

EXACT TRAVELLING WAVE SOLUTIONS FOR THE ($N + 1$)- DIMENSIONAL GENERALIZED CAMASSA-HOLM EQUATION

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Abstract

By using the sine-cosine method and the extended tanh method, we study the $(N + 1)$ - dimensional generalized Camassa-Holm equation. It is shown that this class gives compactons solutions, solitary wave solutions, and periodic wave solutions. It is also found that the qualitative change in the physical structure of solutions depends mainly on the exponent of the wave function $u(x, t)$, positive or negative, and on the coefficient of $(u^n)_{\xi\xi}$ as well.

1. Introduction

In recent years, much attention has been paid on the study of nonlinear wave equations (NLWE) in low dimensions. But, there is little

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work on the high dimensional ones. It is well known that most of high dimensional NLWEs fail the conventional integrability tests, so the natural and important problem is that, are there exact solutions with good properties for the high dimensional NLWEs? To solve this problem, people have made some attempts. Recently, Jibin and Ming [1] have studied the $(n + 1)$ -dimensional sine- and sinh-Gordon equation

$$\sum_{i=1}^n \frac{\partial^2 \phi}{\partial x_i^2} - \frac{\partial^2 \phi}{\partial t^2} = \sin \phi, \quad (1.1)$$

$$\sum_{i=1}^n \frac{\partial^2 \phi}{\partial x_i^2} - \frac{\partial^2 \phi}{\partial t^2} = \sinh \phi, \quad (1.2)$$

and concluded that they had many 21 different explicit exact bounded travelling wave solutions with arbitrary functions, which led to abundant structures of these solutions.

Most recently, Yan [8, 9] has introduced the semi-travelling wave similarity transformation and separation transformation to study the $(N + 1)$ -dimensional generalized Boussinesq equation

$$u_{tt} = u_{xx} + \mu(u^n)_x + u_{xxxx} + \sum_{i=1}^{N-1} u_{x_j x_j}, \quad (1.3)$$

and the $(N + 1)$ -dimensional complex nonlinear Klein-Gordon equation

$$u_{tt} - \sum_{i=1}^N u_{x_j x_j} + \alpha u + \beta |u|^2 u = 0, \quad (1.4)$$

and constructed many kinds of novel exact solutions with arbitrary functions.

In this paper, we will investigate the following two $(N + 1)$ -dimensional nonlinear dispersive variant $\text{CH}(n, n, 2n-1)$ and $\text{CH}(-n, -n, -(2n-1))$ of the generalized Camassa-Holm equation of the form, respectively,

$$u_{tt} = \sum_{i=1}^N (a_i u + b_i u^n + d_i u^{2n-1})_{x_i x_i} + \sum_{i=1}^N k_i (u^n)_{x_i x_i t t}, \quad (1.5)$$

$$u_{tt} = \sum_{i=1}^N (a_i u + b_i u^{-n} + d_i u^{-(2n-1)})_{x_i x_i} + \sum_{i=1}^N k_i (u^{-n})_{x_i x_i t t}. \quad (1.6)$$

When $N = 1$, $d_1 = 0$, Equations (1.5) and (1.6) are called the *nonlinear dispersive variants* CH(n , n) and CH($-n$, $-n$) of the generalized Camassa-Holm equation [7].

The tanh method and the sine-cosine method will be mainly used to back up our analysis. The tanh method, developed by Malfliet [2, 3], is a direct and effective algebraic method for handling many nonlinear equations. The sine-cosine method was proved to be powerful in handling nonlinear problems, with genuine nonlinear dispersion, where compactons and solitary patterns solutions are generated. The two methods will be described briefly, where details can be found in [2-7] and the references therein.

2. Analysis of the two Methods

The sine-cosine method, the tanh method, and the extended tanh method have been applied for a wide variety of nonlinear problems. The main features of the two methods will be reviewed briefly.

For both methods, we first use the wave variable $\xi = \sum_{i=1}^n x_i - ct$ to carry a PDE in two independent variables

$$P(u, u_t, u_{x_1}, \dots, u_{x_n}, u_{tt}, u_{x_1 x_1}, \dots, u_{x_n x_n}, \dots) = 0, \quad (2.1)$$

into an ODE

$$Q(u, u', u'', u''', \dots) = 0. \quad (2.2)$$

Equation (2.2) is then integrated as long as all terms contain derivatives, where integration constants are considered zeros.

2.1. The sine-cosine method. The sine-cosine method admits the use of the solution in the form

$$u(\xi) = \begin{cases} \lambda \cos^\beta(\mu\xi), & |\mu\xi| < \frac{\pi}{2}, \\ 0, & \text{otherwise,} \end{cases} \quad (2.3)$$

or in the form

$$u(\xi) = \begin{cases} \lambda \sin^\beta(\mu\xi), & |\mu\xi| < \pi, \\ 0, & \text{otherwise,} \end{cases} \quad (2.4)$$

where λ , μ , and β are parameters that will be determined.

We substitute (2.3) or (2.4) into the reduced ordinary differential equation obtained above in (2.2), balance the terms of the cosine functions when (2.3) is used, or balance the terms of the sine functions when (2.4) is used, and solving the resulting system of algebraic equations by using the computerized symbolic calculations to obtain all possible values of the parameters λ , μ , and β .

2.2. The tanh method and the extended tanh method. The standard tanh method is introduced in [2, 3], where the tanh is used as a new variable, since all derivatives of a tanh are represented by a tanh itself. We use a new independent variable

$$Y = \tanh(\mu\xi), \quad (2.5)$$

that leads to the change of derivatives:

$$\begin{aligned} \frac{d}{d\xi} &= \mu(1 - Y^2) \frac{d}{dY}, \\ \frac{d^2}{d\xi^2} &= \mu^2(1 - Y^2) \left(-2Y \frac{d}{dY} + (1 - Y^2) \frac{d^2}{dY^2} \right). \end{aligned} \quad (2.6)$$

We then apply the following finite expansion:

$$u(\mu\xi) = S(Y) = \sum_{k=0}^M a_k Y^k, \quad (2.7)$$

and

$$u(\mu\xi) = S(Y) = \sum_{k=0}^M a_k Y^k + \sum_{k=1}^M b_k Y^{-k}, \quad (2.8)$$

where M is a positive integer that will be determined to derive a closed form analytic solution. However, if M is not an integer, a transformation formula is usually used. Substituting (2.5) and (2.6) into the simplified ODE (2.2) results in an equation in powers of Y . To determine the parameter M , we usually balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms. With M determined, we collect all coefficients of powers of Y in the resulting equation, where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters a_k ($k = 0, \dots, M$), μ , and c . Having determined these parameters, knowing that M is a positive integer in most cases, and by using (2.7) or (2.8), we obtain an analytic solution $u(x, t)$ in a closed form.

3. Using the Sine-Cosine Method

3.1. For positive exponents.

We first consider the $(N + 1)$ -dimensional nonlinear dispersive variant CH($n, 2n-1, n$) of the generalized Camassa-Holm equation

$$u_{tt} = \sum_{i=1}^N (a_i u + b_i u^n + d_i u^{2n-1})_{x_i x_i} + \sum_{i=1}^N k_i (u^n)_{x_i x_i t t}, \quad a_i, k_i > 0, \quad n > 1. \quad (3.1)$$

Using the wave variable $\xi = \sum_{i=1}^N x_i - ct$, carries (3.1) into the ODE

$$c^2 u'' = \sum_{i=1}^N (a_i u + b_i u^n + d_i u^{2n-1})'' + c^2 \sum_{i=1}^N k_i (u^n)^{(4)}. \quad (3.2)$$

Integrating (3.2) twice, using the constants of integration to be zero, we find

$$(\alpha_0 - c^2)u + b_0u^n + d_0u^{2n-1} + c^2k_0(u^n)'' = 0, \quad (3.3)$$

where $\alpha_0 = \sum_{i=1}^N \alpha_i > 0$, $k_0 = \sum_{i=1}^N k_i > 0$, $b_0 = \sum_{i=1}^N b_i$, $d_0 = \sum_{i=1}^N d_i$.

Substituting (2.3) into (3.3) gives

$$\begin{aligned} & (\alpha_0 - c^2)\lambda \cos^\beta(\mu\xi) + b_0\lambda^n \cos^{n\beta}(\mu\xi) + d_0\lambda^{2n-1} \cos^{(2n-1)\beta}(\mu\xi) \\ & + c^2k_0(-n^2\mu^2\lambda^n\beta^2 \cos^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\cos^{n\beta-2}(\mu\xi)) = 0. \end{aligned} \quad (3.4)$$

Equation (3.4) is satisfied only, if the following system of algebraic equations holds:

$$\begin{aligned} n\beta - 1 \neq 0, \alpha_0 - c^2 = 0, b_0\lambda^n &= c^2k_0n^2\mu^2\beta^2\lambda^n, \\ (2n-1)\beta = n\beta - 2, d_0\lambda^{2n-1} &= -c^2k_0n\mu^2\lambda^n\beta(n\beta-1). \end{aligned} \quad (3.5)$$

Solving the system (3.5) gives

$$c = \pm\sqrt{\alpha_0}, \beta = \frac{2}{1-n}, \mu = \frac{n-1}{2n} \sqrt{\frac{b_0}{\alpha_0 k_0}}, \lambda = \left(\frac{b_0(3n-1)}{-2nd_0} \right)^{\frac{1}{n-1}}. \quad (3.6)$$

The results (3.6) can be easily obtained, if we also use the sine method (2.4). Combining (3.6) with (2.3) and (2.4), the following periodic solutions

$$u_1(x_1, \dots, x_N, t) = \begin{cases} \left\{ \frac{b_0(3n-1)}{-2nd_0} \csc^2 \left[\frac{n-1}{2n} \sqrt{\frac{b_0}{\alpha_0 k_0}} \left(\sum_{i=1}^N x_i \pm \sqrt{\alpha_0} t \right) \right] \right\}^{\frac{1}{n-1}}, \\ 0 < \sum_{i=1}^N x_i \pm \sqrt{\alpha_0} t < \frac{\pi}{\mu}, b_0 > 0, d_0 < 0, \\ 0, & \text{otherwise,} \end{cases} \quad (3.7)$$

and

$$u_2(x_1, \dots, x_N, t) = \begin{cases} \left\{ \frac{a_0(3n-1)}{-2nd_0} \sec^2 \left[\frac{n-1}{2n} \sqrt{\frac{b_0}{a_0 k_0}} \left(\sum_{i=1}^N x_i \pm \sqrt{a_0} t \right) \right] \right\}^{\frac{1}{n-1}}, \\ 0 < \sum_{i=1}^N x_i \pm \sqrt{a_0} t < \frac{\pi}{2\mu}, b_0 > 0, d_0 < 0, \\ 0, & \text{otherwise} \end{cases} \tag{3.8}$$

are readily obtained.

However, for $b_0 < 0$, we obtain the following solitons solutions

$$u_3(x_1, \dots, x_N, t) = \left\{ \frac{b_0(3n-1)}{2nd_0} \operatorname{csch}^2 \left[\frac{n-1}{2n} \sqrt{\frac{-b_0}{a_0 k_0}} \left(\sum_{i=1}^N x_i \pm \sqrt{a_0} t \right) \right] \right\}^{\frac{1}{n-1}},$$

$$d_0 < 0, b_0 < 0, \tag{3.9}$$

and

$$u_4(x_1, \dots, x_N, t) = \left\{ \frac{b_0(3n-1)}{-2nd_0} \operatorname{sech}^2 \left[\frac{n-1}{2n} \sqrt{\frac{-b_0}{a_0 k_0}} \left(\sum_{i=1}^N x_i \pm \sqrt{a_0} t \right) \right] \right\}^{\frac{1}{n-1}},$$

$$d_0 > 0, b_0 < 0. \tag{3.10}$$

3.2. For negative exponents

We consider the $(N + 1)$ -dimensional nonlinear dispersive variant CH $(-n, -(2n-1), -n)$ of the generalized Camassa-Holm equation

$$u_{tt} = \sum_{i=1}^N (a_i u + b_i u^{-n} + d_i u^{-(2n-1)})_{x_i x_i} + \sum_{i=1}^N k_i (u^{-n})_{x_i x_i t t}, \quad a_i, k_i > 0, n > 1. \tag{3.11}$$

Proceeding as before, we obtain the ODE

$$c^2 u'' = \sum_{i=1}^N (a_i u + b_i u^n + d_i u^{2n-1})'' + c^2 \sum_{i=1}^N k_i (u^n)^{(4)}, \quad (3.12)$$

upon using the wave variable $\xi = \sum_{i=1}^N x_i - ct$ in (3.11). Integrating (3.12) twice, using the constants of integration to be zero, we find

$$(a_0 - c^2)u + b_0 u^{-n} + d_0 u^{-(2n-1)} + c^2 k_0 (u^{-n})'' = 0. \quad (3.13)$$

Substituting (2.3) into (3.13) gives

$$\begin{aligned} & (a_0 - c^2)\lambda \cos^\beta(\mu\xi) + b_0 \lambda^{-n} \cos^{-n\beta}(\mu\xi) + d_0 \lambda^{-(2n-1)} \cos^{-(2n-1)\beta}(\mu\xi) \\ & + c^2 k_0 (-n^2 \mu^2 \lambda^{-n} \beta^2 \cos^{-n\beta}(\mu\xi) + n \mu^2 \lambda^{-n} \beta(n\beta + 1) \cos^{-n\beta-2}(\mu\xi)) = 0. \end{aligned} \quad (3.14)$$

Equation (3.14) is satisfied only, if the following system of algebraic equations holds:

$$\begin{aligned} & n\beta + 1 \neq 0, \quad a_0 - c^2 = 0, \quad b_0 = c^2 k_0 n^2 \mu^2 \beta^2, \\ & -(2n-1)\beta = -n\beta - 2, \quad d_0 \lambda^{-(2n-1)} = -c^2 k_0 n \mu^2 \lambda^{-n} \beta(n\beta + 1). \end{aligned} \quad (3.15)$$

Solving the system (3.15) gives

$$c = \pm\sqrt{a_0}, \quad \beta = \frac{2}{n-1}, \quad \mu = \frac{n-1}{2n} \sqrt{\frac{b_0}{a_0 k_0}}, \quad \lambda = \left(\frac{-2nd_0}{b_0(3n-1)} \right)^{\frac{1}{n-1}}. \quad (3.16)$$

The results (3.16) can be easily obtained, if we also use the sine method (2.4). Combining (3.16) with (2.3) and (2.4), the following compactons solutions

$$u_5(x_1, \dots, x_N, t) = \begin{cases} \left\{ \frac{-2nd_0}{b_0(3n-1)} \sin^2 \left[\frac{n-1}{2n} \sqrt{\frac{b_0}{\alpha_0 k_0}} \left(\sum_{i=1}^N x_i \pm \sqrt{\alpha_0 t} \right) \right] \right\}^{\frac{1}{n-1}}, \\ 0 < \left| \sum_{i=1}^N x_i \pm \sqrt{\alpha_0 t} \right| < \frac{\pi}{\mu}, b_0 > 0, d_0 < 0, \\ 0, & \text{otherwise,} \end{cases} \tag{3.17}$$

and

$$u_6(x_1, \dots, x_N, t) = \begin{cases} \left\{ \frac{-2nd_0}{b_0(3n-1)} \cos^2 \left[\frac{n-1}{2n} \sqrt{\frac{b_0}{\alpha_0 k_0}} \left(\sum_{i=1}^N x_i \pm \sqrt{\alpha_0 t} \right) \right] \right\}^{\frac{1}{n-1}}, \\ 0 < \left| \sum_{i=1}^N x_i \pm \sqrt{A_0 t} \right| < \frac{\pi}{2\mu}, b_0 > 0, d_0 < 0, \\ 0, & \text{otherwise} \end{cases} \tag{3.18}$$

are readily obtained.

However, for $b_0 < 0$, we obtain the following solitary patterns solutions

$$u_7(x_1, \dots, x_N, t) = \left\{ \frac{2nd_0}{b_0(3n-1)} \sinh^2 \left[\frac{n-1}{2n} \sqrt{\frac{-b_0}{\alpha_0 k_0}} \left(\sum_{i=1}^N x_i \pm \sqrt{\alpha_0 t} \right) \right] \right\}^{\frac{1}{n-1}},$$

$$d_0 < 0, b_0 < 0, \tag{3.19}$$

and

$$u_8(x_1, \dots, x_N, t) = \left\{ \frac{-2nd_0}{b_0(3n-1)} \cosh^2 \left[\frac{n-1}{2n} \sqrt{\frac{-b_0}{\alpha_0 k_0}} \left(\sum_{i=1}^N x_i \pm \sqrt{\alpha_0 t} \right) \right] \right\}^{\frac{1}{n-1}},$$

$$d_0 > 0, b_0 < 0. \tag{3.20}$$

4. Using the Extended tanh Method

In this section, we will use the extended tanh method to handle the $(N + 1)$ -dimensional nonlinear dispersive variant CH($n, 2n-1, n$) of the generalized Camassa-Holm and its variants that were examined before.

4.1. For positive exponents.

Balancing the linear term of highest order with the nonlinear term in (3.3), we find

$$M = \frac{2}{n-1}. \quad (4.1)$$

To get a closed form analytic solution, the parameter M should be an integer. A transformation formula

$$u = \phi^{\frac{1}{n-1}}, \quad (4.2)$$

should be used to achieve our goal. This in turn transforms (3.3) to

$$(a_0 - c^2)\phi + b_0\phi^2 + d_0\phi^3 + \frac{c^2k_0n}{(n-1)^2}(\phi)^2 + \frac{c^2k_0n}{n-1}\phi\phi'' = 0. \quad (4.3)$$

Balancing $\phi\phi''$ and ϕ^3 , we find

$$M = 2. \quad (4.4)$$

The extended tanh method allows us to use the substitution

$$\phi(\xi) = S(Y) = A_0 + A_1Y + A_2Y^2 + B_1Y^{-1} + B_2Y^{-2}. \quad (4.5)$$

Substituting (4.5) into (4.3), collecting the coefficients of each power of Y , and solve the resulting system of algebraic equations to find the sets of solutions:

$$\text{Case 1. } A_0 = A_1 = B_1 = B_2 = 0, A_2 = -\frac{b_0(3n-1)}{4d_0n},$$

$$\mu^2 = \frac{2b_0d_0(n-1)^2}{k_0[b_0^2(-3n+1)(n+1) + 16a_0n^2d_0]}, c^2 = \frac{b_0^2(-3n+1)(n+1) + 16a_0n^2d_0}{16d_0n^2}.$$

Case 2. $A_1 = B_1 = B_2 = 0, A_0 = -A_2 = -\frac{b_0(3n-1)}{2d_0n},$

$$\mu^2 = -\frac{b_0(n-1)^2}{4a_0k_0n^2}, c^2 = a_0.$$

Case 3. $A_0 = A_1 = A_2 = B_1 = 0, B_2 = -\frac{b_0(3n-1)}{4d_0n},$

$$\mu^2 = \frac{2b_0d_0(n-1)^2}{k_0[b_0^2(-3n+1)(n+1)+16a_0n^2d_0]}, c^2 = \frac{b_0^2(-3n+1)(n+1)+16a_0n^2d_0}{16d_0n^2}.$$

Case 4. $A_1 = A_2 = B_1 = 0, A_0 = -B_2 = -\frac{b_0(3n-1)}{2d_0n},$

$$\mu^2 = -\frac{b_0(n-1)^2}{4a_0k_0n^2}, c^2 = a_0.$$

Case 5. $A_1 = B_1 = 0, A_0 = 2A_2 = 2B_2 = -\frac{b_0(3n-1)}{8d_0n},$

$$\mu^2 = \frac{b_0d_0(n-1)^2}{2k_0[b_0^2(-3n+1)(n+1)+16a_0n^2d_0]}, c^2 = \frac{b_0^2(-3n+1)(n+1)+16a_0n^2d_0}{16d_0n^2}.$$

Case 6. $A_1 = B_1 = 0, A_0 = -2A_2 = -2B_2 = -\frac{b_0(3n-1)}{4d_0n},$

$$\mu^2 = -\frac{b_0(n-1)^2}{16a_0k_0n^2}, c^2 = a_0.$$

We obtain the solitary patterns solutions:

$$u_9 = \left\{ -\frac{b_0(3n-1)}{4d_0n} \tanh^2 \sqrt{\frac{2b_0d_0(n-1)^2}{k_0[b_0^2(-3n+1)(n+1)+16a_0n^2d_0]}} \left(\sum_{i=1}^N x_i - ct \right) \right\}^{\frac{1}{n-1}},$$

$$b_0 > 0, d_0 < 0, c = \pm \sqrt{\frac{b_0^2(-3n+1)(n+1)+16a_0n^2d_0}{16d_0n^2}}. \tag{4.6}$$

$$u_{10} = \left\{ -\frac{b_0(3n-1)}{2d_0n} \tanh^2 \left[\frac{n-1}{2n} \sqrt{\frac{-b_0}{a_0k_0}} \left(\sum_{i=1}^N x_i \pm \sqrt{a_0}t \right) \right] \right\}^{\frac{1}{n-1}}, \quad b_0 < 0, d_0 > 0. \quad (4.7)$$

$$u_{11} = \left\{ -\frac{b_0(3n-1)}{4d_0n} \coth^2 \sqrt{\frac{2b_0d_0(n-1)^2}{k_0[b_0^2(-3n+1)(n+1) + 16a_0n^2d_0]}} \left(\sum_{i=1}^N x_i - ct \right) \right\}^{\frac{1}{n-1}},$$

$$b_0 > 0, d_0 < 0, c = \pm \sqrt{\frac{b_0^2(-3n+1)(n+1) + 16a_0n^2d_0}{16d_0n^2}}. \quad (4.8)$$

$$u_{12} = \left\{ -\frac{b_0(3n-1)}{2d_0n} \coth^2 \left[\frac{n-1}{2n} \sqrt{\frac{-b_0}{a_0k_0}} \left(\sum_{i=1}^N x_i \pm \sqrt{a_0}t \right) \right] \right\}^{\frac{1}{n-1}}, \quad b_0 < 0, d_0 > 0. \quad (4.9)$$

$$u_{13} = \left\{ -\frac{b_0(3n-1)}{8d_0n} \left(1 + \frac{1}{2} \tanh^2 \mu \left(\sum_{i=1}^N x_i - ct \right) + \frac{1}{2} \coth^2 \mu \left(\sum_{i=1}^N x_i - ct \right) \right) \right\}^{\frac{1}{n-1}},$$

$$b_0 > 0, d_0 < 0, c = \pm \sqrt{\frac{b_0^2(-3n+1)(n+1) + 16a_0n^2d_0}{16d_0n^2}},$$

$$\mu = \pm \sqrt{\frac{b_0d_0(n-1)^2}{2k_0[b_0^2(-3n+1)(n+1) + 16a_0n^2d_0]}}. \quad (4.10)$$

$$u_{14} = \left\{ -\frac{b_0(3n-1)}{4d_0n} \left(1 - \frac{1}{2} \tanh^2 \mu \left(\sum_{i=1}^N x_i - ct \right) - \frac{1}{2} \coth^2 \mu \left(\sum_{i=1}^N x_i - ct \right) \right) \right\}^{\frac{1}{n-1}},$$

$$b_0 < 0, d_0 > 0, c = \pm \sqrt{a_0}, \mu = \pm \left(\frac{n-1}{n} \sqrt{\frac{-b_0}{a_0k_0}} \right), \quad (4.11)$$

and the compactons solutions

$$u_{15} = \left\{ \frac{b_0(3n-1)}{4d_0n} \tan^2 \sqrt{\frac{-2b_0d_0(n-1)^2}{k_0[b_0^2(-3n+1)(n+1)+16a_0n^2d_0]}} \left(\sum_{i=1}^N x_i - ct \right) \right\}^{\frac{1}{n-1}},$$

$$b_0 < 0, d_0 < 0, c = \pm \sqrt{\frac{b_0^2(-3n+1)(n+1)+16a_0n^2d_0}{16d_0n^2}}. \quad (4.12)$$

$$u_{16} = \left\{ \frac{b_0(3n-1)}{2d_0n} \tan^2 \left[\frac{n-1}{2n} \sqrt{\frac{b_0}{a_0k_0}} \left(\sum_{i=1}^N x_i \pm \sqrt{a_0}t \right) \right] \right\}^{\frac{1}{n-1}}, \quad b_0 > 0, d_0 > 0.$$

(4.13)

$$u_{17} = \left\{ \frac{b_0(3n-1)}{4d_0n} \cot^2 \sqrt{\frac{-2b_0d_0(n-1)^2}{k_0[b_0^2(-3n+1)(n+1)+16a_0n^2d_0]}} \left(\sum_{i=1}^N x_i - ct \right) \right\}^{\frac{1}{n-1}},$$

$$b_0 < 0, d_0 < 0, c = \pm \sqrt{\frac{b_0^2(-3n+1)(n+1)+16a_0n^2d_0}{16d_0n^2}}. \quad (4.14)$$

$$u_{18} = \left\{ \frac{b_0(3n-1)}{2d_0n} \cot^2 \left[\frac{n-1}{2n} \sqrt{\frac{b_0}{a_0k_0}} \left(\sum_{i=1}^N x_i \pm \sqrt{a_0}t \right) \right] \right\}^{\frac{1}{n-1}}, \quad b_0 > 0, d_0 > 0.$$

(4.15)

$$u_{19} = \left\{ \frac{b_0(3n-1)}{8d_0n} \left(1 - \frac{1}{2} \tan^2 \mu \left(\sum_{i=1}^N x_i - ct \right) - \frac{1}{2} \cot^2 \mu \left(\sum_{i=1}^N x_i - ct \right) \right) \right\}^{\frac{1}{n-1}},$$

$$b_0 < 0, d_0 < 0, c = \pm \sqrt{\frac{b_0^2(-3n+1)(n+1)+16a_0n^2d_0}{16d_0n^2}},$$

$$\mu = \pm \sqrt{\frac{-b_0d_0(n-1)^2}{2k_0[b_0^2(-3n+1)(n+1)+16a_0n^2d_0]}}. \quad (4.16)$$

$$u_{20} = \left\{ \frac{b_0(3n-1)}{4d_0n} \left(1 + \frac{1}{2} \tan^2 \mu \left(\sum_{i=1}^N x_i - ct \right) + \frac{1}{2} \cot^2 \mu \left(\sum_{i=1}^N x_i - ct \right) \right) \right\}^{\frac{1}{n-1}},$$

$$b_0 > 0, d_0 > 0, c = \pm\sqrt{a_0}, \mu = \pm\left(\frac{n-1}{n}\sqrt{\frac{b_0}{a_0k_0}}\right). \quad (4.17)$$

4.2. For negative exponents

Balancing the linear term of highest order with the nonlinear term in (3.13), we find

$$M = \frac{2}{n+1}. \quad (4.18)$$

To get a closed form solution, M should be an integer. This means that $M = 1$ for $n = 1$. The last case was examined before, then it remains to examine the case, where $M = 1$ for $n = 1$. This means that

$$u(\mu\xi) = S(Y) = A_0 + A_1Y + B_1Y^{-1}. \quad (4.19)$$

Substituting (4.19) into (3.13), collecting the coefficients of each power of Y , and solve the resulting system of algebraic equations to find the set of solution:

$$A_0 = 0, A_1 = -B_1, c^2 = a, \mu^2 = -\frac{b_0 + d_0}{a_0k_0}.$$

We obtain the soliton solution

$$u_{21} = A_1 \left(\tanh \frac{1}{2} \sqrt{\frac{b_0 + d_0}{-a_0k_0}} \left(\sum_{i=1}^N x_i - ct \right) - \coth \frac{1}{2} \sqrt{\frac{b_0 + d_0}{-a_0k_0}} \left(\sum_{i=1}^N x_i - ct \right) \right),$$

$$b_0 + d_0 < 0, c = \pm\sqrt{a_0}, A_1 \in R. \quad (4.20)$$

However, for $b_0 + d_0 > 0$, we obtain the following periodic wave solutions

$$u_{22} = A \left(\tan \frac{1}{2} \sqrt{\frac{b_0 + d_0}{a_0k_0}} \left(\sum_{i=1}^N x_i - ct \right) + \cot \frac{1}{2} \sqrt{\frac{b_0 + d_0}{a_0k_0}} \left(\sum_{i=1}^N x_i - ct \right) \right),$$

$$b_0 + d_0 > 0, c = \pm\sqrt{a_0}, A \in R. \quad (4.21)$$

5. Discussion

In this paper, we used the sine-cosine method and the extended tanh method to study the generalized Camassa-Holm equation. The methods provided solitary wave solutions, compactons solutions, solitary patterns solutions, and triangular periodic solutions. Moreover, the obtained results in this work clearly demonstrate the reliability of the methods that were used.

References

- [1] L. Jibin and L. Ming, Bounded travelling wave solutions for the $(n+1)$ -dimensional sine-and sinh-Gordon equation, *Chaos. Solitons. Fract.* 25 (2005), 1037-1047.
- [2] W. Malfliet, The tanh method: I, Exact solutions of nonlinear evolution and wave equations, *Phys. Scr.* 54 (1996), 563-568.
- [3] W. Malfliet and W. Hereman, The tanh method: II, Perturbation technique for conservative systems, *Phys. Scr.* 54 (1996), 569-575.
- [4] S. Tang et al., Travelling wave solutions for a class of nonlinear fourth order variant of a generalized Camassa-Holm equation, *Appl. Math. Comput.* 210 (2009), 39-47.
- [5] A. M. Wazwaz, A sine-cosine method for handling nonlinear wave equations, *Math. Comput. Model.* 40 (2004), 499-508.
- [6] A. M. Wazwaz, A reliable treatment of the physical structure for the nonlinear equation $K(m, n)$, *Appl. Math. Comput.* 163 (2005), 1081-1095.
- [7] A. M. Wazwaz, A class of nonlinear fourth order variant of a generalized Camassa-Holm equation with compact and noncompact solutions, *Appl. Math. Comput.* 165 (2005), 485-501.
- [8] Z. Yan, Separation transformation and envelope solutions of the higher-dimensional complex nonlinear Klein-Gordon equation, *Phys. Scr.* 75 (2007), 320-322.
- [9] Z. Yan, Similarity transformations and exact solutions for a family of higher-dimensional generalized Boussinesq equations, *Phys. Lett. A* 361 (2007), 223-230.

