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# THEORETICAL AND NUMERICAL STUDY FOR TWO POINT BOUNDARY VALUE PROBLEMS

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## Abstract

In this paper, we consider the numerical solution of two point boundary value problems by using collocation method with quartic and quintic splines as the approximating function. A symptomatic bound on the maximum error shows that in certain cases, the quintic spline can give higher accuracy by a factor of between one and two compared with the quartic spline case. Numerical tests support the theoretical results.

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## 1. Introduction

Consider the general, variable coefficient, second order boundary value problem given by

$$(Lu)(t) = p(t)u''(t) + q(t)u'(t) + r(t)u(t) = f(t, u(t)), \tag{1}$$

with the boundary conditions

$$u(0) = u(1) = 0, \tag{2}$$

some Russian authors, who used ordinary polynomials for approximating solutions [18] have obtained collocation methods for such problem. More, spline functions were introduced with more desirable results [2, 7, 11, 17]. Ahlberg and Ito [1] introduce another equation of the method, especially, in the treatment of boundary conditions for the approximation splines. And the also described for immediate application; shooting methods are discussed by Henrici [9] and Roberts and Shipman [14]; for instance, the problem is solved by using variation techniques in Burden [8]; and commonly used finite difference methods are discussed by many authors. Khalifa and Eilbeck [10] used cubic and quadratic splines for the same problem. Al-Said [3, 4] has demonstrated the use of quadratic spline for obtaining smooth approximations for the solution, and its first derivative of second order obstacle problems and of two point boundary value problems. The use of higher order spline functions and collocation methods with splines as basis functions for solving various second order boundary value problems were demonstrated by different authors [1, 5, 6, 10, 12, 13, 15, 16, 19, 20]. The purpose of this paper shows on theoretical and experimental grounds that splines of odd degree can give better results in certain cases. Specifically, we concentrate on a detail comparison of collocation method with quartic and quintic splines. By obtaining asymptotic error bounds, we show that the quintic spline gives a higher accuracy by a factor of between one and two than quartic spline.

## 2. Error Bounds on Interpolator Splines and their Derivatives

Consider an arbitrary function $f(t) \in C^6(0, 1)$, if we use quartic and quintic, then the bounds of error are

$$\left\| f'' - \overline{u''} \right\|_c \leq \begin{cases} \dfrac{7}{400} h^4 \left\| f^{(6)} \right\| + O(h^6), & \text{quartic}, \\ \dfrac{1}{96} h^4 \left\| f^{(6)} \right\| + O(h^6), & \text{quintic}, \end{cases} \tag{3}$$

as a numerical example, we take $f(t) = \sin(\pi t)$, find $\overline{u}$ the quartic (quintic) spline of interpolation to the solution for various values of $h$, the results are shown in Table 1.

**Table 1.** $\left\| f'' - \overline{u''} \right\|_c$ when $f(t) = \sin(\pi t)$

| $h$ | Quintic | | Quartic | |
|---|---|---|---|---|
| | Numerical | Theory | Numerical | Theory |
| 0.2 | 1.3*E*-6 | 1.6*E*-2 | 1.3*E*-6 | 2.6*E*-2 |
| 0.1 | 7.6*E*-6 | 1.0*E*-3 | 5.7*E*-6 | 1.6*E*-3 |
| 0.05 | 1.8*E*-5 | 6.2*E*-5 | 2.8*E*-5 | 1.0*E*-4 |
| 0.04 | 2.8*E*-5 | 2.5*E*-5 | 4.0*E*-5 | 4.3*E*-5 |

the fact that the quintic spline provides a better approximation of $f''$ at knots collocation points than the quartic spline at mid knots collocation points. Now, we also need bounds $\left\| f' - \overline{u'} \right\|_c$, the maximum norm for the first derivatives over the collocation points can be obtained in a similar way to those for the second derivatives

$$\left\| f' - \overline{u'} \right\|_c \leq \begin{cases} \dfrac{7}{1200} h^4 \left\| f^{(5)} \right\| + O(h^6), & \text{quartic} \\ \dfrac{1}{672} h^6 \left\| f^{(7)} \right\| + O(h^8), & \text{quintic} \end{cases}, \tag{4}$$

we can obtain a bound on $\left\| f - \overline{u} \right\|_c$ in quartic and quintic cases by expanding in a Taylor series about each collocation point, we get

$$\left\| f - \bar{u} \right\|_c \leq \begin{cases} \frac{7}{2400} h^5 \left\| f^{(5)} \right\| + O\left(h^6\right), & \text{quartic}, \\ O\left(h^6\right), & \text{quintic}, \end{cases} \tag{5}$$

### 3. Collocation for Two Point Boundary Value Problem

In this section, we follow closely the approach of [1] for quintic splines. Details are given only of the calculation in the quartic spline case. We define $\tilde{u}(t)$ be the collocating approximate solution in the form

$$\tilde{u}(t) = \sum_{i=-2}^{N+1} c_i B_i(t), \tag{6}$$

where $B_i(t)$ is the basis function. Then, we get the scheme

$$\frac{p_i}{4h^2}\left[c_{i-2} + 4c_{i-1} - 10c_i + 4c_{i+1} + c_{i+2}\right] + \frac{q_i}{8h}\left[- c_{i-2} - 22c_{i-1} + 22c_{i+1} + c_{i+2}\right]$$

$$+ \frac{r_i}{16}\left[c_{i-2} + 76c_{i-1} + 230c_i + 76c_{i+1} + c_{i+2}\right] = f_i, \qquad i = 0(1)N. \tag{7}$$

Collecting these equations, we obtain a matrix equation for unknown coefficient $c$'s of the form

$$Ac = f(c), \tag{8}$$

where $A$ is an $(N + 4) \times (N + 4)$ matrix and $c$ is an $(N + 4)$- dimensional vector with components $c_i$. Following a similar calculation to [10], it is straightforward to show that $A$ is diagonally dominant, if $h$ is sufficiently small and

$$p(t)r(t) < 0,\ t \in (0, 1), \tag{9}$$

since

$$\left| \frac{-10p_i}{4h^2} + \frac{230r_i}{16} \right| - \left\{ \begin{array}{l} \left| \frac{p_i}{4h^2} + \frac{q_i}{8h} + \frac{r_i}{16} \right| + \left| \frac{p_i}{h^2} + \frac{11q_i}{4h} + \frac{76r_i}{16} \right| \\ + \left| \frac{p_i}{h^2} - \frac{11q_i}{4h} + \frac{76r_i}{16} \right| + \left| \frac{p_i}{4h^2} - \frac{q_i}{8h} + \frac{r_i}{16} \right| \end{array} \right\}$$

$$= \begin{cases} -24r > 0, \text{ if } p > 0,\ r < 0, \\ 24r < 0, \text{ if } p < 0,\ r > 0. \end{cases} \tag{10}$$

In this case, the maximum norm of the inverse matrix $A^{-1}$ satisfies the following inequality

$$\left\|A^{-1}\right\|_{\infty} \leq \frac{1}{24 \min|r(x_i)|} = \lambda_c, \tag{11}$$

where suffix $c$ means related to the collocation points, then Equation (8) can be written as

$$c = A^{-1} f(c). \tag{12}$$

## 4. Error Estimates

We discuss the error in detail and also give an explicit formula for the leading term in the error, when both quartic and quintic splines are used. Let $\overline{u}(t)$ be the quartic spline of interpolation to the unique solution of Equations (1) and (2), then

$$\overline{u}(t) = \sum_{i=-2}^{N+1} \overline{c}_i B_i(t), \tag{13}$$

using the linear differential operator $L$ in (1), we get

$$L(u - \overline{u}) = p\left(u'' - \overline{u''}\right) + q\left(u' - \overline{u'}\right) + r(u - \overline{u}), \tag{14}$$

at the collocation points, thus,

$$\|Lu - L\overline{u}\|_c \leq \|p\|_{\infty}\left\|u'' - \overline{u''}\right\| + \|q\|_{\infty}\left\|u' - \overline{u'}\right\|, \tag{15}$$

the right hand side of (15) depends on two the norms and from the results given in Section (2), we get

$$\|Lu - L\overline{u}\|_c \leq \begin{cases} \dfrac{7\| p \|_{\infty}}{400} h^4\left\|u^{(6)}\right\| + \dfrac{7\| q \|_{\infty}}{1200} h^5\left\|u^{(5)}\right\| + O\left(h^6\right), & \text{quartic}, \\ \dfrac{\| p \|_{\infty}}{96} h^4\left\|u^{(6)}\right\| + \dfrac{\| q \|_{\infty}}{672} h^6\left\|u^{(7)}\right\| + O\left(h^6\right), & \text{quintic}, \end{cases} \tag{16}$$

we can say that the second terms are relatively small so suppose that $q(t) = 0,$ so that

$$\|Lu - L\overline{u}\|_c \leq a\, h^4 \|\, p\,\|_\infty \left\|u^{(6)}\right\| + O\left(h^6\right), \tag{17}$$

where $a = \begin{cases} 0.0175, & \text{quartic}, \\ 0.010416666, & \text{quintic}. \end{cases}$

If $g(t)$ represents the error function, so that

$$g_i = L\overline{u}_i - Lu_i, \tag{18}$$

where $g_i$ is the value of $g(t)$ at a collocation points, then

$$\|\, g\,\|_c \leq a\, h^4 \|\, p\,\|_\infty \left\|u^{(6)}\right\| + O\left(h^6\right), \tag{19}$$

the important step in our discussion is to get an estimate for $\|\, u - \widetilde{u}\,\|_c$ by using this inequality

$$\|\, u - \widetilde{u}\,\| \leq \|\, u - \overline{u}\,\| + \|\, \overline{u} - \widetilde{u}\,\|, \tag{20}$$

$$L(\overline{u}_i - \widetilde{u}_i) = f(z_i, \overline{u}_i) - f(z_i, \widetilde{u}_i) + g, \tag{21}$$

where $z_i$ represent the collocation points. Let $A_G$ be a square matrix representing the matrices given by quintic or quartic splines, then

$$L(\overline{u}_i - \widetilde{u}_i) = A_G\, e, \tag{22}$$

thus from (21) and (22), we get

$$A_G\, e = g + \nu, \tag{23}$$

where $\nu = f(z, \overline{u}) - f(z, \widetilde{u}),$ so $e = (A_G)^{-1} g + (A_G)^{-1}\nu,$ and

$$\|\, e\,\|_c < \left\|(A_G)^{-1}\right\| \|\, g\,\|_c + \left\|(A_G)^{-1}\right\| \|\, \nu\,\|_c. \tag{24}$$

Assuming that $f(t, u)$ is Lipschitz condition in $u$

$$|f(t, u_1) - f(t, u_2)| < M|u_1 - u_2|, \tag{25}$$

then

$$\| \nu \|_c = \max |f(z_i, \overline{u}(z_i)) - f(z_i, \widetilde{u}(z_i))| \leq \max l_i |\overline{u}_i - \widetilde{u}_i| \leq R_2 \| B \| \| e \|_c, \tag{26}$$

where $R_2 = \max l_i$, then we get $\| e \|_c \leq \left\| (A_G)^{-1} \right\|_\infty \| g \|_c + R_2 \left\| (A_G)^{-1} \right\|$ $\| B \| \| e \|_c$. Now, let

$$R_1 = \| B \| = \begin{cases} 24, & \text{quartic}, \\ 120, & \text{quintic}, \end{cases}$$

$$\| e \|_c \leq \frac{\lambda_c \| g \|_c}{1 - \lambda_c R_1 R_2}, \tag{27}$$

by substituting from (19) in (27), we get

$$\| e \|_c \leq \frac{\lambda_c a \| p \| h^4 \left\| u^{(6)} \right\|}{1 - \lambda_c R_1 R_2} + O\left(h^6\right), \tag{28}$$

where $$\lambda_c = \begin{cases} \dfrac{1}{24 \min |r(z_i)|}, & \text{quartic}, \\ \dfrac{1}{120 \min |r(z_i)|}, & \text{quintic}, \end{cases}$$

if $h$ is sufficiently small, we can consider the term $\min | r_i |$ to be almost the same, and also let us denote this term by $[r]$, so (28) can be written as

$$\| e \|_c \leq \frac{a}{R_1([r] - R_2)} \| p \| h^4 \left\| u^{(6)} \right\| + O\left(h^6\right), \tag{29}$$

then we get the following relation

$$\| u - \widetilde{u} \|_c \leq \frac{a}{[r] - R_2} \| p \| h^4 \left\| u^{(6)} \right\| + O\left(h^6\right), \tag{30}$$

this relation gives the general formula for the leading term in the approximate solution, in practice, the error may be better than this estimate. For the linear case, when $f$ is to be a function of $t$ only, the constant $R_2 = 0$. From this result, we can state the following theorem.

**Theorem.** *Let the two point boundary value problems* (*TPBVP*) *of the form* (1), (2), *where the coefficient functions* $p(t)$ *and* $r(t)$ *and the forcing function* $f(t, u)$ *satisfy the conditions* $p(t)r(t) < 0,$ *for all t and* (25). *Let* $\tilde{u}(t)$ *be the quartic* (*quintic*) *spline*, *if the true solution* $u(t)$ *of TPBVP satisfies* $u \in C^6(0, 1),$ *then*

$$\| u - \tilde{u} \|_c \leq \frac{a}{[r] - R_2} \| p \| h^4 \left\| u^{(6)} \right\| + O\left(h^6\right).$$

It is clear from this theorem that the upper bound on the maximum norm is less in the quintic spline.

## 5. Numerical Results

In this section, the results of some numerical example is shown. The problem [1, 10] $u''(t) - 100u(t) = 0,$ with the boundary conditions $u(0) = u(1) = 0,$ which has the exact solution given by $u(t) = \cosh(10t - 5) / \cosh(5).$ The errors for both the quintic and quartic are shown in Table 2.

**Table 2.** Errors theory and numerical

| $N$ | Quintic | | | Quartic | | |
|---|---|---|---|---|---|---|
| | $L_2$- error | Max error | Theory error | $L_2$- error | Max error | Theory error |
| 5 | 0.773326$E$-3 | 0.119537$E$-2 | 0.106667$E$-2 | 0.167295$E$-2 | 0.213557$E$-2 | 0.179200$E$-2 |
| 10 | 0.984635$E$-4 | 0.144729$E$-3 | 0.666667$E$-4 | 0.231102$E$-3 | 0.391601$E$-3 | 0.112000$E$-3 |
| 15 | 0.238474$E$-4 | 0.403387$E$-4 | 0.131687$E$-4 | 0.570999$E$-4 | 0.982698$E$-4 | 0.221235$E$-4 |
| 20 | 0.824613$E$-5 | 0.137709$E$-4 | 0.416667$E$-5 | 0.200101$E$-4 | 0.324316$E$-4 | 0.700000$E$-5 |
| 25 | 0.358215$E$-5 | 0.583322$E$-5 | 0.170667$E$-5 | 0.874830$E$-5 | 0.141851$E$-4 | 0.286720$E$-5 |

## 6. Discussion

In this paper, we have shown that the method using quintic spline is superior to the method using quartic spline by a factor in the maximum error between one and two. Khalifa and Eilbeck conjectured that their results would generalized to two point boundary value problems spline of even degree will be superior to spline of odd degree in the collocation method with equally spaced points. But, we prove that from our theoretical and numerical results, quintic spline is superior to the method using quartic spline. So, we can not generalized the results for all degrees even and odd.

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