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ATTRACTORS FOR DOUBLY NONLINEAR PARABOLIC EQUATIONS IN UNBOUNDED DOMAINS

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Abstract

Our aim in this paper is to study the long time behavior of a class of doubly nonlinear parabolic equations in R^N. We prove the asymptotic compactness of the solutions and then establish the existence of the global attractor in $L^2(R^N)$.

1. Introduction

We are interested in the long time behavior of doubly nonlinear parabolic equations of the form

$$\frac{\partial \alpha(u)}{\partial t} - \Delta u + f(u) + \lambda u = g(x) \quad in\ R^N \times R^+. \tag{1.1}$$

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The existence of global attractor for Eq. (1.1) in a bounded domain was studied in [7]. In bounded domains, the asymptotic compactness of the solutions follows from the compactness of the Sobolev embedding. This method cannot be applied to unbounded domain, since in this case the compactness of Sobolev embedding are no longer hold. To overcome the difficulty of the noncompact embedding, several typical methods have been developed [5, 6, 10]. The existence of the global attractor in $L^2(R^N)$ for nondegenerate reaction diffusion with forcing term from $L^2(R^N)$ was, to our knowledge, first proved in [10]. In this paper, using the similar idea, we will show that the solution of (1.1) are uniformly small at infinity for large time. Combining priori estimate method, we obtain the existence of global attractor for Eq. (1.1).

Throughout this paper, we will use $(\cdot)$ and $((\cdot))$ to denote the scalar product in $L^2(R^N)$ and $H^1(R^N)$, $|\cdot|_p$ and $\|\cdot\|$ to denote the norm of $L^p(R^N)$ and $H^1(R^N)$, and $|u|$ the modular (or absolute value) of u.

2. Preliminaries

In this section, we consider the following doubly nonlinear parabolic equation of the form

$$\frac{\partial \alpha(u)}{\partial t} - \Delta u + f(u) + \lambda u = g(x) \quad \text{in } R^N \times R^+, \tag{2.1}$$

$$u(0, x) = u_0(x) \quad \text{in } R^N. \tag{2.2}$$

We assume that $g(x), u_0(x) \in L^2(R^N)$, λ is a positive constant. Furthermore, we make the following assumptions:

$$\alpha \in C^1(R),\ \alpha(0) = 0,\ c_1 \le \alpha'(s) \le c_2,\ c_1, c_2 > 0,\ s \in R; \tag{2.3}$$

$$\alpha_2 |u|^p - k_2 |u|^2 \le f(u)u \le \alpha_1 |u|^p + k_1 |u|^2,\ f'(s) \ge l, \tag{2.4}$$

here $\alpha_1, \alpha_2, k_1, k_2$ are positive constants and $p \ge 0$, $k_2 < \lambda$, $l \in R$, F is the primitive function of f : $F(s) = \int_0^s f(r)dr$.

We set $w = \alpha(u)$, note that $\alpha' \geq c_1 > 0$, hence there exists α^{-1}, the inverse function of α. Eq. (2.1) can be transformed into the following equation:

$$\frac{\partial w}{\partial t} - \Delta(\alpha^{-1}(w)) + f(\alpha^{-1}(w)) + \lambda\alpha^{-1}(w) = g(x) \quad \text{in } R^N \times R^+. \tag{2.5}$$

Under the above assumptions, in a bounded domain $B_R = \{x : |x| \leq R, x \in R^n\}$, we can prove, by using Galerkin approximations method that (2.5) is well-posed in $L^2(B_R)$ (see, e.g., [1, 8, 9]), that is, for any $w_0 \in L^2(B_R)$, there exists a unique solution $w(t) \in C(R^+, L^2(B_R)) \cap L^2(R^+, H_0^1(B_R))$. By Lemma 2.3, the solution of (2.1) in R^N is almost equal to the solution of (2.5) in a bounded domain B_R when R is large enough. Therefore, wet get for any $u_0 \in L^2(R^N)$, Eq. (2.1) exists a unique solution $u(t) \in C(R^+, L^2(R^N)) \cap L^2(R^+, H^1(R^N))$. This establishes the existence of a dynamical system $\{S(t)\}_{t\geq 0}$, which maps $L^2(R^N)$ into $L^2(R^N)$ such that $S(t)u_0 = u(t)$, the solution of problem (2.1)-(2.2); see [1-3, 8, 9].

Hereafter, the same letters c(and, sometimes, c', c'' *or* c''') denotes constants which may change from line to line and even in the same line.

Lemma 2.1. *Assume that* (2.3)-(2.4) *hold,* $g \in L^2(R^N)$. *Then for any bounded set* $D \subset L^2(R^N)$, *there exist constants* $M > 0$, $T_0 > 0$, *for any* $u_0 \in D$, *the solution* $u(t)$ *of* (2.1)-(2.2) *satisfies*

$$|u(t)|_2 \leq M, \int_t^{t+1} |\nabla u(s)|_2^2 ds \leq M,$$

$$\int_t^{t+1} |u(s)|_p^p ds \leq M, \text{ for all } t \geq T_0. \tag{2.6}$$

Proof. Taking the inner product in $L^2(R^N)$ of (2.1) with u, this yields

$$\frac{1}{2}\int_{R^N}\alpha'(u)\frac{d}{dt}|u|^2dx + |\nabla u|_2^2 + (f(u), u) + \lambda(u, u) = (g, u).$$

Since $(g, u) \le |g|_2|u|_2 \le \frac{\lambda - k_2}{2}|u|_2^2 + \frac{|g|_2^2}{2(\lambda - k_2)}$ and using (2.4), we have

$$\int_{R^N}\alpha'(u)\frac{d}{dt}|u|^2dx + 2|\nabla u|_2^2 + 2\alpha_2|u|_p^p + (\lambda - k_2)|u|_2^2 \le c,$$

here $c = \frac{|g|_2^2}{\lambda - k_2}$.

By (2.3), we find

$$\frac{d^+}{dt}|u|_2^2 + \frac{2}{c_1}|\nabla u|_2^2 + \frac{2\alpha_2}{c_1}|u|_p^p + \frac{\lambda - k_2}{c_1}|u|_2^2 \le \frac{c}{c_1}, \tag{2.7}$$

and

$$\frac{d^-}{dt}|u|_2^2 + \frac{2}{c_2}|\nabla u|_2^2 + \frac{2\alpha_2}{c_2}|u|_p^p + \frac{\lambda - k_2}{c_2}|u|_2^2 \le \frac{c}{c_2}. \tag{2.8}$$

Here $\frac{d^+}{dt}|u|_2^2 := \frac{d}{dt}|u|_2^2 \ge 0$, $\frac{d^-}{dt}|u|_2^2 := \frac{d}{dt}|u|_2^2 \le 0$.

Combining (2.7) and (2.8), and note that $c_1 \le c_2$, we obtain

$$\frac{d}{dt}|u|_2^2 + \frac{2}{c_2}|\nabla u|_2^2 + \frac{2\alpha_2}{c_2}|u|_p^p + \frac{\lambda - k_2}{c_2}|u|_2^2 \le \frac{c}{c_1}. \tag{2.9}$$

Therefore, we get

$$\frac{d}{dt}|u|_2^2 + \frac{\lambda - k_2}{c_2}|u|_2^2 \le \frac{c}{c_1}. \tag{2.10}$$

Thanks to the Gronwall inequality, we find

$$|u(t)|_2^2 \le e^{-c't}|u(0)|_2^2 + c'',$$

here $c' = \frac{\lambda - k_2}{c_2}$, $c'' = \frac{cc_2}{c_1(\lambda - k_2)}$.

Obviously for any bounded set $D \subset L^2(R^N)$ and every $u_0 \in D$, there exists $T_0 > 0$, such that

$$|u(t)|_2^2 \le 2c'' \quad \text{for all } t \ge T_0. \tag{2.11}$$

Integrating (2.9) between t and $t+1$, by (2.11) we get for $t \ge T_0$

$$\frac{2}{c_2}\int_t^{t+1} |\nabla u(s)|_2^2 ds + \frac{2\alpha_2}{c_2}\int_t^{t+1} |u(s)|_p^p ds \le |u(t)|_2^2 + \frac{c}{c_1} \le c''' \quad \textit{for all } t \ge T_0,$$

where $c''' = 2c'' + \frac{c}{c_1}$, we immediately get

$$\int_t^{t+1} |\nabla u(s)|_2^2 ds \le \frac{c_2 c'''}{2}, \int_t^{t+1} |u(s)|_p^p ds \le \frac{c_2 c'''}{2\alpha_2} \textit{ for all } t \ge T_0.$$

Taking $M = \max\{(2c'')^{\frac{1}{2}}, \frac{c_2 c'''}{2}, \frac{c_2 c'''}{2\alpha_2}\}$, we conclude the proof of Lemma 2.1.

In what follows, we denote by B the ball of radius M, i.e., $B = \{u(x) \in L^2(R^N) : |u(x)| \le M\}$. Obviously B is a bounded absorbing set in $L^2(R^N)$. By the bounded of B and Lemma 2.1. We know that there exists a constant $T_1(B) > 0$ such that $S(t)B \subset B$ for all $t \ge T_1$.

We now establish the estimates in the space $H^1(R^N)$.

Lemma 2.2. *Assume that* (2.3)-(2.4) *hold,* $g \in L^2(R^N)$, $u_0 \in B$. *Then there exist* $M_1 > 0$ *and* $T_2 > 0$, *for any solution* $u(t)$ *of problem* (2.1)-(2.2) *satisfying*

$$\| u \|^2 \le M_1, |u|_p^p \le M_1 \quad \textit{for} \quad t \ge T_2. \tag{2.12}$$

Proof. Taking the inner product of (2.1) with u_t in $L^2(R^N)$, we get

$$\int_{R^N} \alpha'(u)|u_t|^2 dx + \frac{d}{dt}\int_{R^N} (|\nabla u|^2 + F(u) + \frac{\lambda}{2}|u|^2) dx = (g, u_t).$$

Since $|(g,\ u_t)| \le \int_{R^N} \alpha'(u)u_t^2 dx + \int_{R^N} \frac{|g(x)|^2}{4\alpha'(u)}\, dx \le \int_{R^N} \alpha'(u)u_t^2 dx + \frac{1}{4c_1}$ $|g|_2^2$, therefore, we have

$$\frac{d}{dt}\int_{R^N} (|\nabla u|^2 + F(u) + \frac{\lambda}{2}|\ u\ |^2)dx \le \frac{1}{4c_1}|g|_2^2. \tag{2.13}$$

In the light of (2.4), we can get

$$\alpha_2'|\ u\ |^p - k_2'|\ u\ |^2 \le F(u) \le \alpha_1'|\ u\ |^p + k_1'|\ u\ |^2,\ \alpha_1',\ \alpha_2',\ k_1',\ k_2' > 0. \tag{2.14}$$

Using (2.6), we obtain

$$\int_t^{t+1}\int_{R^N} (|\nabla u|^2 + F(u) + \frac{\lambda}{2}|\ u\ |^2)ds \le c \quad \text{for} \quad \text{some } c > 0, \text{ and } t \ge T_1.$$

In view of the Uniform Cronwall Lemma, we get

$$\int_{R^N} (|\nabla u|^2 + F(u) + \frac{\lambda}{2}|\ u\ |^2) \le M_0, \text{ for some } M_0 > 0 \text{ and } t \ge T_1 + 1,$$

thus from (2.14), we have

$$\|u(t)\|^2 \le M_1,\ |u(t)|_p^p \le M_1, \text{ for some } M_1 > 0, \text{ and } t \ge T_2 := T_1 + 1, \tag{2.15}$$

so that (2.12) satisfied.

We are now in a position to derive the estimate on solution for large time and space variables. The following lemma will be used when we prove the asymptotic compactness of dynamical system $S(t)$ and play a crucial role in the proof of our result.

Lemma 2.3. *Assume that* (2.3)-(2.4) *hold,* $g \in L^2(R^N)$, *and* $u_0 \in B$, *the bounded absorbing set of* $S(t)$. *Then for any* $\varepsilon > 0$, *there exist* $T(\varepsilon)$ *and* $K(\varepsilon)$ *depending on* ε *such that the following holds*

$$\int_{|x|\ge k} |u(t)|^2 dx < \varepsilon \quad \textit{for} \quad t \ge T(\varepsilon), \quad k \ge K(\varepsilon). \tag{2.16}$$

Proof. Choose a smooth function θ such that $0 \le \theta(s) \le 1$ for $s \in R^+$, and $\theta(s) = 0$ for $0 \le s \le 1$, $\theta(s) = 1$ for $s \ge 2$, and there exists a constant c such that $|\theta'(s)| \le c$ for $s \in R^+$.

Multiplying (2.1) by $\theta(\frac{|x|^2}{k^2})u$, and integrating in $L^2(R^N)$, we find

$$\frac{1}{2}\int_{R^N} \alpha'(u)\theta(\frac{|x|^2}{k^2})\frac{d}{dt}|u|^2 dx + \int_{R^N} \nabla u \nabla(\theta(\frac{|x|^2}{k^2})u)dx$$

$$+ \int_{R^N} f(u)\theta(\frac{|x|^2}{k^2})udx + \lambda\int_{R^N} u\theta(\frac{|x|^2}{k^2})|u|^2 dx$$

$$= \int_{R^N} g(x)\theta(\frac{|x|^2}{k^2})udx. \qquad (2.17)$$

We now majorization each term of (2.17)

$$\int_{R^N} \nabla u \nabla(\theta(\frac{|x|^2}{k^2})u)dx = \int_{R^N} \theta(\frac{|x|^2}{k^2})|\nabla u|^2 dx$$

$$+ \int_{R^N} \frac{2x}{k^2} u\theta'(\frac{|x|^2}{k^2})\nabla u dx.$$

We get

$$|\int_{R^N} \frac{2x}{k^2} u\theta'(\frac{|x|^2}{k^2})\nabla u dx|$$

$$\le c\int_{k\le|x|\le\sqrt{2}k} \frac{|x|}{k^2}\theta'(\frac{|x|^2}{k^2})|\nabla u\| u|dx$$

$$\le \frac{c'}{k}|\nabla u|_2|u|_2.$$

Using Lemma 2.1 and Lemma 2.2, for given $\varepsilon > 0$, there exists $K_0(\varepsilon) > 0$, for $k \ge K_0$, $t \ge T_2$ we have

$$\int_{R^N} \nabla u \nabla(\theta(\frac{|x|^2}{k^2})u)dx \ge -\varepsilon. \qquad (2.18)$$

We infer from (2.4) that

$$\int_{R^N} f(u)\theta(\frac{|x|^2}{k^2})udx + \lambda\int_{R^N}\theta(\frac{|x|^2}{k^2})|u|^2dx \geq (\lambda - k_2)\int_{R^N}\theta(\frac{|x|^2}{k^2})|u|^2dx.$$

On the right-hand side of (2.17), we have

$$\begin{aligned}
&|\int_{R^N} g(x)\theta(\frac{|x|^2}{k^2})udx| \\
&= |\int_{|x|\geq k}\theta(\frac{|x|^2}{k^2})gudx| \\
&\leq (\int_{|x|\geq k}\theta(\frac{|x|^2}{k^2})|g|^2dx)^{\frac{1}{2}}(\int_{|x|\geq k}\theta(\frac{|x|^2}{k^2})|u|^2dx)^{\frac{1}{2}} \\
&\leq (\int_{|x|\geq k}|g|^2dx)^{\frac{1}{2}}(\int_{|x|\geq k}\theta(\frac{|x|^2}{k^2})|u|^2dx)^{\frac{1}{2}} \qquad (2.19)\\
&\leq \frac{\lambda - k_2}{2}\int_{|x|\geq k}\theta(\frac{|x|^2}{k^2})|u|^2dx + \frac{1}{2(\lambda - k_2)}\int_{|x|\geq k}|g|^2dx \\
&\leq \frac{\lambda - k_2}{2}\int_{R^N}\theta(\frac{|x|^2}{k^2})|u|^2dx + \frac{1}{2(\lambda - k_2)}\int_{|x|\geq k}|g|^2dx.
\end{aligned}$$

Since $g \in L^2(R^N)$, for given $\varepsilon > 0$, there exists $K_1(\varepsilon) > 0$ such that

$$\frac{1}{2(\lambda - k_2)}\int_{|x|\geq k}|g|^2dx \leq \varepsilon \quad for \quad k \geq K_1. \qquad (2.20)$$

Therefore, from (2.19) and (2.20) we find

$$|\int_{R^N}\theta(\frac{|x|^2}{k^2})ugdx| \leq \frac{\lambda - k_2}{2}\int_{R^N}\theta(\frac{|x|^2}{k^2})|u|^2dx + \varepsilon \quad for \quad k \geq K_1. \quad (2.21)$$

Taking $K = \max\{K_0, K_1\}$, by (2.3) and (2.18)-(2.20), we get for $t \geq T_2$ and $k \geq K$, we have

$$\frac{d}{dt}\int_{R^N}\theta(\frac{|x|^2}{k^2})|u|^2dx+\frac{\lambda-k_2}{c_2}\int_{R^N}\theta(\frac{|x|^2}{k^2})|u|^2dx\le\frac{4\varepsilon}{c_1},$$

then it follows from Gronwall inequality that

$$\int_{R^N}\theta(\frac{|x|^2}{k^2})|u|^2dx\le e^{-\frac{\lambda-k_2}{c_2}t}\int_{R^N}\theta(\frac{|x|^2}{k^2})|u_0|^2dx+\frac{4c_2\varepsilon}{\lambda-k_2}$$

$$\le e^{-\frac{\lambda-k_2}{c_2}t}|u_0|_2^2+\frac{4c_2\varepsilon}{\lambda-k_2}.$$

Obviously there exists $T>0$, for any $t\ge T$, $k\ge K$ we have

$$\int_{|x|\ge k}|u|^2dx\le\int_{R^N}\theta(\frac{|x|^2}{k^2})|u|^2dx\le\frac{8c_2\varepsilon}{\lambda-k_2}.$$

Which conclude the proof of Lemma 2.3.

3. Global Attractors

In this section, we present our main result, that is, the existence of the global attractor for $S(t)$ in $L^2(R^N)$. We need to establish the asymptotic compactness for the dynamical system $S(t)$. Once this kind of compactness is obtained, the existence of the global attractor follows from a standard result which can be stated as follows [2, 4, 8, 9].

Theorem 3.1. *Assume that X is a metric space and $\{S(t)\}_{t\ge0}$ is a semigroup of continuous operators in X. If $\{S(t)\}_{t\ge0}$ has a bounded absorbing set and is asymptotically compact, then $\{S(t)\}_{t\ge0}$ possesses a global attractor which is a compact invariant set and attracts every bounded set in X.*

We now prove the asymptotic compactness for the dynamical system $\{S(t)\}_{t\ge0}$ generated by problem (2.1)-(2.2).

Theorem 3.2. *Assume that* (2.3)-(2.4) *hold and $g\in L^2(R^N)$. Then the semigroup $S(t)$ is asymptotically compact, that is, if $\{u_n\}$ is bounded*

in $L^2(R^N)$ *and* $t_n \to +\infty$ *as* $n \to \infty$, *then* $\{S(t_n)u_n\}$ *is precompact in* $L^2(R^N)$.

Proof. For every $k > 0$, we denote by $\Omega_k = \{x \in R^N : |x| \le k\}$ and for a function w defined on R^N, we denote by ω_{Ω_k} the restriction of ω to Ω_k.

Since $\{u_n\}$ is bounded in $L^2(R^N)$, we can assume that there exists M_0 such that $|u_n|_2 \le M_0$. By Lemma 2.1, we see that there exists a constant T depending on M_0 such that $\{S(t_n)u_n\} \subset B$ for $t_n \ge T$. By Lemma 2.1 and Lemma 2.2, we know that there exist constants $M > 0$ and $T_1 > 0$ such that

$$|S(t_n)u_n|_2 \le M, \quad \|S(t_n)u_n\| \le M \quad \text{for} \quad t_n \ge T_1. \tag{3.1}$$

For any $\varepsilon > 0$, by Lemma 2.3, we know that there exist $T_2 > 0$, $K > 0$, we have

$$|S(t_n)u_n|_{L^2(R^N \setminus \Omega_k)} < \varepsilon \quad \text{for} \quad t_n \ge T_2,\ k \ge K. \tag{3.2}$$

Since $|S(t_n)u_n|_{L^2(\Omega_K)} \le |S(t_n)u_n|_{L^2(R^N)}$, hence

$$|S(t_n)u_n|_{L^2(\Omega_K)} \le M, \quad \|S(t_n)u_n\|_{H^1(\Omega_K)} \le M \text{ for } t_n \ge T_1. \tag{3.3}$$

The embedding $H^1(\Omega_k) \subset L^2(\Omega_k)$ is compact, it is clear $\{S(t_n)u_n\}_{\Omega_k}$ lies in a compact set in $L^2(\Omega_k)$. For given $\varepsilon > 0$, $t \ge T_1$, $k \ge K$, there exist finite elements $z_1, z_2, \ldots, z_m$, and $\{S(t_n)u_n\}_{\Omega_k} \subset \bigcup_{i=1}^m B(z_i, \varepsilon)$, $z_i \in L^2(\Omega_k)$. For $z \in \{S(t_n)u_n\}$, there exists z_i, and

$$|z - z_i|_{L^2(R^N)} \le |z - z_i|_{L^2(\Omega_k)} + |z - z_i|_{L^2(R^N \setminus \Omega_k)} \le 3\varepsilon.$$

That is, $\{S(t_n)u_n\}$ has a finite covering in $L^2(R^N)$, which imply $\{S(t_n)u_n\}$ is precompact in $L^2(R^N)$. Then the proof is complete.

We are now in a position to state our main result.

Theorem 3.3. *Assume that* (2.3)-(2.4) *hold and* $g \in L^2(R^N)$. *The problem* (2.1)-(2.2) *has a global attractor which is a compact invariant set and attracts every bounded set in* $L^2(R^N)$.

Proof. We note that we have established the existence of a bounded absorbing set for $S(t)$ in Lemma 2.1 and the asymptotic compactness in Theorem 3.2, so the existence of the global attractor follows from Theorem 3.1.

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