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# THE STOCHASTIC INTEGRAL WITH RESPECT TO THE SUB-FRACTIONAL BROWNIAN MOTION WITH $H > \frac{1}{2}$

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**Abstract**

We develop a stochastic calculus for the sub-fractional Brownian motion with index $H > \frac{1}{2}$ by using the techniques of the Malliavin calculus. We establish estimates in $L^p$, maximal inequalities for the divergence integral with respect to sub-fractional Brownian motion. We also study the $\frac{1}{H}$ variation of the divergence integral and generalize the result of sub-fractional Brownian motion for this divergence integral.

## 1. Introduction

Recently, the long-range dependence property has become an important aspect of stochastic models in various scientific areas including hydrology, telecommunication, turbulence, image processing, and finance. The best known and most widely used process that exhibits the long-range dependence property is *fractional Brownian motion* (fBm in short). The fBm is a suitable generalization of the standard Brownian motion, but exhibits long-range dependence, self-similarity, and it has stationary increments. Some surveys and complete literatures could be found in Biagini et al. [3], Hu [9], Mishura [10], and Nualart [11]. On the other hand, many authors have proposed to use more general self-similar Gaussian processes and random fields as stochastic models. Such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. Therefore, some generalizations of the fBm has been introduced. However, contrast to the extensive studies on fBm, there has been little systematic investigation on other self-similar Gaussian processes. The main reason for this is the complexity of dependence structures for self-similar Gaussian processes, which do not have stationary increments.

As an extension of Brownian motion, recently, Bojdecki et al. [4] introduced and studied a rather special class of self-similar Gaussian processes, which preserves many properties of the fBm. This process arises from occupation time fluctuations of branching particle systems

with Poisson initial condition. This process is called the *sub-fractional Brownian motion*. The so-called sub-fractional Brownian motion (sub-fBm in short) with index $H \in (0, 1)$ is a mean zero Gaussian process $S^H = \{S_t^H, t \geq 0\}$ with $S_0^H = 0$, and the covariance

$$C_H(t, s) \equiv E[S_t^H S_s^H] = s^{2H} + t^{2H} - \frac{1}{2}[(s+t)^{2H} + |t-s|^{2H}], \qquad (1.1)$$

for all $s, t \geq 0$. For $H = \frac{1}{2}$, $S^H$ coincides with the Brownian motion $B$. $S^H$ is neither a semi-martingale nor a Markov process unless $H = 1/2$, so the Itô approach to the construction of a stochastic integral with respect to sub-fBm is not valid. As a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to $S^H$ (see, for example, Alós et al. [1]). The sub-fBm has properties analogous to those of fBm (self-similarity, long-range dependence), and satisfies the following estimates:

$$[(2 - 2^{2H-1}) \wedge 1](t-s)^{2H} \leq E\left[(S_t^H - S_s^H)^2\right] \leq [(2 - 2^{2H-1}) \vee 1](t-s)^{2H}. \qquad (1.2)$$

Thus, Kolmogorov's continuity criterion implies that sub-fBm is Hölder continuous of order $\gamma$ for any $\gamma < H$. But its increments are not stationary. Therefore, it seems interesting to study sub-fBm.

Fix $t > 0$ and set $t_k^n := \frac{kT}{n}$, where $n$ is a positive integer and $k = 0, 1, \ldots, n$. Then, we have the following result for the $p$ variation of $S^H$ (see [14]):

$$\sum_{k=0}^{n-1} |S_{t_{k+1}^n}^H - S_{t_k^n}^H|^p \xrightarrow[n\to\infty]{L^2} 0, \text{ if } p > \frac{1}{H}, \qquad (1.3)$$

$$\sum_{k=0}^{n-1} |S_{t_{k+1}^n}^H - S_{t_k^n}^H|^p \xrightarrow[n\to\infty]{L^2} \rho_{\frac{1}{H}} T, \text{ if } p = \frac{1}{H}, \qquad (1.4)$$

$$\sum_{k=0}^{n-1} |S^H_{t^n_{k+1}} - S^H_{t^n_k}|^p \xrightarrow[n\to\infty]{P} \infty, \text{ if } p < \frac{1}{H}, \tag{1.5}$$

where $\rho_{\frac{1}{H}} = E(|\xi|^{\frac{1}{H}})$, $\xi \sim N(0, 1)$.

More works for sub-fBm can be found in Bojdecki et al. [5-7], Shen and Yan [12], Tudor [14-17], and Yan and Shen [18], [19].

The aim of this paper is to develop a stochastic calculus with respect to the sub-fBm $S^H$ with index $H > \frac{1}{2}$ by using the techniques of the Malliavin calculus. Unlike some previous works (see, for instance, [1]), we will not use the integral representation of $S^H$ as a stochastic integral with respect to a Wiener process. Instead of this, we will rely on the intrinsic Malliavin calculus with respect to $S^H$. We establish estimates in $L^p$, maximal inequalities for the indefinite integral with respect to sub-fBm for $H > \frac{1}{2}$, where this indefinite integral is defined as the divergence integral in the framework of the Malliavin calculus and study the $\frac{1}{H}$ variation of the divergence integral, which generalize the result (1.4) for this divergence integral.

This note is organized as follows. In Section 2, we review the basic facts on Malliavin calculus that will be used in order to define the stochastic integral. In Section 3, we show the existence of the symmetric stochastic integral. The symmetric integral as the limit of symmetric Riemann sums as those have been done in Chapter 3 of Nualart [11] in the sense of the Malliavin calculus. We derive maximal $L^p$ inequalities for the divergence integral with respect to sub-fBm and prove the $\frac{1}{H}$ variation of the divergence integral $\int_0^{\cdot} u_s \delta S^H_s$ on a time interval $[0, T]$ is equal to $\rho_{\frac{1}{H}} \int_0^T |u_s|^{\frac{1}{H}} ds$, where $\rho_{\frac{1}{H}}$ is a constant depending on $H$.

## 2. Malliavin Calculus for Sub-fBm

Let $(S_t^H, t \in [0, T])$ be a sub-fBm with $\frac{1}{2} < H < 1$, defined on the complete probability space $(\Omega, \mathcal{F}, P)$. It is possible to construct a stochastic calculus of variations with respect to the Gaussian process $S^H$, which will be related to the Malliavin calculus. Some surveys and complete literatures could be found in Alòs et al. [1], Nualart [11]. We recall here the basic definitions and results of this calculus. The crucial ingredient is the canonical Hilbert space $\mathcal{H}$ (is also said to be *reproducing kernel Hilbert space*) associated to the sub-fBm, which is defined as the closure of the linear space $\mathcal{E}$ generated by the indicator functions $\{1_{[0,t]}, t \in [0, T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = C_H(t, s) \equiv t^{2H} + s^{2H} - \frac{1}{2}[(s+t)^{2H} + |t-s|^{2H}].$$

The mapping $1_{[0,s]} \to S_s^H$ can be extended to a linear isometry between $\mathcal{H}$ and the Gaussian space associated with $S^H$. We will denote the isometry by $\varphi \to S^H(\varphi)$. For $\frac{1}{2} < H < 1$, we denote by $\mathcal{S}$ the set of smooth functionals of the form

$$F = f(S^H(\varphi_1), \ldots, S^H(\varphi_n)),$$

where $f \in C_b^\infty(\mathbb{R}^n)$ and $\varphi_i \in \mathcal{H}$. The Malliavin derivative of a functional $F$ as above is given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(S^H(\varphi_1), \ldots, S^H(\varphi_n))\varphi_i,$$

and this operator can be extended to the closure $\mathbb{D}^{m,p}(m \geq 1)$ of $\mathcal{S}$ with respect to the norm

$$\| F \|_{m,p}^{p} \equiv E|F|^{p} + E\| DF \|_{\mathcal{H}}^{p} + \cdots + E\| D^{m}F \|_{\mathcal{H}^{\otimes m}}^{p},$$

where the $m$-th derivative $D^{m}$ is defined by iteration.

The divergence operator $\delta$ is the adjoint operator of $D$. Concretely, a random variable $u \in L^{2}(\Omega, \mathcal{H})$ belongs to the domain of the divergence operator $\delta$ (in symbol $\mathrm{Dom}(\delta)$), if

$$E|\langle DF, u\rangle_{\mathcal{H}}| \le c\| F \|_{L^{2}(\Omega)},$$

for every $F \in \mathcal{S}$. In this case, $\delta(u)$ is given by the duality relationship

$$E(F\delta(u)) = E\langle DF, u\rangle_{\mathcal{H}},$$

for any $F \in \mathbb{D}^{1,2}$, and we have the following integration by parts:

$$F\delta(u) = \delta(Fu) + \langle DF, u\rangle_{\mathcal{H}}, \tag{2.1}$$

for any $u \in \mathrm{Dom}(\delta)$, $F \in \mathbb{D}^{1,2}$ such that $Fu \in L^{2}(\Omega, \mathcal{H})$. It follows that

$$E[\delta(u)^{2}] = E\| u \|_{\mathcal{H}}^{2} + E\langle Du, (Du)^{*}\rangle_{\mathcal{H}\otimes\mathcal{H}},$$

where $(Du)^{*}$ is the adjoint of $Du$ in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$, and

$$\| u \|_{\mathcal{H}}^{2} = \int_{0}^{T}\int_{0}^{T} u_{s}u_{r}\phi_{H}(s, r)dsdr, \tag{2.2}$$

where

$$\phi_{H}(s, r) = \frac{\partial^{2}C_{H}}{\partial s\partial r}(s, r) = H(2H-1)\left(|s-r|^{2H-2} - (s+r)^{2H-2}\right) \ge 0,$$

and for $\varphi : [0, T]^{2} \to \mathbb{R}$, we have

$$\| \varphi \|_{\mathcal{H}\otimes\mathcal{H}}^{2} = \int_{[0,T]^{4}} \varphi(t, s)\varphi(t', s')\phi_{H}(t, t')\phi_{H}(s, s')dtdsdt'ds' < \infty.$$

We denote by $|\mathcal{H}|$ the subspace of $\mathcal{H}$, which is defined as the set of measurable function $f$ on $[0, T]$ with

$$\| f \|^2_{|\mathcal{H}|} := \int_0^T \int_0^T |f(s)||f(r)|\phi_H(s, r)dsdr < \infty. \tag{2.3}$$

We can show that the space $|\mathcal{H}| \subset \mathcal{H}$ is a Banach space for the norm $\| \cdot \|_{|\mathcal{H}|}$ and

$$E(\delta(u))^2 \leq E\| u \|^2_{|\mathcal{H}|} + E\| Du \|^2_{|\mathcal{H}|\otimes|\mathcal{H}|}, \tag{2.4}$$

where

$$\| \varphi \|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} := \int_{[0,T]^4} |\varphi(t, s)||\varphi(t', s')|\phi_H(t, t')\phi_H(s, s')dtdsdt'ds' < \infty.$$

We also will use the notation

$$\delta(u) = \int_0^T u_s dS_s^H,$$

to express the divergence integral of an adapted process $u$. For the divergence integral, we have the following convergence:

If $\{u_n\}$ is a sequence of elements in $\mathrm{Dom}(\delta)$ such that $u_n \to u$ in $L^2(\Omega; \mathcal{H})$, and $\delta(u_n) \to G$ in $L^2(\Omega)$, then we have $u \in \mathrm{Dom}(\delta)$ and $\delta(u) = G$.

We also have the following estimate:

**Lemma 2.1.** *Let* $H > \frac{1}{2}$ *and* $\psi \in L^{\frac{1}{H}}([0, T])$. *Then*

$$\| \psi \|_{|\mathcal{H}|} \leq C_H \| \psi \|_{L^{\frac{1}{H}}([0,T])}, \tag{2.5}$$

*for some constant* $C_H > 0$.

This estimate implies the inclusion $L^{\frac{1}{H}}([0, T]) \subset |\mathcal{H}|$. Indeed,

$$\| \varphi \|^2_{|\mathcal{H}|} = \alpha_H \int_0^T \int_0^T |\psi_s||\psi_t| [|s-t|^{2H-2} - (s+t)^{2H-2}] ds dt$$

$$\leq \alpha_H \int_0^T \int_0^T |\psi_s||\psi_t| |s-t|^{2H-2} ds dt$$

$$\leq \alpha_H \left( \int_0^T |\psi_r|^{\frac{1}{H}} dr \right)^H \left( \int_0^T \left( \int_0^T |\psi_u| |r-u|^{2H-2} du \right)^{\frac{1}{1-H}} dr \right)^{1-H}.$$

The second factor in the above expression, up to a multiplicative constant, it is equal to the $\frac{1}{1-H}$ norm of the left sided fractional integral $I_{0+}^{2H-1} |\psi|$. Finally, it is suffices to apply the Hardy-Littlewood inequality

$$\| I_{0+}^{\alpha} f \|_{L^q(0,\infty)} \leq C_{H,p} \| f \|_{L^p(0,\infty)},$$

where $0 < \alpha < 1$, $1 < p < q < \infty$ satisfy $\frac{1}{q} = \frac{1}{p} - \alpha$, with the particular values $\alpha = 2H - 1$, $q = \frac{1}{1-H}$, and $p = \frac{1}{H}$.

A slight extension of (2.5) implies that

$$\langle |\varphi|, |\psi| \rangle_{\mathcal{H}} \leq C_H^2 \| \varphi \|_{L^{\frac{1}{H}}([0,T])} \| \psi \|_{L^{\frac{1}{H}}([0,T])},$$

for all $\varphi$ and $\psi$ in $\mathcal{H}$. Hence, applying twice the inequality yields the following result: If $\psi$ belongs to $|\mathcal{H}| \otimes |\mathcal{H}|$, then

$$\| \psi \|_{|\mathcal{H}| \otimes |\mathcal{H}|} \leq C_H \| \psi \|_{L^{\frac{1}{H}}([0,T]^2)}. \tag{2.6}$$

For any $p > 1$, we denote by $\mathbb{D}^{1,p}(|\mathcal{H}|)$ denote the subspace of the Sobolev space $\mathbb{D}^{1,p}(\mathcal{H})$, whose elements $u$ are such that $u \in |\mathcal{H}|$ a.s., $Du \in |\mathcal{H}| \otimes |\mathcal{H}|$ a.s., and

$$E[\| u \|^p_{|\mathcal{H}|}] + E[\| Du \|^p_{|\mathcal{H}| \otimes |\mathcal{H}|}] < \infty.$$

We have the inclusion

$$\mathbb{D}^{1,2}(|\mathcal{H}|) \subset \mathbb{D}^{1,2}(\mathcal{H}) \subset \mathrm{Dom}(\delta).$$

We denote by $\mathbb{L}_H^{1,p}$, the set of processes $u$ in $\mathbb{D}^{1,p}(|\mathcal{H}|)$ such that

$$\| u \|^p_{\mathbb{L}_H^{1,p}} := E\| u \|^p_{L^{\frac{1}{H}}([0,T])} + E\| Du \|^p_{L^{\frac{1}{H}}([0,T]^2)} < \infty.$$

From (2.5) and (2.6), we obtain

$$\| u \|_{\mathbb{D}^{1,p}(|\mathcal{H}|)} \leq C_H \| u \|_{\mathbb{L}_H^{1,p}},$$

and, as a consequence, the space $\mathbb{L}_H^{1,2} \subset \mathrm{Dom}(\delta)$. By Meyer's inequalities (see Nualart [11], for example), if $p > 1$, a process $u \in \mathbb{D}^{1,p}(|\mathcal{H}|)$ belongs to the domain of the divergence in $L^p(\Omega)$, and we have

$$E|\delta(u)|^p \leq C_{H,p}(\| Eu \|^p_{|\mathcal{H}|} + E\| Du \|^p_{|\mathcal{H}|\otimes|\mathcal{H}|}).$$

Hence, if $u \in \mathbb{L}_H^{1,p}$, we have

$$E|\delta(u)|^p \leq C_{H,p}(\| Eu \|^p_{L^{\frac{1}{H}}([0,T])} + E\| Du \|^p_{L^{\frac{1}{H}}([0,T]^2)}) \leq C_{H,p} \| u \|^p_{\mathbb{L}_H^{1,p}}. \tag{2.7}$$

## 3. The Stochastic Integral with Respect to the Sub-fBm with $H > \frac{1}{2}$

In the case of an ordinary Brownian motion, the adapted processes in $L^2([0, T] \times \Omega)$ belong to the domain of the divergence operator, and on this set, the divergence operator coincides with the Itô stochastic integral. Actually, the divergence operator coincides with an extension of the Itô stochastic integral introduced by Skorohod [13]. We can ask in which sense, the divergence operator with respect to a sub-fBm $S^H$ can be interpreted as a stochastic integral. Note that the divergence operator provides an isometry between the Hilbert space $\mathcal{H}$ associated with the

sub-fBm $S^H$ and the Gaussian space $H_1(S^H)$, and gives rise to a notion of stochastic integral in the space of deterministic functions $|\mathcal{H}|$ included in $\mathcal{H}$.

Let us introduce the symmetric integral as the limit of symmetric Riemann sums as those have been done in Chapter 3 of Nualart [11] in the framework of the Malliavin calculus. Consider a measurable process $u = \{u_t, t \in [0, T]\}$ such that $\int_0^T |u_t| dt < \infty$ a.s.. Let us define the approximating sequences of processes

$$(\pi^n u)_t = \sum_{i=1}^{n} \Delta_n^{-1} \left( \int_{t_i}^{t_{i+1}} u_s ds \right) 1_{(t_i, t_{i+1}]}(t), \tag{3.1}$$

where $t_i = i\Delta$, $i = 1, \ldots, n$, and $\Delta_n = \frac{T}{n}$.

Let

$$S^n = \sum_{i=1}^{n} \Delta_n^{-1} \left( \int_{t_i}^{t_{i+1}} u_s ds \right) (S^H_{t_{i+1}} - S^H_{t_i}). \tag{3.2}$$

**Definition 3.1.** The symmetric integral of a process $u$ with integrable paths with respect to the sub-fBm is defined as the limit in probability of the sequence $S^n$ as $n \to \infty$, if it exists. We denote this limit by $\int_0^T u_t \circ dS_t^H$.

The following theorem gives sufficient conditions for the existence of the symmetric integral, and provides a representation of the divergence operator as a stochastic integral.

**Theorem 3.1.** *Let* $u = \{u_t, t \in [0, t]\}$ *be a stochastic process in the space* $\mathbb{D}^{1,2}(|\mathcal{H}|)$. *Suppose also that a.s.*

$$\int_0^T \int_0^T |D_s u_t| \left( |t - s|^{2H-2} - (t + s)^{2H-2} \right) ds dt < \infty. \tag{3.3}$$

*Then, the symmetric integral exists and we have*

$$\int_0^T u_t \circ dS_t^H = \delta(u) + \beta_H \int_0^T \int_0^T D_s u_t (|t-s|^{2H-2} - (t+s)^{2H-2}) ds dt, \quad (3.4)$$

*where* $\beta_H = H(2H-1)$.

**Proof.** The proof of the theorem will be decomposed into two steps:

**Step 1.** We claim that

$$\|\pi^n u\|_{|\mathcal{H}|}^2 \leq e_H \| u \|_{|\mathcal{H}|}^2, \quad (3.5)$$

for some positive constant $e_H$. In fact, we have that

$$\|\pi^n u\|_{|\mathcal{H}|}^2 = \beta_H \int_0^T \int_0^T |(\pi^n u)_s| |(\pi^n u)_t| (|t-s|^{2H-2} - (t+s)^{2H-2}) ds dt$$

$$\leq \beta_H \int_0^T \int_0^T |u_s| |u_t| f_n(s,t) ds dt,$$

where

$$f_n(s,t) = \Delta_n^{-2} \sum_{i,j=1}^n 1_{(t_i, t_{i+1}]}(s) 1_{(t_j, t_{j+1}]}(t) \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} (|\sigma-\theta|^{2H-2} - (\sigma+\theta)^{2H-2}) d\sigma d\theta.$$

Therefore, in order to show (3.5), it suffices to check that

$$f_n(s,t) \leq e_H (|t-s|^{2H-2} - (t+s)^{2H-2}).$$

Notice that

$$\beta_H \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} (|\sigma-\theta|^{2H-2} - (\sigma+\theta)^{2H-2}) d\sigma d\theta = E[(S_{t_{i+1}}^H - S_{t_i}^H)(S_{t_{j+1}}^H - S_{t_j}^H)].$$

Thus, for $s, t \in (t_i, t_{i+1}]$

$$f_n(s,t) \frac{1}{(|t-s|^{2H-2} - (t+s)^{2H-2})} \leq \Delta_n^{2H-2} \beta_H^{-1} \frac{1}{\Delta_n^{2H-2} - (2t_i)^{2H-2}}$$

$$\leq \frac{1}{1-2^{2H-2}} \beta_H^{-1},$$

where we use the estimates (1.2) and for $s \in (t_i, t_{i+1}]$, $t \in (t_j, t_{j+1}]$ with $i < j$, we get

$$f_n(s, t)\frac{1}{(|t-s|^{2H-2} - (t+s)^{2H-2})}$$

$$\leq \frac{1}{2\beta_H \Delta_n^2} \frac{1}{(|t-s|^{2H-2} - (t+s)^{2H-2})}$$

$$\times [(t_j - t_{i+1})^{2H} + (t_j - t_{i+1} + 2\Delta_n)^{2H} - 2(t_j - t_{i+1} + \Delta_n)^{2H}]$$

$$\leq \frac{1}{2\beta_H} \max_{k\geq 0}[k^{2H} + (k+2)^{2H} - 2(k+1)^{2H}] \frac{1}{(k+2)^{2H-2} - (2i)^{2H-2}}.$$

Hence (3.5) holds with

$$e_H = \frac{1}{2\beta_H} \max_{k\geq 0}\Big( \frac{2}{1 - 2^{2H-2}},$$

$$[k^{2H} + (k+2)^{2H} - 2(k+1)^{2H}] \frac{1}{(k+2)^{2H-2} - (2i)^{2H-2}} \Big).$$

We can find a sequence of step processes $u_k$ such that $\|u_k - u\|^2_{|\mathcal{H}|} \to 0$, $k \to \infty$. Then

$$\|\pi^n u - u\|_{|\mathcal{H}|} \leq \|\pi^n u - \pi^n u_k\|_{|\mathcal{H}|} + \|\pi^n u_k - u_k\|_{|\mathcal{H}|} + \|u_k - u\|_{|\mathcal{H}|}$$

$$\leq (\sqrt{e_H} + 1)\|u_k - u\|_{|\mathcal{H}|} + \|\pi^n u_k - u_k\|_{|\mathcal{H}|},$$

and letting first $n \to \infty$ and then $k \to \infty$, we have

$$\lim_{n\to\infty} \|\pi^n u - u\|_{|\mathcal{H}|} = 0,$$

a.s., and by dominated convergence theorem, we get

$$\lim_{n\to\infty} E[\|\pi^n u - u\|^2_{|\mathcal{H}|}] = 0.$$

In a similar way, we can show that

$$\|D\pi^n u\|^2_{\mathcal{H}|\otimes|\mathcal{H}|} \leq d_H \| Du \|^2_{\mathcal{H}|\otimes|\mathcal{H}|},$$

for some constant $d_H > 0$, and as a consequence,

$$\lim_{n\to\infty} E[\|D\pi^n u - Du\|^2_{\mathcal{H}|\otimes|\mathcal{H}|}] = 0.$$

Hence, $\pi^n u \to u$ in the norm of the space $\mathbb{D}^{1,2}(|\mathcal{H}|)$.

**Step 2.** By (2.1), we have

$$\sum_{i=1}^{n} \Delta_n^{-1} \left( \int_{t_i}^{t_{i+1}} u_s ds \right) (S^H_{t_{i+1}} - S^H_{t_i}) = \delta(\pi^n u) + A_n(u), \tag{3.6}$$

where

$$A_n u = \sum_{i=1}^{n} \Delta_n^{-1} \int_{t_i}^{t_{i+1}} \langle Du_s, 1_{[t_i, t_{i+1}]} \rangle_{\mathcal{H}} ds.$$

By Step 1, $\delta(\pi^n u)$ converge to $\delta(u)$ in $L^2(\Omega)$ as $n$ tends to infinity. In order to end the theorem, it suffices to prove that $A_n(u)$ converges almost surely to

$$\beta_H \int_0^T \int_0^T D_s u_t (|t-s|^{2H-2} - (t+s)^{2H-2}) ds dt.$$

In fact, we can rewrite

$$A_n(u) = \beta_H \int_0^T \int_0^T D_s u_t g_n(s, t) ds dt,$$

where

$$g_n(s, t) = \sum_{i=1}^{n} 1_{[t_i, t_{i+1}]}(t) \Delta_n^{-1} \int_{t_i}^{t_{i+1}} (|s-\sigma|^{2H-2} - (s+\sigma)^{2H-2}) d\sigma.$$

By dominated convergence theorem, it suffices to prove that

$$g_n(s, t) \leq f_H(|t-s|^{2H-2} - (t+s)^{2H-2}), \tag{3.7}$$

for some constant $f_H > 0$. If $s, t \in (t_i, t_{i+1}]$, then

$$g_n(s, t)\frac{1}{(|t-s|^{2H-2} - (t+s)^{2H-2})}$$

$$\leq \Delta_n^{-1}(2H-1)^{-1}[(t_{i+1}-s)^{2H-1} + (s-t_i)^{2H-1}]\frac{1}{\Delta_n^{2H-2} - (2t_i)^{2H-2}}$$

$$\leq \frac{2}{(2H-1)(1-2^{2H-2})}.$$

On the other hand, for $s \in (t_i, t_{i+1}]$, $t \in (t_j, t_{j+1}]$ with $i < j$, we get

$$g_n(s, t)\frac{1}{(|t-s|^{2H-2} - (t+s)^{2H-2})}$$

$$\leq \frac{1}{2H-1}\sup_{\substack{K\geq 0,\\ \lambda\in[0,1]}}[(k+1+\lambda)^{2H-1} - (k+\lambda)^{2H-1}]\frac{1}{(k+2)^{2H-2} - (2i)^{2H-2}}.$$

Hence, (3.7) holds with

$$f_H = \frac{1}{2H-1}\max\Big(\frac{2}{1-2^{2H-2}}, \sup_{\substack{K\geq 0,\\ \lambda\in[0,1]}}[(k+1+\lambda)^{2H-1} - (k+\lambda)^{2H-1}] \times \frac{1}{(k+2)^{2H-2} - (2i)^{2H-2}}\Big).$$

This completes the proof of the theorem. □

**Remark 3.1.** Suppose that $u = \{u_t, t \in [0, t]\}$ be a stochastic process in the space $\mathbb{D}^{1,2}(|\mathcal{H}|)$ such that condition (3.3) holds. Then, for any $t \in [0, T]$, the process $u1_{[0,t]}$ also belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$ and satisfies (3.3). Therefore, by Theorem 3.1, we can define the indefinite integral $\int_0^t u_s \circ dS_s^H = \int_0^T u_s 1_{[0,t]}(s) \circ dS_s^H$ and the following equality holds:

$$\int_0^t u_s \circ dS_s^H = \delta(u1_{[0,t]}) + \beta_H \int_0^t \int_0^T D_r u_s (|s-r|^{2H-2} - (s+r)^{2H-2}) dr ds.$$

The second summand in this expression is a process with absolutely continuous paths that can be studied by means of usual methods. Therefore, in order to deduce $L^p$ estimates of $\int_0^t u_s \circ dS_s^H$, we can reduce our analysis to the process $\delta(u1_{[0,t]})$. In the following, we will establish $L^p$ maximal estimates for this divergence process. We will make use of the notation $\int_0^t u_s \delta S_s^H := \delta(u1_{[0,t]})$.

**Theorem 3.2.** *Let* $pH > 1$, $u = \{u_t, t \in [0, T]\}$ *be a stochastic process* in $\mathbb{L}_H^{1,p}$. *Then we have*

$$E(\sup_{t\in[0,T]} |\int_0^t u_s \delta S_s^H|^p) \le C[\int_0^T |Eu_s|^p ds + E\int_0^T (\int_0^T |D_s u_r|^{\frac{1}{H}} ds)^{pH} dr],$$

*where the constant* $C > 0$ *depends on p, H, T.*

**Proof.** Let $1 - H < \gamma < 1 - \frac{1}{p}$, using the equality $c_\gamma = \int_r^t (t-\theta)^{-\gamma} (\theta - r)^{\gamma-1} d\theta$, we have

$$\begin{aligned}
|\int_0^t u_s \delta S_s^H|^p &= |c_\gamma^{-1} \int_0^t u_s (\int_s^t (t-r)^{-\gamma} (r-s)^{\gamma-1} dr) \delta S_s^H|^p \\
&= |c_\gamma^{-1} \int_0^t (t-r)^{-\gamma} (\int_0^r u_s (r-s)^{\gamma-1} \delta S_s^H) dr|^p \\
&\le c_\gamma^{-p} \int_0^t |\int_0^r u_s (r-s)^{\gamma-1} \delta S_s^H|^p dr \times (\int_0^t (t-r)^{-\gamma q} dr)^{\frac{p}{q}} \\
&\le c_{\gamma,p} \int_0^t |\int_0^r u_s (r-s)^{\gamma-1} \delta S_s^H|^p dr,
\end{aligned}$$

where we use the Hölder's inequality and the fact $\gamma < 1 - \frac{1}{p}$.

Hence, by (2.7) and Hölder's inequality, taking into account that $1 - H < \gamma$, we have

$$E \sup_{t\in[0,T]} \left|\int_0^t u_s \delta S_s^H\right|^p \le c_{\gamma,p} E \int_0^T \left|\int_0^r u_s (r-s)^{\gamma-1} \delta S_s^H\right|^p dr$$

$$\le c_{\gamma,p} \left\{ \int_0^T \left( \int_0^r (r-s)^{\frac{\gamma-1}{H}} |Eu_s|^{\frac{1}{H}} ds \right)^{pH} dr \right\}$$

$$+ c_{\gamma,p} \left\{ E \int_0^T \left( \int_0^r \int_0^T (r-s)^{\frac{\gamma-1}{H}} |D_\theta u_s|^{\frac{1}{H}} d\theta ds \right)^{pH} dr \right\}$$

$$\le C\left[ \int_0^T |Eu_s|^p ds + E \int_0^T \left( \int_0^T |D_s u_r|^{\frac{1}{H}} ds \right)^{pH} dr \right],$$

where the constant $C > 0$ depends on $p$, $H$, $T$. □

In the last, we consider now the notion of *p-variation* for a continuous process $X = \{X_t,\ 0 \le t < \infty\}$. Fix $T > 0$ and we let $t_i = \dfrac{iT}{n}$, where $n$ is a positive integer and $i = 0, 1, \ldots, n$. Define

$$V_p^n(X; T) := \sum_{j=0}^{n-1} |X_{t_{j+1}} - X_{t_j}|^p,$$

where $p > 0$. The process $X$ is of *p-variation* on $[0, T]$, if $V_p^n(X; T)$ converges in $L^1(\Omega)$ as $n$ tends to infinity. We denote by $V_p(X; T)$, the $p$-variation of $X$ on the interval $[0, T]$.

It is not difficult to prove the following result (see, for example, Guerra and Nualart [8]): Let $X = \{X_t\}_{t\in[0,T]}$ and $Y = \{Y_t\}_{t\in[0,T]}$ be stochastic processes defined on $(\Omega, \mathcal{F}, P)$ such that $EV_{\frac{1}{H}}^n(X; T)$ and $EV_{\frac{1}{H}}^n(Y; T)$ are finite. Then

$$E(|V^n_{\frac{1}{H}}(X;\,T) - V^n_{\frac{1}{H}}(Y;\,T)|)$$

$$\leq \frac{1}{H}\,(E[V^n_{\frac{1}{H}}(X - Y;\,T)])^H[(EV^n_{\frac{1}{H}}(X;\,T))^{1-H} + (EV^n_{\frac{1}{H}}(Y;\,T))^{1-H}]. \quad (3.8)$$

By applying (2.7) and (3.8), one can obtain the following lemma:

**Lemma 3.1.** *Let* $u, v \in \mathbb{L}_H^{1,\frac{1}{H}}$, $X_t = \int_0^t u_s \delta S_s^H$, $Y_t = \int_0^t v_s \delta S_s^H$, *for any* $t \in [0, T]$. *We have*

$$E(|V^n_{\frac{1}{H}}(X;\,T) - V^n_{\frac{1}{H}}(Y;\,T)|) \leq C_H \| u - v \|_{\mathbb{L}_H^{1,\frac{1}{H}}} (\| u \|^{\frac{1-H}{H}}_{\mathbb{L}_H^{1,\frac{1}{H}}} + \| v \|^{\frac{1-H}{H}}_{\mathbb{L}_H^{1,\frac{1}{H}}}),$$

*where* $C_H$ *is a constant depending on* $H$.

By Lemma 3.1, it is easy to see that if $u \in \mathbb{L}_H^{1,\frac{1}{H}}$, $X_t = \int_0^t u_s \delta S_s^H$ for any $t \in [0, T]$, then

$$E(|V^n_{\frac{1}{H}}(X;\,T)|) \leq C_H \| u \|^{\frac{1}{H}}_{\mathbb{L}_H^{1,\frac{1}{H}}}. \quad (3.9)$$

Consider the following spaces of random variables:

$$\mathbb{D}_b^{1,\frac{1}{H}} := \{F \in \mathbb{D}^{1,\frac{1}{H}} : F \text{ is bounded and } E(\int_0^T |D_s F|^{\frac{1}{H}} ds) < \infty\}.$$

Let $\varphi_T$ be the set of bounded step processes $u = \sum_{j=0}^{m-1} F_j 1_{[s_j, s_{j+1})}$, where $m \in \mathbb{N}$, $F_j \in \mathbb{D}_b^{1,\frac{1}{H}}$ for $j = 0, \ldots, m-1$ and $0 = s_0 < s_1 < \ldots < s_m = T$. It is obvious that $\varphi_T$ is dense in $\mathbb{L}_H^{1,\frac{1}{H}}$.

The following theorem study the $\frac{1}{H}$ variation for the indefinite integral $\int_0^t u_s \delta S_s^H$ and generalize result (1.4) for the divergence integral.

**Theorem 3.3.** *Let* $\frac{1}{2} < H < 1$ *and* $u \in \mathbb{L}_H^{1,\frac{1}{H}}$, $X_t = \int_0^t u_s \delta S_s^H$ *for any* $t \in [0, T]$. *We have, as n tends to infinity,*

$$V_{\frac{1}{H}}^n(X; T) \to \rho_{\frac{1}{H}} \int_0^T |u_s|^{\frac{1}{H}} ds, \text{ in } L^1(\Omega), \tag{3.10}$$

*where* $\rho_{\frac{1}{H}} = E(|\xi|^{\frac{1}{H}})$, $\xi \sim N(0, 1)$.

**Proof.** Since $\varphi_T$ is dense in $\mathbb{L}_H^{1,\frac{1}{H}}$, for any $u \in \mathbb{L}_H^{1,\frac{1}{H}}$, we may find a sequence $u^m = \sum_{j=0}^{m-1} F_j^m 1_{[s_j^m, s_{j+1}^m)} \in \varphi_T$ such that

$$\|u - u^m\|_{\mathbb{L}_H^{1,\frac{1}{H}}} \to 0, \text{ as } m \to \infty.$$

Set $X_t^m = \int_0^t u_s^m \delta S_s^H$, then we have

$$\begin{aligned} E|V_{\frac{1}{H}}^n(X; T) - \rho_{\frac{1}{H}} \int_0^T |u_s|^{\frac{1}{H}} ds| &\le E|V_{\frac{1}{H}}^n(X; T) - V_{\frac{1}{H}}^n(X^m; T)| \\ &\quad + E|V_{\frac{1}{H}}^n(X^m; T) - \rho_{\frac{1}{H}} \int_0^T |u_s^m|^{\frac{1}{H}} ds| \\ &\quad + \rho_{\frac{1}{H}} E|\int_0^T (|u_s^m|^{\frac{1}{H}} - |u_s|^{\frac{1}{H}}) ds|. \end{aligned}$$

By Lemma 3.1, we obtain that, as $m$ tends to infinite,

$$\begin{aligned} \sup_n E|V_{\frac{1}{H}}^n(X; T) - V_{\frac{1}{H}}^n(X^m; T)| &\le \sup_n C_H \|u - u^m\|_{\mathbb{L}_H^{1,\frac{1}{H}}} \\ &\quad \times (\|u\|_{\mathbb{L}_H^{1,\frac{1}{H}}}^{\frac{1-H}{H}} + \|u^m\|_{\mathbb{L}_H^{1,\frac{1}{H}}}^{\frac{1-H}{H}}) \to 0. \end{aligned}$$

Applying the mean value theorem to $f(x) = |x|^{\frac{1}{H}}$ with $\frac{1}{2} < H < 1$ and Hölder's inequality, we have that, as $m$ tends to infinite,

$$\rho_{\frac{1}{H}} E\Big|\int_0^T (|u_s^m|^{\frac{1}{H}} - |u_s|^{\frac{1}{H}})ds\Big|$$

$$\leq \frac{\rho_{\frac{1}{H}}}{H} E\Big|\int_0^T |u_s^m - u_s|(|u_s^m|^{\frac{1}{H}-1} + |u_s|^{\frac{1}{H}-1})ds\Big|$$

$$\leq \frac{\rho_{\frac{1}{H}}}{H} \|u - u^m\|_{\mathbb{L}_H^{1,\frac{1}{H}}} (\|u\|_{\mathbb{L}_H^{1,\frac{1}{H}}}^{\frac{1-H}{H}} + \|u^m\|_{\mathbb{L}_H^{1,\frac{1}{H}}}^{\frac{1-H}{H}}) \to 0.$$

In order to finish the proof, it is enough to show that, for any fixed $m$,

$$\lim_{n\to\infty} E|V_{\frac{1}{H}}^n(X^m;\, T) - \rho_{\frac{1}{H}} \int_0^T |u_s^m|^{\frac{1}{H}} ds| = 0.$$

Without loss of generality, we consider that for $n$ large enough, $(s_0^m, s_1^m, \ldots, s_m^m) \in (0, t_0^n, \ldots, t_n^n)$. Let $F \in \mathbb{D}_b^{1,\frac{1}{H}}$. By the equality $\delta(Fu) = F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}$, we have that

$$X_t^m = \sum_{j=0}^{m-1} F_j^m (S^H_{s^m_{j+i}\wedge t} - S^H_{s_j^m \wedge t}) - \beta_H \int_0^t \int_0^T D_s u_r^m [|s-r|^{2H-2} - (s+r)^{2H-2}] ds dr$$

$$:= Y_t^m + Z_t^m.$$

By the triangular inequality, we have that

$$E|V_{\frac{1}{H}}^n(X^m;\, T) - \rho_{\frac{1}{H}} \int_0^T |u_s^m|^{\frac{1}{H}} ds|$$

$$\leq E|V_{\frac{1}{H}}^n(X^m;\, T) - V_{\frac{1}{H}}^n(Y^m;\, T)| + E|V_{\frac{1}{H}}^n(Y^m;\, T) - \rho_{\frac{1}{H}} \int_0^T |u_s^m|^{\frac{1}{H}} ds|$$

$$:= a_n^m + b_n^m.$$

Applying Lemma 3.1 to $a_n^m$, we have that, as $n$ tends to infinite,

$$a_n^m \leq \frac{1}{H} (E[V_{\frac{1}{H}}^n(Z^m;\, T)])^H [(EV_{\frac{1}{H}}^n(X^m;\, T))^{1-H} + (EV_{\frac{1}{H}}^n(Y^m;\, T))^{1-H}] \to 0.$$

Since $\{Z_t^m\}_{[0,T]}$ has bounded variation and hence has zero $\frac{1}{H}$ variation, and by (3.9), $EV_{\frac{1}{H}}^n(X^m;\, T)$ and $EV_{\frac{1}{H}}^n(Y^m;\, T)$ are bounded as $n$ tends to infinite. Using (1.4), we have that, as $n$ tends to infinite,

$$b_n^m = E|V_{\frac{1}{H}}^n(Y^m;\, T) - \rho_{\frac{1}{H}} \int_0^T |u_s^m|^{\frac{1}{H}} ds|$$

$$\leq \sum_{j=0}^{m-1} \Big( \sup_{\omega\in\Omega} |F_j^m(\omega)|^{\frac{1}{H}} \Big) E\Big[\Big| \Big( \sum_{i\in I_j^{m,n}} |S^H_{t_{i+1}^n} - S^H_{t_i^n}|^{\frac{1}{H}} \Big) - \rho_{\frac{1}{H}} |s_{j+1}^m - s_j^m| \Big] \to 0,$$

where $I_j^{m,n} := \{i : t_i^n \in [s_j^m,\, s_{j+1}^m)\}$. This completes the proof. □

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