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BOUNDARY VALUE PROBLEM FOR A CLASS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH *p*-LAPLACIAN OPERATOR

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Abstract

This paper deals with the existence of solution for boundary value problem of fractional differential equation with p-Laplacian operator

$$\begin{cases} D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}u(t))) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u(1) = u(\xi) = 0,\ D_{0+}^{\alpha}u(t)|_{t=0} = 0, \end{cases}$$

where $0 < \gamma < 1$, $2 < \alpha < 3$, D_{0+}^{α} is the standard Riemann-Liouville derivative, and $f : [0, 1] \times R \to R$ is given function, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$. We obtain the existence of solution by means of Schauder fixed-point theorem and an example is given to illustrate the effectiveness of our work.

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1. Introduction

Fractional differential equations have been found to be effective to describe many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic and so on (see, for example, [4-8]). Recently, there are a large number of papers dealing with the boundary value problems of nonlinear fractional differential equations; see [1-3, 9] and the references therein. However, not much has been done for boundary value problems of fractional differential equations with p-Laplacian operator [10, 11]. In many literatures, the order of the differential operator D_{0+}^{α} is $0 < \alpha < 2,$ if $\alpha > 2,$ the Green's function for boundary value problem of fractional differential equation is more complex and the properties of the Green's function are relatively weak. The discusses of the existence for the solution is more difficult and complicated. The aim of this paper is to discuss the existence of solution of the following boundary value problem of fractional differential equation with p-Laplacian operator

$$\begin{cases} D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}u(t))) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u(1) = u(\xi) = 0, \ D_{0+}^{\alpha}u(t)|_{t=0} = 0, \end{cases} \tag{1.1}$$

where $0 < \gamma < 1, 2 < \alpha < 3, D_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative, and $f : [0, 1] \times R \to R$ is given function, $\phi_p(s) = |s|^{p-2}s, \ p > 1,$ $(\phi_p)^{-1} = \phi_q, \frac{1}{p} + \frac{1}{q} = 1.$

The organization of this paper is as follows: In Section 2, we introduce some lemmas and definitions, which will be used later. In Section 3, existence of solution to the boundary value problem (1.1) will be discussed.

2. Basic Definitions and Preliminaries

In this section, we give some basic definitions and lemmas, which are used further in the paper.

Definition 2.1 [8]. The fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to R$ is given by

$$I^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

provided that the integral exists.

Definition 2.2 [8]. The standard Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \to R$ is given by

$$D^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} y(s) ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integral part of number α, provided that the right converges.

Lemma 2.1 [1]. *Assume that* $u \in C(0, 1) \cap L(0, 1)$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0, 1) \cap L(0, 1)$. *Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N},$$

where $c_i (i = 1, 2, \cdots, N)$ *are real numbers and N is the smallest integer greater than or equal to* α.

Lemma 2.2. *Let* $y \in C[0, 1]$ *and* $2 < \alpha < 3$. *Then* $u(t)$ *is a solution of the following equations*:

$$D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = 0,\ u(1) = u(\xi) = 0, \tag{2.2}$$

if and only if $u(t)$ *satisfies the integral equation*

$$u(t) = \int_0^1 G(t, s) y(s) ds, \tag{2.3}$$

where

$$G(t, s) = \begin{cases} \dfrac{[\xi^{\alpha-2}t^{\alpha-1} - \xi^{\alpha-1}t^{\alpha-2}](1-s)^{\alpha-1} - [\xi^{\alpha-2} - \xi^{\alpha-1}](t-s)^{\alpha-1} + [t^{\alpha-2} - t^{\alpha-1}](\xi - s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})}, \\ \qquad\qquad 0 \leqslant s \leqslant t \leqslant 1,\ s \leqslant \xi, \\ \dfrac{[\xi^{\alpha-2}t^{\alpha-1} - \xi^{\alpha-1}t^{\alpha-2}](1-s)^{\alpha-1} + [t^{\alpha-2} - t^{\alpha-1}](\xi - s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})}, & 0 < t \leqslant s \leqslant \xi \leqslant 1, \\ \dfrac{[\xi^{\alpha-2}t^{\alpha-1} - \xi^{\alpha-1}t^{\alpha-2}](1-s)^{\alpha-1} - [\xi^{\alpha-2} - \xi^{\alpha-1}](t-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})}, & 0 \leqslant \xi \leqslant s \leqslant t \leqslant 1, \\ \dfrac{[\xi^{\alpha-2}t^{\alpha-1} - \xi^{\alpha-1}t^{\alpha-2}](1-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})}, & 0 \leqslant t \leqslant s \leqslant 1,\ \xi \leqslant s. \end{cases}$$

Proof. Suppose that $u(t)$ is a solution of BVP (2.1) ~ (2.2), by Lemma 2.1, we have

$$u(t) = -I_{0+}^{\alpha} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3},$$

$$= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}, \quad c_1,\ c_2,\ c_3 \in R.$$

By (2.2), there are

$$C_1 = \int_0^1 \frac{(\xi^{\alpha-2} - \xi^{\alpha-1})(1-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})} y(s)ds + \int_0^1 \frac{\xi^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})} y(s)ds$$

$$- \int_0^{\xi} \frac{(\xi - s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})} y(s)ds,$$

$$C_2 = -\int_0^1 \frac{\xi^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})} y(s)ds + \int_0^{\xi} \frac{(\xi - s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})} y(s)ds,$$

$$C_3 = 0.$$

Therefore, we have

$$u(t) = \int_0^t \frac{-(\xi^{\alpha-2} - \xi^{\alpha-1})(t-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})} y(s)ds$$

$$+ \int_0^1 \frac{[\xi^{\alpha-2}t^{\alpha-1} - \xi^{\alpha-1}t^{\alpha-2}](1-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})} y(s)ds$$

$$+\int_0^{\xi} \frac{[t^{\alpha-2} - t^{\alpha-1}](\xi - s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})} y(s)ds,$$

$$= \int_0^1 G(t,\ s)y(s)ds. \tag{2.4}$$

Conversely, if $u(t)$ is a solution of integral equation (2.3), from (2.4) and Remark 2.1 in [1] ($D^{\alpha}t^{\alpha-m} = 0,\ m = 1, 2, \cdots, N,$ where N is the smallest integer greater than or equal to α), we obtain

$$D_{0+}^{\alpha}u(t) = D_{0+}^{\alpha}[\int_0^t \frac{-(\xi^{\alpha-2} - \xi^{\alpha-1})(t - s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})} y(s)ds]$$

$$+[\int_0^1 \frac{\xi^{\alpha-2}(1-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})} y(s)ds]D_{0+}^{\alpha}t^{\alpha-1}$$

$$-[\int_0^1 \frac{\xi^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})} y(s)ds]D_{0+}^{\alpha}t^{\alpha-2}$$

$$+[\int_0^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})} y(s)ds]D_{0+}^{\alpha}t^{\alpha-2}$$

$$-[\int_0^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})} y(s)ds]D_{0+}^{\alpha}t^{\alpha-1},$$

$$= -D_{0+}^{\alpha}I^{\alpha}y(t) = -y(t).$$

That is, $D_{0+}^{\alpha}u(t) + y(t) = 0.$ A simple computation shows $u(0) = u(1) = u(\xi) = 0.$ The proof is completed.

3. Main Result

In this section, we shall argument the existence of solution of boundary value problem (1.1).

Let $E = C[0, 1]$ be endowed with the norm $\| u \|_E = \max_{0 \leqslant t \leqslant 1} |u(t)|$, then $(E, \| u \|_E)$ is a Banach space.

Lemma 3.1. *Suppose that* $f : [0, 1] \times R \to R$ *is continuous, then BVP* (1.1) *is equivalent to the following integral equation*:

$$u(t) = \int_0^1 G(t, s)\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau))d\tau\right)ds. \tag{3.1}$$

Proof. Using Lemma 2.1 and note that $D_{0+}^{\alpha}u(t)|_{t=0} = 0$, we obtain

$$\begin{aligned}\phi_p(D_{0+}^{\alpha}u(t)) &= -I_{0+}^{\gamma}f(t, u(t)) + C_0 t^{\gamma-1}\\ &= -\frac{1}{\Gamma(\gamma)}\int_0^t (t-\tau)^{\gamma-1} f(\tau, u(\tau))d\tau.\end{aligned}$$

Then,

$$D_{0+}^{\alpha}u(t) = -\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^t (t-\tau)^{\gamma-1} f(\tau, u(\tau))d\tau\right),$$

therefore, BVP (1.1) is equivalent to the following problem:

$$\begin{cases} D_{0+}^{\alpha}u(t) + \phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^t (t-\tau)^{\gamma-1} f(\tau, u(\tau))d\tau\right) = 0,\\ 0 < t < 1,\ 2 < \alpha < 3,\\ u(0) = u(1) = u(\xi) = 0. \end{cases}$$

By Lemma 2.2, BVP (1.1) is equivalent to the integral equation (3.1).

Let $T : E \to E$ be the operator defined as

$$Tu(t) = \int_0^1 G(t, s)\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau))d\tau\right)ds. \tag{3.2}$$

Then by Lemma 3.1, the fixed point of operator T coincides with the solution of boundary value problem (1.1).

Theorem 3.1. *Let* $f : [0, 1] \times R \to R$ *is continuous and there exist two non-negative constants* $C > 0$, $\beta \neq 1$ *such that*

$$|f(t, u)| \leqslant \phi_p(C|u(t)|^{\beta}).$$

Then problem (1.1) has at least one solution.

Proof. Let $A = \dfrac{C(2 + \frac{\xi^2}{1-\xi})}{[\Gamma(\gamma+1)]^{q-1}\Gamma(\alpha+1)}$, $P = \{u | u \in E, \| u \| \leqslant r, t \in [0, 1]\}$. Clearly, P is the ball in the Banach space E.

Firstly, we prove that $T(P) \subseteq P$. For convenience, the proof is divided into two cases:

Case 1. If $\beta < 1$, choose $r \geqslant A^{\frac{1}{1-\beta}}$. For any $u(t) \in P$, we obtain

$$\begin{aligned}
&|\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau))d\tau\right)| \\
&\leqslant \phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} \phi_p(C|u(t)|^{\beta})d\tau\right) \\
&\leqslant \phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} \phi_p(Cr^{\beta})d\tau\right) \\
&= Cr^{\beta}\phi_q\left(\frac{s^{\gamma}}{\Gamma(\gamma+1)}\right) \leqslant \frac{Cr^{\beta}}{[\Gamma(\gamma+1)]^{q-1}}.
\end{aligned}$$

By (2.4) and (3.2), we have

$$| Tu(t) | = |\int_0^1 G(t, s)\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau))d\tau\right)ds|$$

$$\leqslant \left|\int_0^t \frac{-(\xi^{\alpha-2}-\xi^{\alpha-1})(t-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2}-\xi^{\alpha-1})}\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1}f(\tau, u(\tau))d\tau\right)ds\right|$$

$$+\left|\int_0^1 \frac{[\xi^{\alpha-2}t^{\alpha-1}-\xi^{\alpha-1}t^{\alpha-2}](1-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2}-\xi^{\alpha-1})}\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1}f(\tau, u(\tau))d\tau\right)ds\right|$$

$$+\left|\int_0^\xi \frac{[t^{\alpha-2}-t^{\alpha-1}](\xi-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2}-\xi^{\alpha-1})}\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1}f(\tau, u(\tau))d\tau\right)ds\right|$$

$$\leqslant \frac{Cr^\beta}{[\Gamma(\gamma+1)]^{q-1}}\left\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right.$$

$$\left.+\int_0^\xi \frac{(t^{\alpha-2}-t^{\alpha-1})(\xi-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2}-\xi^{\alpha-1})}ds\right\}$$

$$\leqslant \frac{Cr^\beta}{[\Gamma(\gamma+1)]^{q-1}}\left\{\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{\alpha-1}}{\Gamma(\alpha+1)}+\frac{(t^{\alpha-2}-t^{\alpha-1})\xi^\alpha}{\Gamma(\alpha+1)(\xi^{\alpha-2}-\xi^{\alpha-1})}\right\}$$

$$\leqslant \frac{Cr^\beta(2+\frac{\xi^2}{1-\xi})}{[\Gamma(\gamma+1)]^{q-1}\Gamma(\alpha+1)}\leqslant r.$$

Therefore, $\| Tu \| \leqslant r.$

Case 2. If $\beta > 1$, choose $0 < r \leqslant (\frac{1}{A})^{\frac{1}{\beta-1}}$. For any $u(t) \in P$, repeating arguments similar to the above, we also obtain $\| Tu \| \leqslant r$. Consequently, we have $T(P) \subseteq P$.

Now, we prove that T is completely continuous. In view of the continuity of functions $f(t, u)$ and $G(t, s)$, it is easy to see that the operator T is continuous.

Set $L = \max_{t\in[0,1],\, u\in P} | f(t, u)|$. For any $u(t) \in P$ and let $t_2, t_1 \in [0, 1]$ be such that $t_2 < t_1$, then we have

$$|Tu(t_2) - Tu(t_1)|$$

$$\leqslant \int_0^1 |G(t_2, s) - G(t_1, s)| \phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau))d\tau\right)ds$$

$$\leqslant \left(\frac{L}{\Gamma(\gamma+1)}\right)^{q-1}\int_0^1 |G(t_2, s) - G(t_1, s)|ds$$

$$\leqslant \left(\frac{L}{\Gamma(\gamma+1)}\right)^{q-1}\left\{\int_0^{t_2}\frac{[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]}{\Gamma(\alpha)}ds + \int_{t_2}^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right.$$

$$+ \int_0^1 \frac{\xi^{\alpha-2}(t_1^{\alpha-1} - t_2^{\alpha-1})(1-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})}ds + \int_0^1 \frac{\xi^{\alpha-1}(t_1^{\alpha-2} - t_2^{\alpha-2})(1-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})}ds$$

$$\left.+ \int_0^\xi \frac{(t_2^{\alpha-2} - t_1^{\alpha-2})(\xi-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})}ds + \int_0^\xi \frac{(t_1^{\alpha-1} - t_2^{\alpha-1})(\xi-s)^{\alpha-1}}{\Gamma(\alpha)(\xi^{\alpha-2} - \xi^{\alpha-1})}ds\right\}$$

$$\leqslant \left(\frac{L}{\Gamma(\gamma+1)}\right)^{q-1}\frac{1}{\Gamma(\alpha+1)}$$

$$\times\left\{(t_1^\alpha - t_2^\alpha) + \frac{(\xi^{\alpha-1} - \xi^\alpha)(t_1^{\alpha-2} - t_2^{\alpha-2})}{(\xi^{\alpha-2} - \xi^{\alpha-1})} + \frac{(\xi^{\alpha-2} - \xi^\alpha)(t_1^{\alpha-1} - t_2^{\alpha-1})}{(\xi^{\alpha-2} - \xi^{\alpha-1})}\right\}.$$

Clearly, using the fact that the functions t^α, $t^{\alpha-1}$, and $t^{\alpha-2}$ are uniformly continuous on the interval $[0, 1]$, we conclude that $T(P)$ is an equicontinuous set. Obviously, it is uniformly bounded since $T(P) \subseteq P$. Thus, T is completely continuous. The Schauder fixed-point theorem implies the existence of solution in P for boundary value problem (1.1). We complete the proof.

Remark 3.1. If $\beta = 1$ in Theorem 3.1, then boundary value problem (1.1) has a solution, if and only if $A \leqslant 1$.

Corollary 3.1. *Suppose that* $f(t, u)$ *is bounded and continuous on* $[0, 1] \times R$, *then there exists a solution for problem* (1.1).

Example. Consider the boundary value problem of nonlinear fractional differential equation with p-Laplacian operator

$$\begin{cases} D_{0+}^{\frac{1}{2}}(\phi_p(D_{0+}^{\frac{5}{2}}u(t))) + \dfrac{t}{1+t}|u(t)|^{\beta} = 0, \quad 0 < t < 1, \\ u(0) = u(1) = u(\dfrac{1}{2}) = 0, \; D_{0+}^{\alpha}u(0) = 0. \end{cases} \tag{3.3}$$

Let $p = q = 2$, we have $|f(t, u)| \leqslant \phi_2(\frac{1}{2}|u(t)|^{\beta}) = \frac{1}{2}|u(t)|^{\beta}$. By simple computation, we obtain that $A = \frac{4}{3\pi}$. By Theorem 3.1, the existence of solution is obvious for $\beta > 1$ or $\beta < 1$. If $\beta = 1$, Remark 3.1 implies that system (3.3) has a solution.

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