tt} + a(u^n)_{xx} + bu + cu^3 = 0, \tag{1}$$

where a, b and c are constant, $n \geq 1$ is positive integer. The equation reduces to the following reaction duffing equation for $n = 1$

$$u_{tt} + au_{xx} + bu + cu^3 = 0. \tag{2}$$

Equation (2) can be reduces to many famous NPDE_s, such as ϕ^4 equation, Klein-Gorden equation, Duffing equation, Landau-Ginburg-Higgs equation and so on [1, 4, 6]. By using trigonometric function method, Yang et al. [6] considered some travelling wave solutions for Equation (2). By using the tanh-method, Xie et al. [3] considered some travelling wave solutions for Equation (2). But, the author does not make further research the dynamical behavior of solutions of Equation (2). It is very important to understand the dynamical behavior of solutions for the travelling wave equation of Equation (1). To answer this question, we shall consider the bifurcations of travelling wave solutions of (1) in the five -parameter space (a, b, c, n, λ).

Let $u(x, t) = \phi(\xi) = \phi(x - \lambda t)$, substituting $\phi(x - \lambda t)$ into (1), we obtain

$$n(a + \lambda^2)\phi^{n-1}\phi'' + n(n-1)(a + \lambda^2)\phi^{n-2}(\phi')^2 + b\phi + c\phi^3 = 0, \tag{3}$$

where "/" is the derivative with respect to ξ, λ is the wave speed and $\xi = x - \lambda t$.

Clearly, (3) is equivalent to the following two-dimensional systems for $a + \lambda^2 \neq 0$:

$$\frac{d\phi}{d\xi} = y, \; \frac{dy}{d\xi} = -\frac{1}{n(a + \lambda^2)\phi^{n-1}}\left(n(n-1)(a + \lambda^2)\phi^{n-2}y^2 + b\phi + c\phi^3\right). \tag{4}$$

System (4) has the first integrals

$$y^2 = \frac{1}{n(a + \lambda^2)\phi^{2(n-1)}}\left(\frac{-2b}{n+1}\phi^{n+1} + \frac{-2c}{n+3}\phi^{n+3} + h\right) \tag{5}$$

and

$$H(\phi,\ y) = n(a + \lambda^2)\phi^{2(n-1)}y^2 + \frac{2b}{n+1}\phi^{n+1} + \frac{2c}{n+3}\phi^{n+3} = h. \tag{6}$$

System (4) is a planar dynamical system defined in the 5-parameter space (a, b, c, n, λ). Because the phase orbits defined by the vector fields of (4) determine all travelling wave solutions, we will investigate bifurcations of phase portraits of this system as these parameters are varied. The bifurcation theory of dynamical systems (see [2]) is very important and useful.

For different n and a fixed a and λ, we shall investigate the bifurcations of phase portraits of (4) in the phase plane (ϕ, y) as the parameters b, c are changed. It is easy to see that the right-hand side of the second equation in (4) is generally not continuous when $n \neq 1$, $\phi = 0$. In other words, on such straight lines in the phase plane (ϕ, y), the function ϕ''_ξ is not well-defined. It implies that smooth Systems (1) sometimes have non-smooth travelling wave solutions (see [5]).

2. Bifurcations of Phase Portraits of System (4)

In this section, we study all possible periodic annuluses defined by the vector fields of (4) when the parameters *b, c* are varied. Systems (4) has the same phase orbits as the following systems

$$\frac{d\phi}{d\tau} = n(a + \lambda^2)\phi^{n-1}y, \quad \frac{dy}{d\tau} = -(n(n-1)(a + \lambda^2)\phi^{n-2}y^2 + b\phi + c\phi^3) \tag{7}$$

except for the straight line $\phi = 0$, where $d\xi = n(a + \lambda^2)\phi^{n-1}d\tau$. Now, the straight lines $\phi = 0$ is an integral invariant straight line of (7).

For $bc < 0$, there exist 3 equilibrium points of (4) at $S(0, 0)$ and $A_{\pm}(\pm\sqrt{-\frac{b}{c}},\ 0)$; for $bc > 0$, there exist one equilibrium points of (4) at $S(0, 0)$; when $n = 3$, in the straight line $\phi = 0$ there exist two equilibrium points at $S_{\pm}(0, Y_{\pm})$, where $Y_{\pm} = \pm\sqrt{\dfrac{-6(a+\lambda^2)}{b}}$ for $b(a + \lambda^2) < 0$.

Let $M(\phi_i, y_i)$ be the coefficient matrix of the linearized system of (4) at an equilibrium point (ϕ_i, y_i). Then, we have $Trace\ (M(\phi_i, 0)) = 0$, $J(M(\phi_i, 0)) = n(a + \lambda^2)\phi_i^{n-1}(b + 3c\phi_i^2)$, for $H(\phi, y)$ defined by (6), we have $H(0, 0) = 0$, $H(\pm\sqrt{-\dfrac{b}{c}}, 0) = \dfrac{4b}{(n+1)(n+3)}\left(\pm\sqrt{-\dfrac{b}{c}}\right)^{n+1}$.

For a fixed h, the level curve $H(\phi, y) = h$ defined by (6) determines a set of invariant curves of (4), which contains different branches of curves. As h is varied, it defines different families of orbits of (4) with different dynamical behaviors.

From the above analysis we obtain the different phase portraits of (4) shown in Figure 1 and Figure 2.

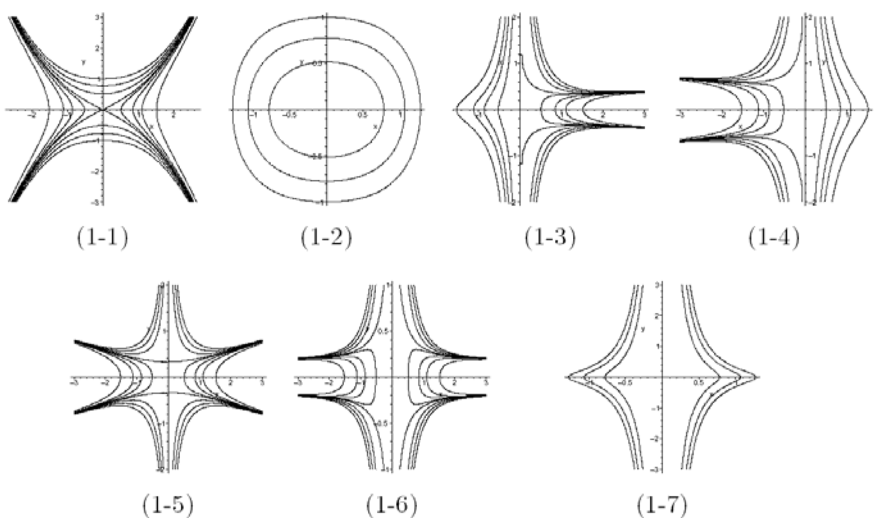

Figure 1. The phase portraits of (4) for $bc > 0$, $(a + \lambda^2) \neq 0$, $k \in Z_+$.

(1-1) $n = 1,\ b < 0,\ c < 0,\ a + \lambda^2 > 0$ or $b > 0,\ c > 0,\ a + \lambda^2 < 0.$

(1-2) $n = 1,\ b > 0,\ c > 0,\ a + \lambda^2 > 0$ or $b < 0,\ c < 0,\ a + \lambda^2 < 0.$

(1-3) $n = 2k,\ b < 0,\ c < 0,\ a + \lambda^2 > 0$ or $b > 0,\ c > 0,\ a + \lambda^2 < 0.$

(1-4) $n = 2k,\ b > 0,\ c > 0,\ a + \lambda^2 > 0$ or $b < 0,\ c < 0,\ a + \lambda^2 < 0.$

(1-5) $n = 3,\ b < 0,\ c < 0,\ a + \lambda^2 > 0$ or $b > 0,\ c > 0,\ a + \lambda^2 < 0.$

(1-6) $n = 2k + 3,\ b < 0,\ c < 0,\ a + \lambda^2 > 0$ or $b > 0,\ c > 0,\ a + \lambda^2 < 0.$

(1-7) $n = 2k + 1,\ b > 0,\ c > 0,\ a + \lambda^2 > 0$ or $b < 0,\ c < 0,\ a + \lambda^2 < 0.$

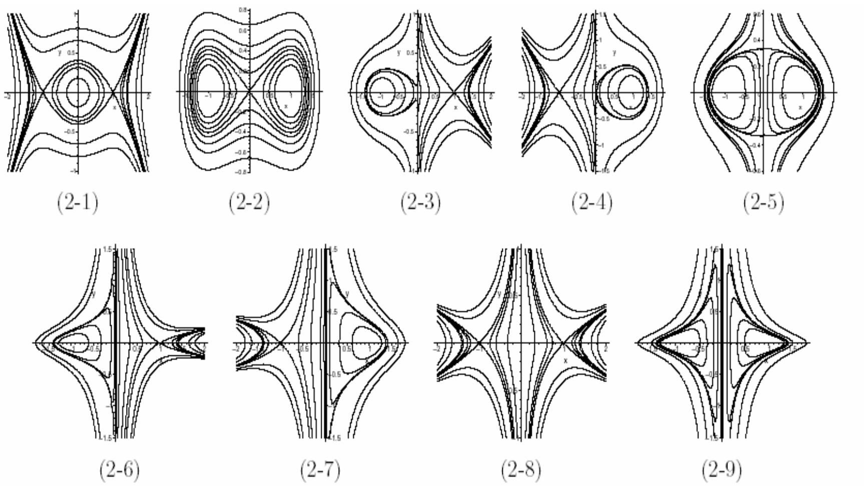

Figure 2. The phase portraits of (4) for $bc < 0,\ (a + \lambda^2) \neq 0,\ k \in Z_+$.

(2-1) $n = 1, b > 0, c < 0, a + \lambda^2 > 0$ or $b < 0, c > 0, a + \lambda^2 < 0.$

(2-2) $n = 1,\ b < 0,\ c > 0,\ a + \lambda^2 > 0$ or $b > 0,\ c < 0,\ a + \lambda^2 < 0.$

(2-3) $n = 2,\ b > 0,\ c < 0,\ a + \lambda^2 > 0$ or $b < 0,\ c > 0,\ a + \lambda^2 < 0.$

(2-4) $n = 2,\ b < 0,\ c > 0,\ a + \lambda^2 > 0$ or $b > 0,\ c < 0,\ a + \lambda^2 < 0.$

(2-5) $n = 3,\ b < 0,\ c > 0,\ a + \lambda^2 > 0$ or $b > 0,\ c < 0,\ a + \lambda^2 < 0.$

(2-6) $n = 2(k+1),\ b > 0,\ c < 0,\ a + \lambda^2 > 0\ \ b < 0,\ c > 0,\ a + \lambda^2 < 0.$

(2-7) $n = 2(k+1),\ b < 0,\ c > 0,\ a + \lambda^2 > 0$ or $b > 0,\ c < 0,\ a + \lambda^2 < 0.$

(2-8) $n = 2k+1,\ b > 0,\ c < 0,\ a + \lambda^2 > 0$ or $b < 0,\ c > 0,\ a + \lambda^2 < 0.$

(2-9) $n = 2k+3,\ b < 0,\ c > 0,\ a + \lambda^2 > 0$ or $b > 0,\ c < 0,\ a + \lambda^2 < 0.$

3. Exact Explicit Parametric Representations of Travelling Wave Solutions of System (1)

In this section, by using (5), (6) and the first equations of (4) to do integrations respectively, we give some exact explicit parametric representations of travelling wave solutions of (1).

1. Suppose that $n = 1,\ b < 0,\ c > 0,\ a + \lambda^2 > 0$ or $b > 0,\ c < 0,\ a + \lambda^2 < 0.$ In this case, we have the phase portrait of (4) shown in Figure 2(2-2). Notice that $H(0, 0) = 0 = h.$ We see from (6) that the arch curve connecting $S(0, 0)$ has the algebraic equation

$$y^2 = \frac{c}{2(a+\lambda^2)}\phi^2\left(\frac{-2b}{c} - \phi^2\right). \tag{8}$$

Thus, by using (8) and the first equation of (4) to calculate, we obtain the parametric representation of this arch as follows:

$$\phi = \frac{2\exp\left(\pm\sqrt{\frac{-b}{a+\lambda^2}}\xi\right)}{\exp\left(\pm 2\sqrt{\frac{-b}{a+\lambda^2}}\xi\right) - \frac{c}{2b}}. \tag{9}$$

It follows that for $n = 1,\ b < 0,\ c > 0,\ a + \lambda^2 > 0$ or $b > 0,\ c < 0,\ a + \lambda^2 < 0,$ Equation (1) has the solutions with the parametric representations

$$u(x, t) = \frac{2\exp\left(\pm\sqrt{\frac{-b}{a+\lambda^2}}(x-ct)\right)}{\exp\left(\pm 2\sqrt{\frac{-b}{a+\lambda^2}}(x-ct)\right) - \frac{c}{2b}}. \tag{10}$$

We will show in Section 4 that (10) gives rise to two solitary wave solutions of peak type and valley type of Equation (1).

2. Suppose that $n = 1, b > 0, c < 0, a + \lambda^2 > 0$ or $n = 1, b < 0, c > 0, a + \lambda^2 < 0.$ In this case, we have the phase portrait of (4) shown in Figure 2 (2-1). Notice that $H\left(\pm\sqrt{\frac{-b}{c}}, 0\right) = \frac{-b^2}{2c}$. We see from (6) that the arch curve connecting A_- and A_+ has the algebraic equation

$$y^2 = \frac{-c}{2(a + \lambda^2)}\left(\phi^2 + \frac{b}{c}\right)^2. \tag{11}$$

It follows that for $n = 1, b > 0, c < 0, a + \lambda^2 > 0$ or $n = 1, b < 0, c > 0, a + \lambda^2 < 0,$ Equation (1) has the solutions with the parametric representations

$$u(x, t) = \pm\sqrt{\frac{-b}{c}}\tanh\left(\sqrt{\frac{b}{2(a + \lambda^2)}}(x - ct)\right). \tag{12}$$

We will show in Section 4 that (12) gives rise to a kink and an anti-kink wave solutions of Equation (1).

3. Suppose that $n = 2, b > 0, c < 0, a + \lambda^2 > 0$ or $n = 2, b < 0, c > 0, a + \lambda^2 < 0.$ In this case, we have the phase portrait of (4) shown in Figure 2 (2-3). Notice that $H(0, 0) = 0 = h.$ We see from (6) that the arch curve has the algebraic equation

$$y^2 = \frac{-c}{5(a + \lambda^2)}\phi\left(\phi + \sqrt{\frac{-5b}{6c}}\right)\left(\phi - \sqrt{\frac{-5b}{6c}}\right). \tag{13}$$

Thus, by using the first equation of (4) and (13) to calculate, we obtain the parametric representation of this arch as follows:

$$\phi = -\sqrt{\frac{-5b}{6c}}cn^2(\Omega_1\xi, k), \tag{14}$$

where $cn(x, k)$ is the Jacobin elliptic function with the modulo k and $\Omega_1 = \left(\frac{-bc}{120(a + \lambda^2)^2}\right)^{\frac{1}{4}}, k^2 = \frac{1}{2}.$

It follows that for $n = 2,\ b > 0,\ c < 0,\ a + \lambda^2 > 0$ or $n = 2,\ b < 0,\ c > 0,\ a + \lambda^2 < 0,$ Equation (1) has the solutions with the parametric representations

$$u(x, t) = -\sqrt{\frac{-5b}{6c}}cn^2(\Omega_1(x - ct), k). \tag{15}$$

4. Suppose that $n = 2, b < 0, c > 0, a + \lambda^2 > 0$ or $n = 2, b > 0, c < 0, \alpha + \lambda^2 < 0.$ In this case, we have the phase portrait of (4) shown in Figure 2 (2-4). Similar to the Case 3, Equation (1) has the solutions with the parametric representations

$$u(x, t) = \sqrt{\frac{-5b}{6c}}cn^2(\Omega_1(x - ct), k). \tag{16}$$

5. Suppose that $n = 3, b < 0, c > 0, a + \lambda^2 > 0$ or $n = 3, b > 0, c < 0, a + \lambda^2 < 0.$ In this case, we have the phase portrait of (4) shown in Figure 2 (2-5). Notice that $H(0, 0) = 0 = h.$ We see from (6) that the arch curve has the algebraic equation

$$y^2 = \frac{c}{9(a + \lambda^2)}\left(\frac{-3b}{2c} - \phi^2\right). \tag{17}$$

It follows that for $n = 3, b < 0, c > 0, a + \lambda^2 > 0$ or $n = 3, b > 0, c < 0, a + \lambda^2 < 0,$ Equation (1) has the solutions with the parametric representations

$$u(x, t) = \pm\sqrt{\frac{-3b}{2c}}\cos\left(\frac{1}{3}\sqrt{\frac{c}{a + \lambda^2}}(x - ct)\right),\ 0 < x - ct < \frac{3\pi}{2}\sqrt{\frac{a + \lambda^2}{c}}. \tag{18}$$

We will show in Section 4 that (18) gives rise to a periodic cusp wave solution of Equation (1).

6. Suppose that $n = 1, b > 0, c > 0, a + \lambda^2 > 0$ or In this case, we have the phase portrait of (4) shown in Figure 1 (1-2). Corresponding to that $H(\phi, y) = h, h \in (0, +\infty)$ or $h \in (-\infty, 0)$ defined by (6), System (4) has a family of periodic solutions enclosing the center $(0, 0),$ these orbits

determine uncountable infinite many periodic wave solutions of Equation (1). We see from (6) that the arch curve has the algebraic equation

$$y^2 = -\frac{c}{2(a+\lambda^2)}\left(\phi^2 + \frac{b-\sqrt{b^2+2ch}}{c}\right)\left(\phi^2 + \frac{b+\sqrt{b^2+2ch}}{c}\right). \tag{19}$$

Thus, by using the first equation of (4) and (19), we obtain

$$\frac{d\phi}{\sqrt{\left(\frac{-b+\sqrt{b^2+2ch}}{c} - \phi^2\right)\left(\phi^2 + \frac{b+\sqrt{b^2+2ch}}{c}\right)}} = \pm\sqrt{\frac{c}{2(a+\lambda^2)}}d\xi. \tag{20}$$

Making the transformations $\phi^2(\xi) = \frac{-b+\sqrt{b^2+2ch}}{c}(1-t^2)$, (20) becomes

$$\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \pm\Omega_2 d\xi. \tag{21}$$

Thus, we obtain the parametric representation of this arch as follows:

$$\phi = \pm\sqrt{\frac{-b+\sqrt{b^2+2ch}}{c}}cn(\Omega_2\xi, k_2), \tag{22}$$

where $\Omega_2 = \frac{(b^2+2ch)^{\frac{1}{4}}}{\sqrt{a+\lambda^2}}$, $k_2^2 = \frac{-b+\sqrt{b^2+2ch}}{2\sqrt{b^2+2ch}}$.

It follows that for $n = 1, b > 0, c > 0, a+\lambda^2 > 0$ or $n = 1, b < 0, c < 0, a+\lambda^2 < 0$, Equation (1) has a family of periodic solutions with the parametric representations

$$u(x, t) = \pm\sqrt{\frac{-b+\sqrt{b^2+2ch}}{c}}cn(\Omega_2(x-ct), k_2). \tag{23}$$

For $n = 1, b < 0, c < 0, a+\lambda^2 < 0$, we also can give exact explicit travelling wave solutions (6) by using elliptic functions, we omit it.

4. The Existence of Smooth Travelling Wave Solutions of (1)

In this section, we use the results of Section 2 to discuss the existence of smooth and non-smooth solitary wave solutions and periodic wave solutions. We first consider the existence of smooth solitary wave solutions and periodic wave solutions.

Theorem 4.1. 1. *Suppose that* $n = 1,\ b < 0,\ c > 0, a + \lambda^2 > 0$ *or* $b > 0,\ c < 0,\ a + \lambda^2 < 0$. *Then,*

(i) *corresponding to a branch of the curves* $H(\phi,\ y) = h_1$ *defined by* (6), *Equation* (1) *has a smooth kink wave solution and a smooth anti-kink wave solution* (*see Figure* 2 (2-1));

(ii) *corresponding to a branch of the curves* $H(\phi,\ y) = h,\ h \in (h_2, h_1)$ *defined by* (6), *Equation* (1) *has a smooth family of periodic wave solution* (*see Figure* 2 (2-1)).

2. *Suppose that* $n = 1,\ b < 0,\ c > 0,\ a + \lambda^2 > 0$ *or* $b > 0,\ c < 0,\ a + \lambda^2 < 0$. *Then,*

(iii) *corresponding to a branch of the curves* $H(\phi,\ y) = h_2$ *defined by* (6), *Equation* (1) *has a smooth solitary wave solutions of peak type and a smooth solitary wave solution of valley type* (*see Figure* 2 (2-2)).

(iv) *corresponding to a branch of the curves* $H(\phi,\ y) = h,\ h \in (h_1,\ h_2)$ *defined by* (6), *Equation* (1) *has two smooth families of periodic wave solutions* (*see Figure* 2 (2-2)).

3. *Suppose that* $n = 1,\ b > 0,\ c > 0,\ a + \lambda^2 > 0$ *or* $b < 0,\ c < 0,\ a + \lambda^2 < 0$. *Then, corresponding to a branch of the curves* $H(\phi,\ y) = h,\ h \in (0, +\infty)$ *or* $h \in (-\infty, 0)$ *defined by* (6), *Equation* (1) *has a smooth family of wave solutions* (*see Figure* 1 (1-2)).

4. *Suppose that* $n = 2,\ b > 0,\ c < 0,\ a + \lambda^2 > 0$ *or* $b < 0,\ c > 0,\ a + \lambda^2 < 0$. *Then, corresponding to a branch of the curves* $H(\phi,\ y) = h,$

$h \in (h_2, h_1)$ *defined by* (6), *Equation* (1) *has a smooth family of wave solutions* (*see Figure* 2 (2-3)).

5. *Suppose that* $n = 2, b < 0, c > 0, a + \lambda^2 > 0$ *or* $b > 0, c < 0, a + \lambda^2 < 0.$ *Then, corresponding to a branch of the curves* $H(\phi, y) = h, h \in (h_1, h_2)$ *defined by* (6), *Equation* (1) *has a smooth family of periodic wave solutions* (*see Figure* 2 (2-4)).

We shall describe what types of non-smooth solitary wave and periodic wave solutions can appear for our System (1) which correspond to some orbits of (7) near the straight line $\phi = 0.$ To discuss the existence of cusp waves, we need to use the following lemma relating to the singular straight line [5].

Lemma 4.2. *The boundary curves of a periodic annulus are the limit curves of closed orbits inside the annulus*; *If these boundary curves contain a segment of the singular straight line* $\phi = 0$ *of* (4), *then along this segment and near this segment*, *in very short time interval* $y = \phi_\xi$ *jumps rapidly*.

Base on Lemma 4.2 *and Figure* 2, *we have the following result*.

Theorem 4.3. 1. *Suppose that* $n = 3, b < 0, c > 0, a + \lambda^2 > 0$ *or* $b > 0, c < 0, a + \lambda^2 < 0.$ *Then*,

(i) *corresponding to the arch curve* $H(\phi, y) = h_2$ *defined by* (6), *Equation* (1) *has two periodic cusp wave solutions* (*see Figure* 2 (2-5));

(ii) *corresponding a branch of the curves* $H(\phi, y) = h, h \in (h_1, h_2)$ *defined by* (6), *Equation* (1) *has two families of periodic wave solutions*. *When* h *varies from* h_1 *to* h_2, *these periodic travelling waves will gradually lose their smoothness*, *and evolve from smooth periodic travelling waves to periodic cusp travelling waves*, *finally approach a periodic cusp wave of valley type and a periodic cusp wave of peak type defined by* $H(\phi, y) = h_2$ *of* (6) (*see Figure* 2 (2-5)).

2. *Suppose that* $n = 2(k+1) > 2$, $b > 0$, $c < 0$, $a + \lambda^2 > 0$ *or* $b < 0$, $c > 0$, $a + \lambda^2 < 0$. *Then, corresponding to the arch curve* $H(\phi, y) = h$, $h \in (h_2, h_1)$ *defined by* (6), *Equation* (1) *has a family periodic wave solutions. When* h *varies from* h_2 *to* h_1, *these periodic travelling waves will gradually lose their smoothness, and evolve from smooth periodic travelling waves to periodic cusp travelling waves* (*see Figure* 2 (2-6)).

3. *Suppose that* $n = 2(k+1) > 2$, $b < 0$, $c > 0$, $a + \lambda^2 > 0$ *or* $b > 0$, $c < 0$, $a + \lambda^2 < 0$. *Then, corresponding to the arch curve* $H(\phi, y) = h$, $h \in (h_3, h_2)$ *defined by* (6), *Equation* (1) *has a family periodic wave solutions. When* h *varies from* h_3 to h_2, *these periodic travelling waves will gradually lose their smoothness, and evolve from smooth periodic travelling waves to periodic cusp travelling waves* (*see Figure* 2 (2-7)).

4. *Suppose that* $n = 2k + 3 > 3$, $b < 0$, $c > 0$, $a + \lambda^2 > 0$ *or* $b > 0$, $c < 0$, $a + \lambda^2 < 0$. *Then, corresponding to the arch curve* $H(\phi, y) = h$, $h \in (h_2, h_1)$ *defined by* (6), *Equation* (1) *has two families periodic wave solutions. When* h *varies from* h_3 *to* h_2, *these periodic travelling waves will gradually lose their smoothness, and evolve from smooth periodic travelling waves to periodic cusp travelling waves* (*see Figure* 2 (2-9)).

References

[1] M. J. Ablavitz et al., Solitons, Nonlinear, Evolution Equations and Inverse Scatting, Cambridge University Press, New York, 1991.

[2] S. N. Chow and J. K. Hale, Method of Bifurcation Theory, Springer-Verlag, New York, 1981.

[3] Xie Fuding et al., Chaos, Solitons and Fractals 19 (2004), 1113-1117.

[4] C. H. Gu et al., Soliton Theory and its Appliation, Zhejiang Science and Technology Press, Zhejiang, 1990.

[5] Li Jibin and Liu Zhenrong, Travelling wave solutions for a class of nonlinear dispersive equations, Chin. Ann. Math. 23B (2002), 397-418.

[6] Z. Y. Yang et al., Acta Physica Sinica (in Chinese) 48(01) (1999), 0001-05.

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