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A NEW ITERATIVE METHOD FOR SOLVING A SYSTEM OF GENERALIZED RELAXED COCOERCIVE VARIATIONAL INEQUALITIES IN BANACH SPACES

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Abstract

In this paper, a new system of generalized nonlinear relaxed cocoercive variational inequalities problems in the framework of Banach spaces is studied. Using the sunny nonexpansive retraction method, a new three-step iterative scheme for this system is proposed, under suitable conditions, some strong convergence theorems are proved.

1. Introduction and Preliminaries

In the recent years, based on the projection methods in Hilbert spaces and the generalized projection methods in Banach spaces, many iterative schemes have been proposed for finding the solutions of the system of generalized nonlinear variational inequality (SNVI) problems in Hilbert spaces [5, 6, 10, 11, 15-18] and in Banach spaces [1, 7, 8, 9, 12].

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In this paper, we introduce a new iterative scheme to approximate a solution of system of generalized nonlinear relaxed cocoercive variational inequality problems in the framework of Banach spaces by using the sunny nonexpansive retraction method.

Throughout this paper, we assume that X is a real Banach space, K is a nonempty closed convex subset of X.

Let $T_1, T_2, T_3 : K \times K \to X$ be nonlinear mappings. We consider the following system of generalized nonlinear variational inequality (SNVI) problems. Find $x^*, y^*, z^* \in K$ such that

$$\begin{cases} \langle \rho T_1(y^*, x^*) + x^* - y^*, J(x - x^*) \rangle \geq 0, \\ \langle \tau T_2(z^*, y^*) + y^* - z^*, J(x - y^*) \rangle \geq 0, \quad \forall x \in K, \rho > 0, \tau > 0, \sigma > 0, \\ \langle \sigma T_3(x^*, z^*) + z^* - x^*, J(x - z^*) \rangle \geq 0, \end{cases} \tag{1.1}$$

where J is the normalized duality mapping of X.

Now, we consider some special cases of the SNVI problem (1.1) as follows:

(I) If $T_1 = T_2 = T_3 = T$ is a univariate mapping, then the SNVI problem (1.1) reduces to the following SNVI problem. Find $x^*, y^*, z^* \in K$ such that

$$\begin{cases} \langle \rho T y^* + x^* - y^*, J(x - x^*) \rangle \geq 0, \\ \langle \tau T z^* + y^* - z^*, J(x - y^*) \rangle \geq 0, \quad \forall x \in K, \rho > 0, \tau > 0, \sigma > 0. \\ \langle \sigma T x^* + z^* - x^*, J(x - z^*) \rangle \geq 0, \end{cases} \tag{1.2}$$

(II) If $T_1 = T_2 = T_3 = T$ and $X = H$ is a Hilbert space, then the SNVI problem (1.1) reduces to the following SNVI problem. Find $x^*, y^*, z^* \in K$ such that

$$\begin{cases} \langle \rho T(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \\ \langle \tau T(z^*, y^*) + y^* - z^*, x - y^* \rangle \geq 0, \quad \forall x \in K, \rho > 0, \tau > 0, \sigma > 0, \\ \langle \sigma T(x^*, z^*) + z^* - x^*, x - z^* \rangle \geq 0, \end{cases} \tag{1.3}$$

which was basically due to He [10].

(III) If $\sigma = 0$, then the SNVI problem (1.3) reduces to the following SNVI problem. Find $x^*, y^* \in K$ such that

$$\begin{cases} \langle \rho T(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \\ \langle \tau T(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in K, \rho > 0, \tau > 0, \end{cases} \tag{1.4}$$

which was basically due to Chang et al. [5] and Verma [17].

(IV) If T is a univariate mapping, then the SNVI problem (1.4) reduces to the following SNVI problem. Find $x^*, y^* \in K$ such that

$$\begin{cases} \langle \rho T y^* + x^* - y^*, x - x^* \rangle \geq 0, \\ \langle \tau T x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in K, \rho > 0, \tau > 0, \end{cases} \tag{1.5}$$

which was basically due to Verma [15, 18].

(V) If $\sigma = \tau = 0$, then the SNVI problem (1.5) reduces to the following standard nonlinear variational inequality (NVI) problem. Find $x^* \in K$ such that

$$\langle T x^*, x - x^* \rangle \geq 0, \quad \forall x \in K. \tag{1.6}$$

For the sake of convenience, we first recall some notations and conclusions. A Banach space X is said to be *smooth*, if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.7}$$

exists for each $x, y \in U = \{x \in E : \|x\| = 1\}$.

A Banach space X is said to be *uniformly smooth*, if the limit (1.7) is attained uniformly for $(x, y) \in U \times U$.

A Banach space X is said to be *uniformly convex*, if for each $\epsilon \in (0, 2]$, there exists $\delta(\epsilon) > 0$ such that $\|x\| \leq 1, \|y\| \leq 1$, and $\|x - y\| \geq \epsilon$ implies $\|x + y\| \leq 2(1 - \delta(\epsilon))$, where $\delta : [0, 2] \to [0, 1]$ is the *modulus of convexity of* X defined by

$$\delta(\epsilon) = \inf \{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\}.$$

Proposition 1.1 [4]**.** (i) *A Banach space X is uniformly smooth, if and only if the normalized duality mapping J is single-valued and norm-to-norm uniformly continuous on each bounded subset of X.*

(ii) *Every uniformly convex Banach space is reflexive.*

The modulus of smoothness of X is defined by

$$\varphi(t) = \sup\{\frac{1}{2}(\|x + y\| - \|x - y\|) - 1 : x, y \in X, \|x\| = 1, \|y\| = t\},$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a function. It is known that X is uniformly smooth, if and only if $\lim_{t\to 0} \frac{\varphi(t)}{t} = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space X is said to be *q-uniformly smooth*, if there exists a constant $k > 0$ such that $\varphi(t) \leq kt^q$, for all $t > 0$. For example, see [3, 14] for more details. It is well known that all Hilbert spaces are 2-uniformly smooth. For each $q > 1$, the generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\},$$

for all $x \in X$. In particular, $J = J_2$ is called the *normalized duality mapping*.

Lemma 1.1 [19]**.** *Let q be a given real number with* $1 < q \leq 2$ *and let X be a q-uniformly smooth Banach space. Then*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + 2\|ky\|^q,$$

for all $x, y \in X$*, where k is the q-uniformly smoothness constant of X.*

Recall that if K is a nonempty closed convex subset of X, and D is a subsets of K, then a mapping $Q : K \to D$ is said to be *sunny*, provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in K$ and $t \geq 0$, whenever $x + t(x - Q(x)) \in K$. A sunny nonexpansive retraction is a sunny and

nonexpansive retraction. Sunny nonexpansive retractions play an important role in our argument. They are characterized by the following lemma.

Lemma 1.2 [13]**.** *Let X be a smooth Banach space, then $Q_K : X \to K$ is a sunny nonexpansive retraction, if and only if the following inequality holds*:

$$\langle y - Q_K y, J(Q_K y - x)\rangle \geq 0, \quad \forall x \in K \text{ and } y \in X.$$

It is well known that if X is a Hilbert space, then a sunny nonexpansive retraction Q_K is coincident with the metric projection from X onto K. Let K be a nonempty closed convex subset of a smooth Banach space X, let $y \in X$, and let $z \in K$. Then, we have from Lemma 1.2 that

$$z = Q_K y \Leftrightarrow \langle y - z, J(z - x)\rangle \geq 0, \quad \forall x \in K,$$

where Q_K is a sunny nonexpansive retraction from X onto K. See also [2].

2. Algorithms

In this section, we introduce some algorithms, which can be applied to the convergence analysis in the context of the approximation solvability of the SNVI problem (1.1) and others.

Algorithm 2.1. For initial points $x_0, y_0, z_0 \in K$, compute the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ by the iterative process

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Q_K[y_n - \rho T_1(y_n, x_n)], \\ y_n = (1 - \beta_n)x_n + \beta_n Q_K[z_n - \tau T_2(z_n, y_n)], \quad \forall n \geq 0, \\ z_n = (1 - \gamma_n)x_n + \gamma_n Q_K[x_n - \sigma T_3(x_n, z_n)], \end{cases} \tag{2.1}$$

where Q_K is the sunny nonexpansive retraction and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0, 1]$.

(I) If $T_1 = T_2 = T_3 = T$ is a univariate mapping, then the Algorithm (2.1) reduces to the following.

Algorithm 2.2. For initial points $x_0, y_0, z_0 \in K$, compute the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ by the iterative process

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n Q_K[y_n - \rho T(y_n)], \\ y_n = (1-\beta_n)x_n + \beta_n Q_K[z_n - \tau T(z_n)], \\ z_n = (1-\gamma_n)x_n + \gamma_n Q_K[x_n - \sigma T(x_n)], \end{cases} \quad \forall n \geq 0, \tag{2.2}$$

where Q_K is the sunny nonexpansive retraction and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0, 1]$.

(II) If $\beta_n = 1$, $\gamma_n = 1$, then the Algorithm (2.1) reduces to the following.

Algorithm 2.3. For initial points $x_0, y_0, z_0 \in K$, compute the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ by the iterative process

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n Q_K[y_n - \rho T_1(y_n, x_n)], \\ y_n = Q_K[z_n - \tau T_2(z_n, y_n)], \\ z_n = Q_K[x_n - \sigma T_3(x_n, z_n)], \end{cases} \quad \forall n \geq 0, \tag{2.3}$$

where Q_K is the sunny nonexpansive retraction and $\{\alpha_n\}$ is sequence in $[0, 1]$.

3. Main Results

We first give the following definitions.

Definition 3.1. A two-variable mapping $T : K \times K \to X$ is said to be (η, ξ)- *relaxed cocoercive*, if there exist constants $\eta, \xi \geq 0$ such that for all $x, y \in K$,

$$\langle T(x, u) - T(y, v), J(x - y)\rangle \geq -\eta\|T(x, u) - T(y, v)\|^2 + \xi\|x - y\|^2, \quad \forall u, v \in K.$$

Definition 3.2. A mapping $T : K \times K \to X$ is said to be μ- *Lipschitz continuous* in the first variable, if there exist constant $\mu \geq 0$ such that for all $x, y \in K$,

$$\|T(x, u) - T(y, v)\| \leq \mu\|x - y\|, \quad \forall u, v \in K.$$

In order to prove our results, we need the following lemma:

Lemma 3.1 [6]. *Let* $\{a_n\}$, $\{b_n\}$, *and* $\{c_n\}$ *be three nonnegative real sequences satisfying the following conditions*:

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n + c_n, \quad \forall n \geq n_0,$$

where n_0 *is some nonnegative integer,* $\lambda_n \in (0, 1)$ *with* $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$, *and* $\sum_{n=1}^{\infty} c_n < \infty$. *Then* $a_n \to 0\,(n \to \infty)$.

Based on Algorithm 2.1, now, we study the approximation solvability of the SNVI problem (1.1). We have the following main result.

Theorem 3.1. *Let* X *be a uniformly convex,* 2*-uniformly smooth Banach space, and* K *be a nonempty closed and convex subset of* X, *let* $T_i : K \times K \to X$ *be* (η_i, ξ_i)*-relaxed cocoercive and* μ_i*-Lipschitz continuous mappings in the first variable for all* $i = 1, 2, 3$. *Suppose that* $(x^*, y^*, z^*) \in K \times K \times K$ *is a solution to the SNVI problem* (1.1) *and that* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *are the sequences generated by Algorithm* (2.1). *If* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0, 1]$ *satisfying the following condition*:

(C1) $\beta_n \to 1$ *and* $\gamma_n \to 1 (n \to \infty)$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $0 < \rho < \dfrac{\xi_1 - \eta_1\mu_1^2}{k^2\mu_1^2}$, $0 < \tau < \dfrac{\xi_2 - \eta_2\mu_2^2}{k^2\mu_2^2}$, $0 < \sigma < \dfrac{\xi_3 - \eta_3\mu_3^2}{k^2\mu_3^2}$,

where k *is the* 2*-uniformly smoothness constant of* X.

Then, the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to x^*, y^*, and z^*, respectively.

Proof. Since (x^*, y^*, z^*) is a solution to the SNVI problem (1.1), it follow that

$$\begin{cases} x^* = Q_K[y^* - \rho T_1(y^*, x^*)], \ \rho > 0, \\ y^* = Q_K[z^* - \tau T_2(z^*, y^*)], \ \tau > 0, \\ z^* = Q_K[x^* - \sigma T_3(x^*, z^*)], \ \sigma > 0. \end{cases}$$

Hence from Algorithm (2.1), we have that

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &= \|(1 - \alpha_n)x_n + \alpha_n Q_k[y_n - \rho T_1(y_n, x_n)] - (1 - \alpha_n)x^* - \alpha_n Q_k[y^* - \rho T_1(y^*, x^*)]\| \\ &\le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|y_n - y^* - \rho[T_1(y_n, x_n) - T_1(y^*, x^*)]\|. \end{aligned} \tag{3.1}$$

Since T_1 is (η_i, ξ_i)-relaxed cocoercive and μ_i-Lipschitz continuous in the first variable, by Lemma 1.1, we have

$$\begin{aligned} &\|y_n - y^* - \rho[T_1(y_n, x_n) - T_1(y^*, x^*)]\|^2 \\ &\quad \le \|y_n - y^*\|^2 - 2\rho\langle T_1(y_n, x_n) - T_1(y^*, x^*), J(y_n - y^*)\rangle \\ &\qquad + 2k^2\rho^2\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 \\ &\quad \le \|y_n - y^*\|^2 - 2\rho(-\eta_1\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 + \xi_1\|y_n - y^*\|^2) \\ &\qquad + 2k^2\rho^2\mu_1^2\|y_n - y^*\|^2 \\ &\quad \le (1 + 2\rho\eta_1\mu_1^2 - 2\rho\xi_1 + 2k^2\rho^2\mu_1^2)\|y_n - y^*\|^2. \end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1) and simplifying the resultant result, we obtain

$$\|x_{n+1} - x^*\| \le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|y_n - y^*\|, \tag{3.3}$$

where $\theta_1 = (1 + 2\rho\eta_1\mu_1^2 - 2\rho\xi_1 + 2k^2\rho^2\mu_1^2)^{\frac{1}{2}} < 1$ (by the condition (C3)).

Next, we consider

$$\begin{aligned}
&\|y_n - y^*\| \\
&= \|(1-\beta_n)x_n + \beta_n Q_k[z_n - \tau T_2(z_n, y_n)] - (1-\beta_n)y^* - \beta_n Q_k[z^* - \tau T_2(z^*, y^*)]\| \\
&\leq (1-\beta_n)\|x_n - y^*\| + \beta_n\|z_n - z^* - \tau[T_2(z_n, y_n) - T_2(z^*, y^*)]\|. \qquad (3.4)
\end{aligned}$$

Similarly, we can show that

$$\|z_n - z^* - \tau[T_2(z_n, y_n) - T_2(z^*, y^*)]\| \leq \theta_2\|z_n - z^*\|, \qquad (3.5)$$

where $\theta_2 = (1 + 2\tau\eta_2\mu_2^2 - 2\tau\xi_2 + 2k^2\tau^2\mu_2^2)^{\frac{1}{2}} < 1$ (by the condition (C3)). It follows from (3.4) and (3.5) that

$$\|y_n - y^*\| \leq (1-\beta_n)\|x_n - y^*\| + \beta_n\theta_2\|z_n - z^*\|. \qquad (3.6)$$

Now, we make an estimation for $\|z_n - z^*\|$, it follows from (2.1) that

$$\begin{aligned}
&\|z_n - z^*\| \\
&= \|(1-\gamma_n)x_n + \gamma_n Q_k[x_n - \sigma T_3(x_n, z_n)] - (1-\gamma_n)z^* - \gamma_n Q_k[x^* - \sigma T_3(x^*, z^*)]\| \\
&\leq (1-\gamma_n)\|x_n - z^*\| + \gamma_n\|x_n - x^* - \sigma[T_3(x_n, z_n)] - T_3(x^*, z^*)]\|. \qquad (3.7)
\end{aligned}$$

Similarly, we can show that

$$\|x_n - x^* - \sigma[T_3(x_n, z_n) - T_3(x^*, z^*)]\| \leq \theta_3\|x_n - x^*\|, \qquad (3.8)$$

where $\theta_3 = (1 + 2\sigma\eta_3\mu_3^2 - 2\sigma\xi_3 + 2k^2\sigma^2\mu_3^2)^{\frac{1}{2}} < 1$ (by the condition (C3)).

Substituting (3.8) into (3.7) and simplifying, we have

$$\begin{aligned}
\|z_n - z^*\| &\leq (1-\gamma_n)\|x_n - z^*\| + \gamma_n\theta_3\|x_n - x^*\| \\
&\leq (1-\gamma_n)\|x_n - x^*\| + (1-\gamma_n)\|x^* - z^*\| + \gamma_n\|x_n - x^*\|
\end{aligned}$$

$$= \|x_n - x^*\| + (1 - \gamma_n)\|x^* - z^*\|. \tag{3.9}$$

Substituting (3.9) into (3.6) and simplifying, we obtain

$$\begin{aligned}\|y_n - y^*\| &\le (1 - \beta_n)\|x_n - y^*\| + \beta_n\theta_2\|x_n - x^*\| + \beta_n\theta_2(1 - \gamma_n)\|x^* - z^*\| \\ &\le (1 - \beta_n)\|x_n - x^*\| + (1 - \beta_n)\|x^* - y^*\| \\ &\quad + \beta_n\|x_n - x^*\| + (1 - \gamma_n)\|x^* - z^*\| \\ &= \|x_n - x^*\| + (1 - \beta_n)\|x^* - y^*\| + (1 - \gamma_n)\|x^* - z^*\| \\ &\le \|x_n - x^*\| + [(1 - \beta_n) + (1 - \gamma_n)]M, \end{aligned} \tag{3.10}$$

where $M = \max\{\|x^* - y^*\|, \|x^* - z^*\|\}$.

Finally, it follows from (3.3) and (3.10) that

$$\begin{aligned}\|x_{n+1} - x^*\| &\le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|x_n - x^*\| + \alpha_n\theta_1 M[(1 - \beta_n) + (1 - \gamma_n)] \\ &= (1 - (1 - \theta_1)\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1 M[(1 - \beta_n) + (1 - \gamma_n)]. \end{aligned} \tag{3.3}$$

We take $a_n = \|x_n - x^*\|$, $\lambda_n = (1 - \theta_1)\alpha_n$, $b_n = \alpha_n\theta_1 M[(1 - \beta_n) + (1 - \gamma_n)]$, and $c_n = 0$, in Lemma 3.1, by conditions (C1) and (C2), we know that $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$, and $\sum_{n=1}^{\infty} c_n = 0$. Therefore, all condition in Lemma 3.1 are satisfied, and so $\|x_n - x^*\| \to 0 (n \to \infty)$, i.e., $x_n \to x^* (n \to \infty)$.

Moreover, it follows from (3.9) and (3.10) that $z_n \to z^*$, $y_n \to y^*$ $(n \to \infty)$. This completes the proof.

Remark 3.1. Theorem 3.1 extends the main results in Chang et al. [5], Verma [18], and He [10] from Hilbert spaces to Banach spaces.

From Theorem 3.1, we obtain the following results immediately.

Theorem 3.2. *Let X be a uniformly convex, 2-uniformly smooth Banach space, and K be a nonempty closed and convex subset of X, let $T : K \to X$ be univariate (η, ξ)-relaxed cocoercive and μ-Lipschitz*

continuous mapping. Suppose that $(x^*, y^*, z^*) \in K \times K \times K$ *form a solution set for the SNVI problem* (1.2) *and that* $\{x_n\}, \{y_n\}$, *and* $\{z_n\}$ *are the sequences generated by Algorithm* (2.2). *If* $\{\alpha_n\}, \{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0, 1]$ *satisfying the following condition*:

(C1) $\beta_n \to 1$ *and* $\gamma_n \to 1$ $(n \to \infty)$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $0 < \rho, \tau, \sigma < \dfrac{\xi - \eta\mu^2}{k^2\mu^2}$, *where* k *is the* 2*-uniformly smoothness constant of* X.

Then, the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ converge strongly to x^*, y^*, and z^*, respectively.

Remark 3.2. Theorem 3.2 extends the main results of Verma [18] from Hilbert spaces to Banach spaces.

Theorem 3.3. *Let* X *be a uniformly convex*, 2*-uniformly smooth Banach space*, *and* K *be a nonempty closed and convex subset of* X, *let* $T_i : K \times K \to X$ *be* (η_i, ξ_i)*-relaxed cocoercive and* μ_i*-Lipschitz continuous mappings in the first variable for all* $i = 1, 2, 3$. *Suppose that* $(x^*, y^*, z^*) \in K \times K \times K$ *form a solution set for the SNVI problem* (1.1) *and that* $\{x_n\}, \{y_n\}$, *and* $\{z_n\}$ *are the sequences generated by Algorithm* (2.3). *If* $\{\alpha_n\}$ *is sequence in* $[0, 1]$ *satisfying the following condition*:

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C2) $0 < \rho < \dfrac{\xi_1 - \eta_1\mu_1^2}{k^2\mu_1^2}$, $0 < \tau < \dfrac{\xi_2 - \eta_2\mu_2^2}{k^2\mu_2^2}$, $0 < \sigma < \dfrac{\xi_3 - \eta_3\mu_3^2}{k^2\mu_3^2}$,

where k *is the* 2*-uniformly smoothness constant of* X.

Then, the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ converge strongly to x^*, y^*, and z^*, respectively.

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