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A JENSEN-TYPE INEQUALITY FOR STATES ON UNITAL $*$- ALGEBRAS

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Abstract

A Jensen-type inequality for states on unital $*$- algebras is given in an abstract algebraic setting.

1. Introduction

Jensen's inequality is known as a useful tool in many fields of mathematical sciences, and various generalizations are discussed. In this paper, we give a Jensen-type inequality for states on unital $*$- algebras in an abstract algebraic setting. A pair of a unital $*$- algebra and a state on it, is called an *algebraic probability space* (cf. [1]). The notion of algebraic probability is obtained by considering a generalization of algebra of random variables in probability theory. The main motivation for considering such a generalization is to study noncommutative extensions of probability theory (cf. [1], [2], [5]). The two main types of methods for

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the studies are functional analytic and abstract algebraic, our studies are focus on an abstract algebraic viewpoint (cf. [3]) for applications to combinatorics.

In this section, we take a general view of main result. First, we review a state on a unital $*$- algebra.

Definition 1.1. Let $\mathcal{A}$ be a unital $*$- algebra and φ be a functional $\varphi : \mathcal{A} \to \mathbb{C}$. We say that φ is a state on $\mathcal{A}$, if φ satisfies the properties:

(1) $\varphi(\lambda a + \mu b) = \lambda\varphi(a) + \mu\varphi(b), \quad \forall\lambda, \mu \in \mathbb{C}, \forall a, b \in \mathcal{A},$

(2) $\varphi(a^* a) \geq 0, \quad \forall a \in \mathcal{A},$

(3) $\varphi(1_{\mathcal{A}}) = 1,$

where $1_{\mathcal{A}}$ is a unit element of $\mathcal{A}$.

In algebraic probability, an element $a \in \mathcal{A}$ is called an *algebraic random variable* and $\varphi(a)$ is called *the mean of* a.

We show the following in Section 2.

Theorem 1.2. *Let* $\mathcal{A}$ *be a unital* $*$- *algebra and* φ *be a state on* $\mathcal{A}$. *For any* $a \in \mathcal{A}$, *we have*

$$\varphi((a^* a)^n) \geq \varphi(a^* a)^n, \qquad n = 1, 2, \cdots. \tag{1}$$

Here, let us recall that the set of states of an algebra $\mathcal{A}$ forms a convex set. Then, we have the following inequality for any $a \in \mathcal{A}$ from (1)

$$\sum_{i=1}^{\infty} \lambda_i \varphi_i((a^* a)^n) \geq \left(\sum_{i=1}^{\infty} \lambda_i \varphi_i(a^* a)\right)^n, \quad n = 1, 2, \cdots, \tag{2}$$

where $\{\varphi_i\}_i$ are states on $\mathcal{A}$ and $\{\lambda_i\}_i$ are real numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1, \lambda_i \geq 0$. If each φ_i is multiplicative and putting $\varphi_i(a^* a) = x_i$, then (2) is represented as the following.

$$\sum_{i=1}^{\infty} \lambda_i x_i^n \geq \left(\sum_{i=1}^{\infty} \lambda_i x_i \right)^n, \qquad x_i \geq 0. \tag{3}$$

(3) is a special case of the Jensen's inequality, so we can interpret (1) in this sense as a Jensen-type (inequality).

2. Proof of Theorem 1.2

We prove the statement by mathematical induction. If $\varphi(a^*a) = 0$, then clearly the statement holds, i.e., we consider in case of $\varphi(a^*a) \gneq 0$. If $n = 1$, then (1) is trivial. Let k be any natural number. We assume the following inequality holds for any $s \leq k$.

$$\varphi((a^*a)^s) \geq \varphi(a^*a)^s.$$

First step. If $k = 2p - 1 (p = 1, 2, \cdots)$, then we have

$$\varphi(((a^*a)^p - \varphi(a^*a)^p 1_{\mathcal{A}})^* ((a^*a)^p - \varphi(a^*a)^p 1_{\mathcal{A}}))$$
$$= \varphi((a^*a)^{2p}) + \varphi(a^*a)^{2p} - 2\varphi((a^*a)^p)\varphi(a^*a)^p \geq 0, \tag{4}$$

by the positivity of state. Since $p \leq k$, it holds

$$\varphi((a^*a)^p) \geq \varphi(a^*a)^p,$$

so we have

$$-\varphi(a^*a)^{2p} \geq \varphi(a^*a)^{2p} - 2\varphi((a^*a)^p)\varphi(a^*a)^p. \tag{5}$$

Then, it follows from (4) and (5) that

$$\varphi((a^*a)^{2p}) \geq \varphi(a^*a)^{2p}.$$

Since $2p = k + 1$, we get

$$\varphi((a^*a)^{k+1}) \geq \varphi(a^*a)^{k+1}.$$

Second step. If $k = 2p$, $(p = 1, 2, \cdots)$, then we have

$$\varphi((a(a^*a)^p - a\varphi(a^*a)^p)^*(a(a^*a)^p - a\varphi(a^*a)^p))$$

$$= \varphi((a^*a)^{2p+1}) + \varphi(a^*a)^{2p+1} - 2\varphi((a^*a)^{p+1})\varphi(a^*a)^p \geq 0, \tag{6}$$

by the positivity of state. Since $p \leq k$, it holds

$$\varphi((a^*a)^{p+1}) \geq \varphi(a^*a)^{p+1},$$

so we have

$$-\varphi(a^*a)^{2p+1} \geq \varphi(a^*a)^{2p+1} - 2\varphi((a^*a)^{p+1})\varphi(a^*a)^p. \tag{7}$$

Then, it follows from (6) and (7) that

$$\varphi((a^*a)^{2p+1}) \geq \varphi(a^*a)^{2p+1}.$$

Since $2p + 1 = k + 1$, we get

$$\varphi((a^*a)^{k+1}) \geq \varphi(a^*a)^{k+1}. \tag{8}$$

Thus, the statement is proved for every natural numbers $n = 1, 2, \cdots$.

3. Some Applications

In this section, we give some related inequalities as applications of (1). Let $\mathcal{A}$ be a unital $*$-algebra and φ be a state on $\mathcal{A}$. It follows from Theorem 1.2 and the Cauchy-Schwartz inequality for functionals that the following inequality.

$$|\varphi(a^*b)|^{2n} \leq \varphi((a^*a)^n)\varphi((b^*b)^n), \quad n = 1, 2, \cdots. \tag{9}$$

If the case $n = 1$, the Cauchy-Schwartz inequality recovers. If $b = 1_{\mathcal{A}}$ in (9), then we have

$$|\varphi(a)|^{2n} \leq \varphi((a^*a)^n), \qquad n = 1, 2, \cdots.$$

If the case $n = 1$ in $\varphi((a^*a)^n) - |\varphi(a)|^{2n}$, i.e., $\varphi(a^*a) - |\varphi(a)|^2$ is called *the variance of a*. If a is a selfadjoint element, i.e., $a^* = a$, then we have

$$\varphi(a)^{2n} \leq \varphi(a^{2n}), \qquad n = 1, 2, \cdots.$$

Corollary 3.1. *Let* $\forall(z_1, \cdots, z_n), (w_1, \cdots, w_n) \in \mathbb{C}^n$ *and* $m \in \mathbb{N}$. *Then we have*

$$\left|\sum_{i=1}^{n} \alpha_i \bar{z}_i w_i\right|^{2m} \leq \sum_{i=1}^{n} \alpha_i |z_i|^{2m} \sum_{i=1}^{n} \alpha_i |w_i|^{2m}, \tag{10}$$

where $\sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0$.

Proof. An n-dimensional complex Euclidean space $\mathbb{C}^n$ becomes $*$- algebra define by

$$z + w = (z_1 + w_1, \cdots, z_n + w_n),$$

$$\lambda z = (\lambda z_1, \cdots, \lambda z_n),$$

$$zw = (z_1 w_1, \cdots, z_n w_n),$$

$$z^* = (\bar{z}_1, \cdots, \bar{z}_n), \qquad \forall \lambda \in \mathbb{C}.$$

It is well known that a state φ on $*$- algebra $\mathbb{C}^n$ is described as following.

$$\varphi(z) = \sum_{i=1}^{n} \alpha_i z_i, \qquad \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0. \tag{11}$$

Then, we obtain (10), if z, w, φ are substituted for (9). □

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