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A SECOND-DERIVATIVE-FREE VARIANT OF HALLEY'S METHOD WITH SIXTH-ORDER CONVERGENCE

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Abstract

In recent years, the modified Halley's methods have been one of the popular iterative methods to find approximate solutions to the roots of nonlinear equation. In this paper, we propose a new method without the second derivative to modify the Halley's method. The present iterative method is of sixth-order convergence and can be viewed as an improvement of the recent works [9, 10]. Several numerical examples are given to illustrate the efficiency and performance of this method.

1. Introduction

Iterative methods are usually the only choice for finding approximate solutions to nonlinear equation $f(x) = 0$ in numerical analysis. In recent years, various iterative methods based on the Taylor series, decomposition and quadrature formulae [1-4, 7-10, 12] have been

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developed. Newton's method is an important and basic method [11]. Recall that the Newton iteration is defined by the equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This method converges quadratically, which is faster than the bisection and the secant methods.

The Halley's method is another well-known iterative method, which is written as

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{L_f(x_n)}{1 - \frac{1}{2}L_f(x_n)}\right)\frac{f(x_n)}{f'(x_n)},$$

where $L_f(x) = f''(x)f(x)/f'(x)^2$. The method is of third-order convergence. To improve the efficiency of the preceding methods, a predictor-corrector Halley's method with sixth-order convergence is proposed by [9]. Once the sixth-order convergence becomes effective, that is, the values of this method sequence are sufficiently close to the root, the convergence is very rapid. However, one of the drawbacks of the Halley's method and its improvement involves the second derivative of the function. The second derivative is difficult to evaluate. To overcome this disadvantage, a number of methods have been proposed. Chun [2] has presented and analyzed a one-parameter fourth-order family of the variants of Chebyshev-Halley methods. The family includes a well-known Jarratt's fourth-order method as particular one. Noor et al. [10] have modified the Halley's method by using the finite difference scheme and proved the iterative method is of fifth-order convergence. The common key step of the above two methods is replacing the second derivative of the function f by its finite difference scheme. Zhou [12] proposes another method by replacing the second derivative by the combination of two evaluations of the function and one evaluation of the first derivative, and gets a modified Halley's method with fourth-order convergence. For other related iterative methods, one may refer to [7, 8].

Motivated by these recent works in this direction, we construct a new modified Halley iteration without the second derivative. We prove that the new method is of sixth-order convergence, which is better than the method in [10], and has the same order of convergence as the predictor-corrector Halley's method proposed by [9]. The advantage of the new method is that, it is of higher order convergence and does not require to evaluate the second derivative.

2. Modified Halley's Methods

Consider the nonlinear equation of the type

$$f(x) = 0. \tag{2.1}$$

We assume that α is a simple root of the Equation (2.1). Using Newton's method as predictor and Halley's method as a corrector, Noor and Noor [9] have obtained the following two step method, which is of sixth-order convergence.

Algorithm 2.1. For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{2f(y_n)f'(y_n)}{2f'(y_n)^2 - f(y_n)f''(y_n)}.$$

In order to implement the algorithm, one has to find the second derivative. Thus, Noor et al. [10] have considered replacing $f''(y_n)$ by $(f'(y_n) - f'(x_n))/(y_n - x_n)$. This idea is very important and plays a significant role in developing some iterative methods free from second derivatives. Then, the following algorithm is obtained.

Algorithm 2.2. For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{2f(x_n)f(y_n)f'(y_n)}{2f(x_n)f'(y_n)^2 - f(y_n)f'(x_n)^2 + f(y_n)f'(x_n)f'(y_n)}.$$

This algorithm does not require the second derivative, and it involves the evaluations of the function and the evaluations of the first derivative. But, this method has only fifth-order convergence, which is worse than Algorithm 2.1. Is there a sixth-order iterative method which does require only the evaluations of the function and the evaluations of its derivative? Now, let us first consider the approximation of $f''(y_n)$

$$f''(y_n) \approx \frac{2}{y_n - x_n}\left\{2f'(y_n) + f'(x_n) - 3\frac{f(y_n) - f(x_n)}{y_n - x_n}\right\} \equiv P_f(x_n, y_n), \quad (2.2)$$

which implies

$$L_f(y_n) = \frac{f''(y_n)f(y_n)}{f'(y_n)^2} \approx \frac{P_f(x_n, y_n)f(y_n)}{f'(y_n)^2} \equiv H_f(x_n, y_n). \quad (2.3)$$

Similar to Algorithm 2.1, we replace $f''(y_n)$ by $P_f(x_n, y_n)$, then we construct the following iterative scheme:

Algorithm 2.3. For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \left(1 + \frac{1}{2}\frac{H_f(x_n, y_n)}{1 - \frac{1}{2}H_f(x_n, y_n)}\right)\frac{f(y_n)}{f'(y_n)}.$$

It is obvious that this method involves only the evaluations of the function and the evaluations of its derivative, and we can verify that the present method is of sixth-order convergence.

3. Convergence Analysis

In this section, we give the analysis of convergence of Algorithm 2.3.

Theorem 3.1. *Let* α *be a simple zero of sufficiently differentiable function f. If* x_0 *is sufficiently close to* α, *then Algorithm* 2.3 *has sixth-order convergence.*

Proof. Let $e_n = x_n - \alpha (n = 0, 1, 2, \ldots)$ and $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$ $(k = 2, 3, \ldots)$. By Taylor's expansion, we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O(e_n^7)], \tag{3.1}$$

and

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + O(e_n^7)]. \tag{3.2}$$

Then, it follows that

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= \alpha + c_2 e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ &\quad + (-8c_2^4 + 20c_2^2c_3 - 10c_2c_4 - 6c_3^2 + 4c_5)e_n^5 \\ &\quad + (16c_2^5 - 52c_2^3c_3 + 33c_2c_3^2 + 28c_2^2c_4 - 13c_2c_5 - 17c_3c_4 + 5c_6)e_n^6 \\ &\quad + O(e_n^7). \end{aligned} \tag{3.3}$$

By Equation (3.3), we have

$$\begin{aligned} f(y_n) = f'(\alpha)[&c_2 e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ &+ (-12c_2^4 + 24c_2^2c_3 - 10c_2c_4 - 6c_3^2 + 4c_5)e_n^5 \end{aligned}$$

$$+ (28c_2^5 - 73c_2^3c_3 + 37c_2c_3^2 + 34c_2^2c_4 - 13c_2c_5 - 17c_3c_4 + 5c_6)e_n^6$$

$$+ O(e_n^7)], \tag{3.4}$$

and

$$f'(y_n) = f'(\alpha)[1 + 2c_2^2e_n^2 + (-4c_2^3 + 4c_2c_3)e_n^3 + (8c_2^4 - 11c_2^2c_3 + 6c_2c_4)e_n^4$$

$$+ (-16c_2^5 + 28c_2^3c_3 - 20c_2^2c_4 + 8c_2c_5)e_n^5$$

$$+ (32c_2^6 - 68c_2^4c_3 + 56c_2^3c_4 - 26c_2^2c_5 + 10c_2c_6 - 16c_2c_3c_4 + 12c_3^3)e_n^6$$

$$+ O(e_n^7)]. \tag{3.5}$$

Dividing (3.4) by (3.5) gives

$$\frac{f(y_n)}{f'(y_n)} = [c_2e_n^2 + 2(-c_2^2 + c_3)e_n^3 + (3c_2^3 - 7c_2c_3 + 3c_4)e_n^4$$

$$+ (-4c_2^4 + 16c_2^2c_3 - 6c_3^2 - 10c_2c_4 + 4c_5)e_n^5$$

$$+ (6c_2^5 - 73c_2^2c_3 + 41c_2^3c_3 + 29c_2c_3^2 + 22c_2^2c_4 - 17c_3c_4$$

$$- 13c_2c_5 + 5c_6)e_n^6 + O(e_n^7)]. \tag{3.6}$$

According to (3.1), (3.2), (3.4), and (3.5), we have

$$H_f(x_n, y_n) = 2c_2^2e_n^2 + 4(-c_2^3 + c_2c_3)e_n^3 + (2c_2^4 - 8c_2^2c_3 + 4c_2c_4)e_n^4$$

$$+ (8c_2^5 - 8c_2^3c_3 + 12c_2c_3^2 - 12c_2^2c_4 - 4c_3c_4 + 4c_2c_5)e_n^5$$

$$+ (-24c_2^6 - 146c_2^3c_3 + 218c_2^4c_3 - 90c_2^2c_3^2 + 24c_3^3 + 12c_2^3c_4$$

$$+ 32c_2c_3c_4 - 6c_4^2 - 16c_2^2c_5 - 8c_3c_5 + 4c_2c_6)e_n^6 + O(e_n^7). \tag{3.7}$$

Combining Equations (3.3), (3.6), (3.7), and the iterative scheme of Algorithm 2.3, we have

$$e_{n+1} = (c_2^5 - 74c_2^3c_3 + 73c_2^2c_3 + c_2^2c_4)e_n^6 + O(e_n^7),$$

from which, it follows that Algorithm 2.3 has sixth-order convergence. This completes the proof.

It is worth noting that, our method is of sixth-order and converges very rapidly even though it requires only the evaluations of the function and the evaluations of its first derivative. From the definition of the efficiency index, one can see this method has a high efficiency index that is equal to $\sqrt[4]{6} \approx 1.565,$ which is better than the ones of Newton's method $\sqrt{2} \approx 1.414$ and the Chebyshev-Halley methods $\sqrt[3]{3} \approx 1.442.$

4. Numerical Results

We present some numerical results for various functions to illustrate the efficiency of the new developed iterative methods in this paper. Let $|f(x_{n+1})| < \epsilon$ be the stopping criteria for computer programs, where the tolerance ϵ has been chosen equal to 10^{-14}. We use the following functions, most of which are the same as in [5, 6], and display the approximate zeros x^*.

$$f_1(x) = x^3 + 4x^2 - 10, \quad x^* = 1.365230013414097.$$

$$f_2(x) = e^{-x} + \cos x, \quad x^* = 1.746139530408012.$$

$$f_3(x) = (5x - 1)/(4x), \quad x^* = 0.2.$$

$$f_4(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \quad x^* = -1.207647827130919.$$

$$f_5(x) = e^x \sin x + \log(x^2 + 1), \quad x^* = 0.$$

$$f_6(x) = x^2 - e^x - 3x + 2, \quad x^* = 0.257530285439861.$$

$$f_7(x) = \sin^2 x - x^2 + 1, \quad x^* = 1.404491648215341.$$

$$f_8(x) = \cos x - x, \quad x^* = 0.739085133215161.$$

Displayed in Table 1 is the number of function evaluation (NEF) required such that $|f(x_{n+1})| < \epsilon$ for various methods. From Table 1, we conclude that the variant of Halley's method (VHM) requires the less NEFs as compared to the classical methods including Newton's method (NM), Chebyshev's method (CVM), Cauchy's method (CYM), and Halley's method (HM). Therefore, the new method improves the computational efficiency greatly.

Table 1. Comparison of various iterative methods

$f(x)$	x_0	NM	CVM	CYM	HM	VHM
f_1	1	12	12	9	9	8
	2	12	12	9	9	8
f_2	1	8	9	9	9	8
	2.5	10	12	9	12	8
f_3	0.25	10	9	9	3	8
	0.15	12	9	12	3	8
f_4	-1	10	12	9	9	8
	-1.45	12	12	12	9	8
f_5	-0.8	10	12	15	9	8
	1.2	10	12	12	12	8
f_6	2.2	10	12	15	12	8
	-2.5	10	12	12	12	8
f_7	1.2	10	12	12	12	8
	2.5	12	15	15	15	8
f_8	0.1	10	12	9	9	8
	2.5	10	12	12	12	8

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