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SOLITARY WAVE SOLUTIONS FOR A GENERALIZED FORM OF THE BONA-SMITH EQUATION

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Abstract

By using the sine-cosine method and the extended tanh-method, we study a generalized form of the Bona-Smith equation. It is shown that, this class gives solitary wave solutions and periodic wave solutions. The power of these methods is investigated and confirmed.

1. Introduction

It is well-known that searching for explicit solutions for nonlinear evolution equation, by using different methods, such as inverse scattering method, Darboux transformation method, Hirota bilinear method, bifurcation method of dynamic systems, Fan-expansion method, and so on (see [1, 4-6, 9, 11] and the references therein). Practically, there is no unified technique that can be employed to handle all types of nonlinear differential equations.

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In this paper, we consider a generalized form of the Bona-Smith equation [3]

$$\begin{cases} \eta_t + u_x + (\eta u)_x - \frac{1}{3}\eta_{xxt} = 0, \\ u_t + \eta_x + uu_x - \frac{1}{3}(u_t + \eta_x)_{xx} = 0. \end{cases} \tag{1.1}$$

This system, derived and studied by Bona and Smith, is somewhat anomalous as it is obtained by appending an extra term of higher order to the second equation. There is a complete existence and uniqueness theory for the Cauchy problem [3], and a non-standard numerical approximation was analyzed by Winther [10]. Various initial-and boundary-value problems for this system, along with their standard Galerkin-finite element discretization, are analyzed in [2].

The sine-cosine and the extended tanh algorithms, that provide a systematic framework for many nonlinear dispersive equations, will be employed to back up our analysis to determine compactions and solitary patterns solutions.

2. Analysis of the Two Methods

In what follows, the methods will be reviewed briefly. Full details can be found in [3, 7, 8].

A PDE

$$P(u, u_t, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots) = 0, \tag{2.1}$$

can be converted to an ODE

$$Q(u, u', u'', u''', \dots) = 0, \tag{2.2}$$

upon using a wave variable $\xi = x - ct$. Equation (2.2) is then integrated as long as all terms contain derivatives, where integration constants are considered zeros.

2.1. The sine-cosine method

The sine-cosine method admits the use of the solution in the form

$$u(x, t) = \begin{cases} \lambda \cos^{\beta}(\mu\xi), & |\mu\xi| < \frac{\pi}{2}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

or in the form

$$u(x, t) = \begin{cases} \lambda \sin^{\beta}(\mu\xi), & |\mu\xi| < \frac{\pi}{2}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.4}$$

where λ, μ, and β are parameters that will be determined.

We substitute (2.3) or (2.4) into the reduced ordinary differential equation obtained above in (2.2), balance the terms of the cosin functions, when (2.3) is used, or balance the terms of the sine functions, when (2.4) is used, and solving the resulting system of algebraic equations by using the computerized symbolic calculations to obtain all possible valuse of the parameters λ, μ, and β.

2.2. The extended tanh method

The standard tanh method introduced in [7, 8], where the tanh is used as a new variable, since all derivatives of a tanh are represented by a tanh itself. We use a new independent variable

$$Y = \tanh(\mu\xi), \tag{2.5}$$

that leads to the change of derivatives:

$$\frac{d}{d\xi} = \mu(1 - Y^2)\frac{d}{dY},$$

$$\frac{d^2}{d\xi^2} = \mu^2(1 - Y^2)\left(-2Y\frac{d}{dY} + (1 - Y^2)\frac{d^2}{dY^2}\right). \tag{2.6}$$

We then apply the following finite expansion:

$$u(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k, \tag{2.7}$$

and

$$u(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}, \tag{2.8}$$

where M is a positive integer that will be determined to derive a closed form analytic solution. However, if M is not an integer, a transformation formula is usually used. Substituting (2.5) and (2.6) into the simplified ODE (2.2) results in an equation in powers of Y. To determine the parameter M, we usually balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms. With M determined, we collect all coefficients of powers of Y in the resulting equation, where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters $a_k(k = 0, \ldots, M)$, μ, and c. Having determined these parameters, knowing that M is a positive integer in most cases, and by using (2.7) or (2.8), we obtain an analytic solution $u(x, t)$ in a closed form.

3. Using the Sine-Cosine Method

Let $u(x, t) = \phi(\xi)$, $\eta(x, t) = \psi(\xi)$, $\xi = x - ct$, where c is the wave speed. Then (1.1) becomes to

$$\begin{cases} -c\psi' + \phi' + (\phi\psi)' + \frac{1}{3}c\psi''' = 0, \\ -c\phi' + \psi' + \phi\phi' - \frac{1}{3}(-c\phi' + \psi')'' = 0, \end{cases} \tag{3.1}$$

where "$'$" is the derivative with respect to ξ.

$$\begin{cases} -c\psi + \phi + \phi\psi + \frac{1}{3}c\psi'' = g_1, \\ -c\phi + \psi + \frac{1}{2}\phi^2 - \frac{1}{3}(-c\phi'' + \psi'') = g_2. \end{cases} \tag{3.2}$$

Let $\phi = a_0 + a_1\psi$, we have

$$\begin{cases} -c\psi + (1 + \psi)(a_0 + a_1\psi) + \frac{1}{3}c\psi'' = g_1, \\ -c(a_0 + a_1\psi) + \psi + \frac{1}{2}(a_0 + a_1\psi)^2 - \frac{1}{3}(ca_1 - 1)\psi'' = g_2. \end{cases} \tag{3.3}$$

Multiply the first equation of (3.3) by $-(ca_1 - 1)$ and multiply the second equation of (3.3) by c on both sides, we have

$$\begin{cases} -\left[-c\psi + (1 + \psi)(a_0 + a_1\psi) + \frac{1}{3}c\psi''\right](ca_1 - 1) = -g_1(ca_1 - 1), \\ \left[-c(a_0 + a_1\psi) + \psi + \frac{1}{2}(a_0 + a_1\psi)^2 - \frac{1}{3}(ca_1 - 1)\psi''\right]c = cg_2. \end{cases} \tag{3.4}$$

Comparing the coefficients with the same powers of $\psi^i (i = 0, 1, 2)$ in two equations of (3.4), we obtain the following parameter conditions:

$$-(ca_1 - 1)a_0 + (ca_1 - 1)g_1 = -c^2 a_0 + \frac{1}{2} ca_0^2 - cg_2,$$

$$-(ca_1 - 1)(-c + a_0 + a_1) = -c^2 a_1 + c + ca_0 a_1,$$

$$-(ca_1 - 1)a_1 = \frac{1}{2} ca_1^2.$$

Solve the system of algebraic equations to find the sets of solutions:

$$a_0 = -2c + \frac{2}{3c},\ a_1 = \frac{2}{3c},\ g_1 = -12c^3 + 4c - \frac{1}{3c} + 3cg_2. \tag{3.5}$$

When $g_1 = a_0 = -2c + \frac{2}{3c}$, substituting (3.5) into the first equation of (3.3), we have

$$\left(\frac{4}{3c} - 3c\right)\psi + \frac{2}{3c}\psi^2 + \frac{1}{3}c\psi'' = 0. \tag{3.6}$$

If ψ is a solution of (3.6) satisfying the conditions (3.5), then ψ and the corresponding $\phi = a_0 + a_1\psi$ must be a travelling wave solution of system (1.1); we shall consider the travelling wave solutions of the system (3.6) under these conditions.

Substituting (2.3) into (3.6) yields

$$\left(\frac{4}{3c} - 3c\right)\lambda \cos^{\beta}(\mu\xi) + \frac{2}{3c}\lambda^2 \cos^{2\beta}(\mu\xi) + \frac{1}{3}c\mu^2\beta\lambda\left[-\beta\cos^{\beta}(\mu\xi) + (\beta - 1)\cos^{\beta-2}\right] = 0. \tag{3.7}$$

Equation (3.7) is satisfied only, if the following system of algebraic equations holds:

$$\lambda \neq 0,\ \beta - 1 \neq 0,\ \beta - 2 = 2\beta,$$

$$\left(\frac{4}{3c} - 3c\right) - \frac{1}{3}c\mu^2\beta^2 = 0, \quad \frac{2}{3c}\lambda + \frac{1}{3}c\mu^2\beta(\beta - 1) = 0. \tag{3.8}$$

Solving the system (3.2) give

$$\beta = -2,\ \mu^2 = \frac{1}{4c^2}(4 - 9c^2),\ \lambda = -\frac{3}{4}(4 - 9c^2). \tag{3.9}$$

The results (3.9) can be easily obtained, if we also use the sine method (2.4). Combining (3.9) with (2.3) and (2.4), for $4 - 9c^2 > 0$, the following periodic wave solutions:

$$\begin{cases} \psi_1 = -\frac{3}{4}(4 - 9c^2)\sec^2 \frac{1}{2c}\sqrt{4 - 9c^2}\,\xi, \\ \phi_1 = \frac{2}{3c}\left[1 - c^2 - \frac{3}{4}(4 - 9c^2)\sec^2 \frac{1}{2c}\sqrt{4 - 9c^2}\,\xi\right], \ |\,\xi\,| < \frac{|c|\pi}{\sqrt{4-9c^2}}, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{3.10}$$

and

$$\begin{cases} \psi_2 = -\frac{3}{4}(4 - 9c^2)\csc^2 \frac{1}{2c}\sqrt{4 - 9c^2}\,\xi, \\ \phi_2 = \frac{2}{3c}\left[1 - c^2 - \frac{3}{4}(4 - 9c^2)\csc^2 \frac{1}{2c}\sqrt{4 - 9c^2}\,\xi\right], \ |\,\xi\,| < \frac{|c|\pi}{\sqrt{4-9c^2}}, \\ 0, \qquad \text{otherwise}. \end{cases} \tag{3.11}$$

However, for $4 - 9c^2 < 0$, we obtain the following solitary wave solutions:

$$\begin{cases} \psi_3 = -\frac{3}{4}(4 - 9c^2)\operatorname{sech}^2 \frac{1}{2c}\sqrt{9c^2 - 4}\,\xi, \\ \phi_3 = \frac{2}{3c}\left[1 - c^2 - \frac{3}{4}(9c^2 - 4)\operatorname{sech}^2 \frac{1}{2c}\sqrt{4 - 9c^2}\,\xi\right], \end{cases} \tag{3.12}$$

and

$$\begin{cases} \psi_4 = -\frac{3}{4}(4 - 9c^2)\operatorname{csch}^2 \frac{1}{2c}\sqrt{9c^2 - 4}\,\xi, \\ \phi_4 = \frac{2}{3c}\left[1 - c^2 - \frac{3}{4}(9c^2 - 4)\operatorname{csch}^2 \frac{1}{2c}\sqrt{4 - 9c^2}\,\xi\right]. \end{cases} \tag{3.13}$$

4. Using the Extended tanh Method

Balancing ψ'' and ψ^2 in Equation (3.6) gives $M = 2$. The extended tanh method allows us to use the substitution

$$\phi(\xi) = S(Y) = A_0 + A_1 Y + A_2 Y^2 + B_1 Y^{-1} + B_2 Y^{-2}. \tag{4.1}$$

Substituting (4.1) into (3.6), collecting the coefficients of each power of Y, and solve the resulting system of algebraic equations to find the sets of solutions:

Case 1: $A_1 = B_1 = B_2 = 0,\ A_0 = -A_2 = -3 + \frac{27}{4}c^2,\ \mu^2 = \frac{9c^2 - 4}{4c^2},\ c^2 = c^2.$

Case 2: $A_1 = B_1 = B_2 = 0,\ A_0 = -\frac{1}{3}A_2 = 1 - \frac{9}{4}c^2,\ \mu^2 = \frac{-9c^2 + 4}{4c^2},\ c^2 = c^2.$

Case 3: $A_1 = B_1 = A_2 = 0,\ A_0 = -B_2 = -3 + \frac{27}{4}c^2,\ \mu^2 = \frac{9c^2 - 4}{4c^2},\ c^2 = c^2.$

Case 4: $A_1 = B_1 = A_2 = 0,\ A_0 = -\frac{1}{3}B_2 = 1 - \frac{9}{4}c^2,\ \mu^2 = \frac{-9c^2 + 4}{4c^2},\ c^2 = c^2.$

Case 5: $A_1 = B_1 = 0,\ A_2 = B_2 = -\frac{1}{2}A_0 = \frac{3}{4} - \frac{27}{16}c^2,\ \mu^2 = \frac{9c^2 - 4}{16c^2},\ c^2 = c^2.$

Case 6: $A_1 = B_1 = 0,\ A_2 = B_2 = \frac{3}{2}A_0 = -\frac{3}{4} + \frac{27}{16}c^2,\ \mu^2 = \frac{-9c^2 + 4}{16c^2},\ c^2 = c^2.$

We obtain the following solitary wave solutions:

$$\begin{cases} \psi_5 = \left(-3 + \frac{27}{4}c^2\right)\left(1 - \tanh^2 \frac{1}{2c}\sqrt{9c^2 - 4}\xi\right), \\ \phi_5 = \frac{2}{3c}\left[1 - c^2 + \left(-3 + \frac{27}{4}c^2\right)\left(1 - \tanh^2 \frac{1}{2c}\sqrt{9c^2 - 4}\xi\right)\right],\ 9c^2 - 4 > 0, \end{cases} \tag{4.2}$$

$$\begin{cases} \psi_6 = \left(1 - \frac{9}{4}c^2\right)\left(1 - 3\tanh^2 \frac{1}{2c}\sqrt{-9c^2 + 4}\xi\right), \\ \phi_6 = \frac{2}{3c}\left[1 - c^2 + \left(1 - \frac{9}{4}c^2\right)\left(1 - 3\tanh^2 \frac{1}{2c}\sqrt{-9c^2 + 4}\xi\right)\right],\ -9c^2 + 4 > 0, \end{cases} \tag{4.3}$$

$$\begin{cases} \psi_7 = \left(-3 + \frac{27}{4}c^2\right)\left(1 - \coth^2 \frac{1}{2c}\sqrt{9c^2 - 4}\xi\right), \\ \phi_7 = \frac{2}{3c}\left[1 - c^2 + \left(-3 + \frac{27}{4}c^2\right)\left(1 - \coth^2 \frac{1}{2c}\sqrt{9c^2 - 4}\xi\right)\right],\ 9c^2 - 4 > 0, \end{cases} \tag{4.4}$$

$$\begin{cases} \psi_8 = \left(1 - \frac{9}{4}c^2\right)\left(1 - 3\coth^2 \frac{1}{2c}\sqrt{-9c^2 + 4}\xi\right), \\ \phi_8 = \frac{2}{3c}\left[1 - c^2 + \left(1 - \frac{9}{4}c^2\right)\left(1 - 3\coth^2 \frac{1}{2c}\sqrt{-9c^2 + 4}\xi\right)\right], \ -9c^2 + 4 > 0, \end{cases} \tag{4.5}$$

$$\begin{cases} \psi_9 = \left(-\frac{3}{2} + \frac{27}{8}c^2\right)\left(1 - \frac{1}{2}\tanh^2 \frac{1}{4c}\sqrt{9c^2 - 4}\xi - \frac{1}{2}\coth^2 \frac{1}{4c}\sqrt{9c^2 - 4}\xi\right), \\ \phi_9 = \frac{2}{3c}\left[1 - c^2 + \left(-\frac{3}{2} + \frac{27}{8}c^2\right)\left(1 - \frac{1}{2}\tanh^2 \frac{1}{4c}\sqrt{9c^2 - 4}\xi - \frac{1}{2}\coth^2 \frac{1}{4c}\sqrt{9c^2 - 4}\xi\right)\right], \\ \quad 9c^2 - 4 > 0, \end{cases} \tag{4.6}$$

$$\begin{cases} \psi_{10} = \left(-\frac{1}{2} + \frac{9}{8}c^2\right)\left(1 + \frac{3}{2}\tanh^2 \frac{1}{4c}\sqrt{4 - 9c^2}\xi + \frac{3}{2}\coth^2 \frac{1}{4c}\sqrt{4 - 9c^2}\xi\right), \\ \phi_{10} = \frac{2}{3c}\left[1 - c^2 + \left(-\frac{1}{2} + \frac{9}{8}c^2\right)\left(1 + \frac{3}{2}\tanh^2 \frac{1}{4c}\sqrt{4 - 9c^2}\xi + \frac{3}{2}\coth^2 \frac{1}{4c}\sqrt{4 - 9c^2}\xi\right)\right], \\ \quad 4 - 9c^2 > 0. \end{cases} \tag{4.7}$$

Moreover, we obtain the following periodic wave solutions:

$$\begin{cases} \psi_{11} = \left(-3 + \frac{27}{4}c^2\right)\sec^2 \frac{1}{2c}\sqrt{4 - 9c^2}\xi, \\ \phi_{11} = \frac{2}{3c}\left[1 - c^2 + \left(-3 + \frac{27}{4}c^2\right)\sec^2 \frac{1}{2c}\sqrt{4 - 9c^2}\xi\right], \ 4 - 9c^2 > 0, \end{cases} \tag{4.8}$$

$$\begin{cases} \psi_{12} = \left(1 - \frac{9}{4}c^2\right)\left(1 + 3\tan^2 \frac{1}{2c}\sqrt{9c^2 - 4}\xi\right), \\ \phi_{12} = \frac{2}{3c}\left[1 - c^2 + \left(1 - \frac{9}{4}c^2\right)\left(1 + 3\tan^2 \frac{1}{2c}\sqrt{9c^2 - 4}\xi\right)\right], \ 9c^2 - 4 > 0, \end{cases} \tag{4.9}$$

$$\begin{cases} \psi_{13} = \left(-3 + \frac{27}{4}c^2\right)\csc^2 \frac{1}{2c}\sqrt{4 - 9c^2}\xi, \\ \phi_{13} = \frac{2}{3c}\left[1 - c^2 + \left(-3 + \frac{27}{4}c^2\right)\csc^2 \frac{1}{2c}\sqrt{4 - 9c^2}\xi\right], \ 4 - 9c^2 > 0, \end{cases} \tag{4.10}$$

$$\begin{cases} \psi_{14} = \left(1 - \frac{9}{4}c^2\right)\left(1 + 3\cot^2 \frac{1}{2c}\sqrt{9c^2 - 4}\xi\right), \\ \phi_{14} = \frac{2}{3c}\left[1 - c^2 + \left(1 - \frac{9}{4}c^2\right)\left(1 + 3\cot^2 \frac{1}{2c}\sqrt{9c^2 - 4}\xi\right)\right], \ 9c^2 - 4 > 0, \end{cases} \tag{4.11}$$

$$\begin{cases} \psi_{15} = \frac{1}{4}\left(-3 + \frac{27}{4}c^2\right)\left(\sec^2 \frac{1}{4c}\sqrt{4-9c^2}\xi + \csc^2 \frac{1}{4c}\sqrt{4-9c^2}\xi\right), \\ \phi_{15} = \frac{2}{3c}\left[1 - c^2 + \frac{1}{4}\left(-3 + \frac{27}{4}c^2\right)\left(\sec^2 \frac{1}{4c}\sqrt{4-9c^2}\xi + \csc^2 \frac{1}{4c}\sqrt{4-9c^2}\xi\right)\right], \\ \quad 4 - 9c^2 > 0, \end{cases} \tag{4.12}$$

$$\begin{cases} \psi_{16} = \left(-\frac{1}{2} + \frac{9}{8}c^2\right)\left(1 - \frac{3}{2}\tan^2 \frac{1}{4c}\sqrt{9c^2-4}\xi - \frac{3}{2}\cot^2 \frac{1}{4c}\sqrt{9c^2-4}\xi\right), \\ \phi_{16} = \frac{2}{3c}\left[1 - c^2 + \left(-\frac{1}{2} + \frac{9}{8}c^2\right)\left(1 - \frac{3}{2}\tan^2 \frac{1}{4c}\sqrt{9c^2-4}\xi - \frac{3}{2}\cot^2 \frac{1}{4c}\sqrt{9c^2-4}\xi\right)\right], \\ \quad 9c^2 - 4 > 0. \end{cases} \tag{4.13}$$

5. Discussion

The sine-cosine method and the extended tanh method were used to investigate the generalized Bona-Smith equation. The study revealed solitary wave solutions and periodic wave solutions for all examined variants. Moreover, the obtained results in this work clearly demonstrate the reliability of the methods that were used.

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