Journal of Pure and Applied Mathematics: Advances and Applications
Volume 1, Number 1, 2009, Pages 13-19

STABILITY OF LINEAR AND NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS WITH TIME VARYING COEFFICIENTS

M. M. A. EL-SHEIKH[a] , E. F. LASHEIN[b] and M. M. HEKAL[b]

[a]Department of Mathematics
Faculty of Science
Minufiya University
Shebeen El-Koom
Egypt

[b]Department of Physics and Engineering Mathematics
Faculty of Engineering
Tanta University
Tanta, Egypt
e-mail: hh_hh1463@yahoo.com

Abstract

Sufficient conditions for uniform stability of the zero solution of linear and nonlinear second order differential equations. We use Liapunov functions technique.

1. Introduction

In this paper we study stability properties of the second order differential equation described by the equation

$$u'' + \beta(t)u = 0, \tag{1.1}$$

2000 Mathematics Subject Classification: 35L70.

Keywords and phrases: stability, second order differential equations, Liapunov functions.

Received November 19, 2008

where $\beta : R^+ \to R$ is differentiable. We also consider an important general form of nonlinear second order differential equation of the form

$$u'' + f(t, u') + \beta(t)u + g(t, u, u') = 0, \tag{1.2}$$

where $f : R^+ \times R \to R$ and $g : R^+ \times R^2 \to R$ are continuous.

Linear and nonlinear second order differential equations were studied by many mathematicians where many important results were obtained, refer for example to ([2]-[8]). Vladimirescu [8] studied the second order differential equation

$$u'' + 2f(t)u' + u + g(t, u) = 0. \tag{1.3}$$

He proved the uniform stability of the zero solution of equation (1.3) when f and g satisfy the following conditions

(i) $f \in C^1(R^+)$ and $f(t) > 0$ for $t \geq t_0$,

(ii) there exist a positive constant $\alpha \geq 0$ and $k \in (0, 1)$ such that $|f'(t) + f^2(t)| \leq kf(t)$ for all $t \in [\alpha, \infty)$,

(iii) $g \in C(R^+ \times R)$ and g is locally Lipshitzian in u,

(iv) there exists a positive constant $M \geq 0$ and $\alpha > 1$ such that $|g(t, u)| \leq M f(t) |u|^{\alpha}$.

Recently Vladimirescu et al. [7] considered a more general type of the equation (1.3) described by the equation

$$u'' + 2f(t)u' + \beta(t)u + g(t, u) = 0. \tag{1.4}$$

They proved the uniform stability of the zero solution of equation (1.4) for the above conditions (i- iii) in addition to the condition

(v)$\beta \in C^1(R^+)$, β is decreasing and $\beta(t) \geq \beta_0 \geq k^2$ with $t \in R^+$, where β_0 is constant.

In this paper we use Liapunov function to prove the uniform stability of the equation (1.1). Moreover we study the nonlinear second order

differential equation with the more general form with time varying coefficients described by equation (1.2). And we give sufficient conditions which guarantee the uniform stability of the zero solution of equation (1.2). Our results extend and improve the results obtained in [7] and [8]. In Section 2 we give the statement and proof of the main results. Illustrative examples are given in Section 3.

2. Main Results

In this section we give two theorems with new sufficient conditions for uniform stability of the zero solution of the linear and nonlinear differential equations (1.1) and (1.2) respectively. We use Liapunov theorems in our proof.

Theorem 2.1. *Assume that there exist positive constants* b_1, b_2 *and* b_3 *such that for* $t \geq t_0$, *suppose that the function* $\beta(t)$ *satisfies the following conditions* :-

$$(\mathrm{H}_1) : \beta(t) \in C^1(R^+) \text{ and } 0 < b_1 \leq \beta(t) \leq b_2,$$

$$(\mathrm{H}_2) : 0 \leq b_3 \leq \beta'(t).$$

Then the zero solution of equation (1.1) *is uniformly stable.*

Proof. Let the system equivalent form of equation (1.1) is

$$y' = A(t)y, \tag{2.1}$$

where $y = (y_1, y_2)^T \in R^2$, $(y_1 = u(t)$ and $y_2 = u'(t))$ is the solution of equation (2.1) starting at t_0 in y_0, $y_0 = (y_1(t_0), y_2(t_0))^T$ and $A(t) = \begin{bmatrix} 0 & 1 \\ -\beta(t) & 0 \end{bmatrix}$. Let

$$V(t, y) \equiv \frac{1}{2} y_1^2(t) + \frac{1}{2\beta(t)} y_2^2(t), \tag{2.2}$$

defined on a cylinder $[t_0, \infty) \times G$, where $G \subset R^2$ be a neighborhood of the origin. From condition (H_1), we have $1/\beta(t) \geq 1/b_2$. Hence,

$$V(t, y) = \frac{1}{2} y_1^2(t) + \frac{1}{2\beta(t)} y_2^2(t) \geq \frac{1}{2} y_1^2(t) + \frac{1}{2b_2} y_2^2(t) \equiv W(y) \text{ for } y \neq 0 \text{ in } G.$$

Clearly $W(0) = 0$ and $W(y) > 0$ for $y \neq 0$, then $W(y)$ is positive definite function on G. From (H_1), $V(t, y) \geq W(y)$ for (t, y) in $[t_0, \infty) \times G$ and $V(t, 0) = 0$. Thus $V(t, y)$ is a positive definite function. Now the first derivative of V along the solution curves of the equation (2.1) is

$$V'(t, y) = -\frac{\beta'(t)}{2[\beta(t)]^2} y_2^2 \leq 0.$$

So $V(t, y)$ is a Liapunov function for equation (2.1). Now we prove that $V(t, y)$ is a decrescent Liapunov function at the origin. Since from condition (H_1), we have $1/\beta(t) \leq 1/b_1$. Then

$$V(t, y) = \frac{1}{2} y_1^2(t) + \frac{1}{2\beta(t)} y_2^2(t) \leq \frac{1}{2} y_1^2(t) + \frac{1}{2b_1} y_2^2(t) \equiv U(y), (t, y) \in [t_0, \infty) \times G.$$

Clearly $U(y) > 0$ for all $y \in G - \{0\}$ and $U(0) = 0$ consequently $U(y)$ is a positive definite function on G, hence we deduce that $V(t, y)$ is a decrescent Liapunov function at the origin. Applying Theorem 10.9 ([9] pp. 330), the zero solution of equation (2.1), is uniformly stable.

Now we consider the differential equation (1.2) where $\beta(t)$ is bounded. We give sufficient conditions to prove that the zero solution is uniformly stable.

Theorem 2.2. *Let the function* $\beta(t)$ *satisfies the conditions* (H_1) *and* (H_2). *Assume that the functions f and g are locally Lipshitzian in* u' *and in* u, u' *respectively. Assume that there exist a continuous nonnegative function* $\gamma(t)$, *where* $\int_{t_0}^{\infty} \gamma(t)dt < \infty$, *such that the functions f and g satisfy the following conditions*:

(H_3) : $f(t, u') \in C^0(R^+ \times R)$, $f(t, 0) = 0$ *and* $|f(t, u')| \leq \gamma(t)\,|u'|$

(H_4) : $g(t, u, u') \in C^0(R^+ \times R^2)$, $g(t, 0, 0) = 0$ *and* $|g(t, u, u')| \leq \gamma(t)\,|(u, u')^T|$.

Then the zero solution of equation (1.2) *is uniformly stable.*

Proof. The system form equivalent to the differential equation (1.2) is

$$x' = A(t)x + h(t, x), \tag{2.3}$$

where $x = (x_1, x_2)^T$, $x_1 = u(t)$ is the solution of equation (1.2) starting at $x_0 = (x_1(t_0), x_2(t_0))^T$ when $t = t_0$ and $x_2 = u'(t)$.

And $h : R^+ \times R^2 \to R^2$, $h(t, x) = \begin{bmatrix} 0 \\ -[f(t, x_1) + g(t, x_1, x_2)] \end{bmatrix}$.

Since equations (2.1) and (2.3) are system forms corresponding to equations (1.1) and (1.2) respectively. Now from conditions, (H_1) and (H_2) we deduce that the zero solution of equation (1.1) (equivalently the System (2.1)) is uniformly stable. Let $Y(t)$ be a fundamental solution matrix of System (2.1). Applying Theorem 1 ([1] pp. 54) we deduce that there exists a constant $K \geq 0$ such that $|Y(t)Y^{-1}(s)| \leq K$ for $t_0 \leq s \leq t \leq \infty$. Using the above results and conditions (H_3) and (H_4) we can apply Theorem 6 ([1] pp. 64) to deduce that the zero solution of (2.3) is uniformly stable. Then the proof is completed.

Remark. Theorem 2.2 extend and improve the work had been done in [7] and [8]. Theorem 2.2 not only nonlinear damping terms are considered but also we are not in needed to the important condition of decreasing the function $\beta(t)$ as stated in [7].

3. Applications

Example 3.1. Consider the linear second order differential equation

$$u''(t) + (10 - \exp(-t))\;\; u(t) = 0.$$

Let $b_1 = 9$, $b_2 = 10$, $b_3 = 0$. Then the coefficients of Example 3.1 satisfy the conditions of Theorem 2.1, and hence the zero solution of Example 3.1 is uniformly stable. A numerical solution of Example 3.1 is shown in figures $(1-a, 1-b)$.

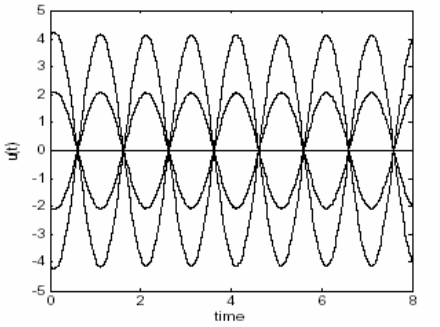

fig. (1-a)

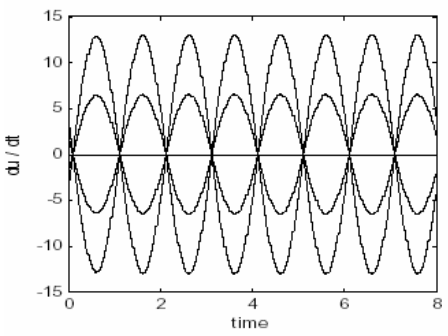

fig. (1-b)

Example 3.2. Consider the nonlinear second order differential equation

$$u''(t) + \exp(-t)\,u'(t))^{3/2} + (10 - \exp(-t))u(t) + \frac{(u(t))^2 + (u'(t))^3}{1+t^2} = 0.$$

The coefficients of Example 3.2 satisfy the conditions of Theorem 2.2 and hence its zero solution is uniformly stable. A numerical solution of Example 3.2 is shown in figures $(2-a, 2-b)$.

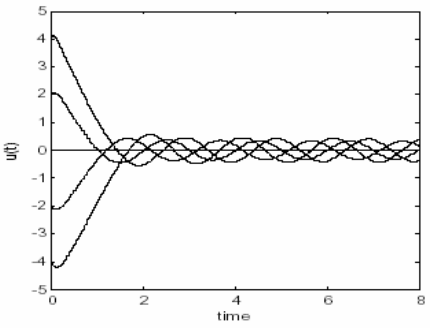

fig. (2-a)

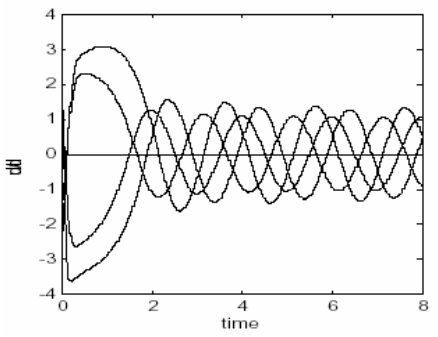

fig. (2-b)

References

[1] W. A. Coppel, Stability and asymptotic behavior of differential equations, Printed in the United State of America (1965).

[2] L. H. Duc, A. Ilchmann, S. Siegmund and P. Taraba, On stability of linear time-varying second-order differential equations. Quarterly of Appl. Maths 64 (2006), 137-151.

[3] L. Hatvani, Integral conditions on the asymptotic stability for the damped linear oscillator with small damping, Proc. Amer. Math. Soc. 124 (1996), 415-422.

[4] A. O. Ignatyev, Stability of a linear oscillator with variable parameters, Electronic J. Differential Equations 17 (1997), 1-6.

[5] E. Juan and N. Valdes, A note on the asymptotic stability for in the whole of non-autonomous systems, Revista Colombianna de Mathematics 33 (1999), 1-8.

[6] N. M. Linh and V. N. Phat, Exponential stability of nonlinear time-varying differential equations and applications, Electronic J. Differential Equation 34 (2001), 1-13.

[7] G. Morosanu and C. Vladimirescu, Stability for damped nonlinear oscillator. Nonlinear Analysis 60 (2005), 303-310.

[8] C. Vladimirescu, Stability for damped oscillator, Annals of University of Craiova, Math. Comp. Sci. Ser. 32 (2005), 227-237.

[9] H. K. Wilson, Ordinary differential equations, Edwardsvill, III (1970).

■