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EXTENSION PROPERTY OF AUTOMORPHISM OF MODULES

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Abstract

In this paper we generalize Schupp's result for groups to modules. For an injective module, every automorphism satisfies the extension property. We characterize the automorphisms of a module M satisfies the extension property.

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1. Introduction

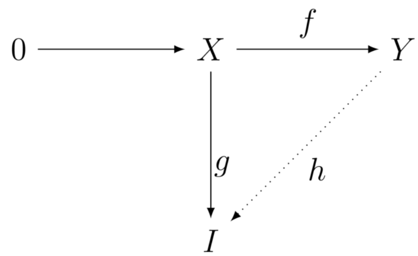
The study of modules by properties of their endomorphisms is a classical research subject. In [10], Schupp showed that the extension property in the category of groups characterizes the inner automorphisms. In [6], Dugas and Gobel gave another simpler proof of Schupp's result, using only the elementary theory of groups. Ben Yakoub showed that the result of Schupp is not valid, in general, for algebras on a commutative ring. It is not yet known whether this result holds true for algebras (of finite dimensions) on a commutative field (see [3]). In [1], Abdelalim et al. construct a sequence of abelian p-groups $(A_k)_{k\geq 1}$ and a sequence of automorphisms $(\alpha_k)_{k\geq 1}$ of $(A_k)_{k\geq 1}$ such that $\alpha_k\lambda = \lambda\alpha$, where λ be a monomorphism from A to A', where A and A' are abelian p-groups, and $\alpha \in Aut(A)$ which is written in the form $\alpha = \pi id_A + \rho$, where π is an invertible p-adic number and $\rho \in Hom(A, A^1)$, where $A^1 = \bigcap_{n\geq 1} nA$ is the first Ulm subgroup of A. Let R be a ring, if M is an R-module and $\alpha \in Aut_R(M)$, we say the α satisfies the extension property if for all R-module N and any monomorphism $\lambda : M \to N$, then there exists $\widetilde{\alpha} \in Aut_R(N)$ such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & N \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \widetilde{\alpha}} \\ M & \xrightarrow[\lambda]{} & N \end{array}$$

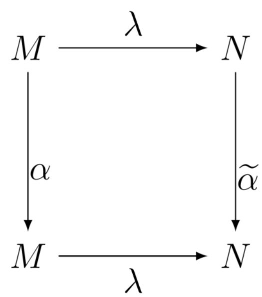
We show that every automorphism of injective module satisfies the Extension Property. And we characterize the automorphism of a free R-module over Dedekind domain having the extension property.

2. Main Results

Definition 2.1 ([9]). An R-module I is called injective if for any injective homomorphism f of X onto Y, and any homomorphism g of X to I, there exists an homomorphism h of Y to I, such that: $g = hf$ (i.e., there exists $h : Y \to I$ such that the following diagram is commutative.



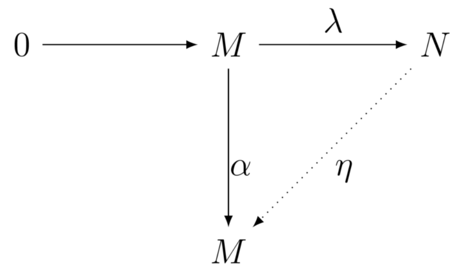
Definition 2.2. Let R be a ring, if M is an R-module and $\alpha \in Aut_R(M)$, we say the α satisfies the extension property if for all R-module N and any monomorphism $\lambda : M \to N$, then there exists $\tilde{\alpha} \in Aut_R(N)$ such that the following diagram is commutative:



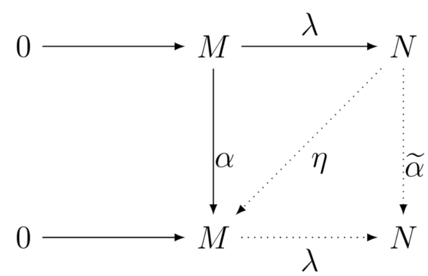
In other words, $\tilde{\alpha}\lambda = \lambda\alpha$.

Proposition 2.3. *Every automorphism* α *of an injective R-module M satisfies the extension property*.

Proof. Let N be an R-module and λ from M to N be a monomorphism, since M is an injective module, then there exists η from N to M such that the following diagram is commutative:



We take $\tilde{\alpha} = \lambda\eta$. Then the following diagram is commutative with exact rows:



Since α is a automorphism then by short five lemma, $\tilde{\alpha}$ is a automorphism. Hence $\tilde{\alpha} \in Aut(N)$ such that $\tilde{\alpha}\lambda = \lambda\eta\lambda = \lambda\alpha$. Then α satisfies the extension property.

□

Definition 2.4 ([9])**.** Let M be an R-module over a domain R. If $r \in R$ and $m \in M$, then we say that m is divisible by r if there is some $m' \in M$ with $m = r.m'$. We say that M is a divisible module if each $m \in M$ is divisible by every nonzero $r \in R$.

Lemma 2.5 ([5], Proposition VII.5.1)**.** *For each integral domain R the following properties are equivalent*:

(i) *R is a Dedekind ring*.

(ii) *Each divisible R-module is injective*.

Corollary 2.6. *Let R be a Dedekind domain and M be a divisible R-module.*

Then every automorphism α of M satisfies the extension property.

Proof. We can apply Proposition 2.3 and Lemma 2.5.

□

Definition 2.7 ([9]). A ring R is left hereditary if every left ideal is projective, a ring R is right hereditary if every right ideal is projective. A Dedekind ring is a hereditary domain.

Lemma 2.8 ([9], Theorem 4.13). *If R is left hereditary, then every submodule A of a free left R-module F is isomorphic to a direct sum of left ideals.*

Recall that $M^1 = \bigcap_{n \geq 0} p^n M$ is the first Ulm submodule of M.

Lemma 2.9. *Let M be a free R-module with a well ordered base $\{x_i / i \in I\}$ and let $x \in M$ such that $M = < x > \oplus M'$, where M' is a submodule of M.*

If $\alpha(x) \in M$ such that $\alpha(x) = rx + m'$, where $r \in R$ and $m' \in (M')^1$, then r is independent of x.

Proof. Let $(i, j) \in I^2$, with $i \neq j$. We can write $M = < x_i > \bigoplus M'$ with $x_j \in M'$. It is easy to see that $< x_i + x_j > \bigoplus M' = M$, so we have:

$$\begin{cases} \alpha(x_i) = r_i x_i + m_1, & \text{where } m_1 \in (M')^1, \\ \alpha(x_j) = r_j x_j + m_2, & \text{where } m_2 \in (M')^1, \\ \alpha(x_i + x_j) = r(x_i + x_j) + m_3, & \text{where } m_3 \in (M')^1, \end{cases}$$

then $(r - r_i)x_i + (r - r_j)x_j \in \bigcap_{n \geq 0} p^n M'$, then $(r - r_i)x_i + (r - r_j)x_j \in p^n M'$, $\forall n \geq 0$,

then $\exists m \in M'$ such that $(r - r_i)x_i + (r - r_j)x_j = p^n m'$, $\forall n \geq 0$,

we take $m' = c_{i_1} + \dots + c_{i_n}$, where $c_{i_k} \in C_{i_k}$, $1 \leq k \leq n$, $\{i_1, \dots, i_n\} \subseteq I$ and $p^k = o(c_{i_k})$ then $(r - r_i)x_i + (r - r_j)x_j = 0$ then $(r - r_i)x_i = (r - r_j)$ $x_j = 0$ and like M is free R-module then M is torsion-free module, then $r - r_i = r - r_j = 0$ then $r = r_i = r_j$.

□

Lemma 2.10. *Let R be a Dedekind domain, M be a free R-module and $\alpha \in Aut_R(M)$ has the extension property, if $< x >$ is a direct summand of M, then there exists $r \in R$ invertible such that then* $\alpha(x) - rx \in M^1$.

Proof. Let M be a free R-module with a well ordered base $\{x_i / i \in I\}$ and let $x \in M$ such that $M =< x > \oplus M'$, where M' is a submodule of M.

Since $\alpha(x) \in M$ then there exists $r_x \in R$ such that

$$\alpha(x) = r_x x + m', \text{ where } m' \in M'.$$

Prove that $m' \in (M')^1$, where $(M')^1$ is the first Ulm submodule of M', assume the contrary: $m' \notin (M')^1$.

Since R is Dedekind domain then R is a hereditary ring, then by Lemma 2.8, M' is a direct sum of ideals of R, $M' = \bigoplus_{i \in I} C_i$.

Then $\exists n \geq 1$ such that $m' = c_{i_1} + \dots + c_{i_n}$, where $c_{i_k} \in C_{i_k}$, $1 \leq k \leq n$, $\{i_1, \dots, i_n\} \subseteq I$ and $c_{i_n} \neq 0$, with $p^k = o(c_{i_k})$.

Consider the R-module N such that $N = \langle y \rangle \oplus M'$, with $y \in M$ and $y \neq 0$.

We define a homomorphism λ from M to N by:

$$\lambda(x) = p^n y \text{ and } \lambda(c_{i_k}) = c_{i_k} \;\; \forall\, k \in \{1, \ldots, n\}.$$

$\star$ λ is clearly a monomorphism. Indeed, if $a \in \ker(\lambda)$ then $\lambda(a) = 0$, i.e., $\exists\,(r_x, m') \in R \times M'$ such that $a = r_x x + m'$ and $0 = \lambda(r_x x + m') = r_x p^n y + m'$ then $0 = r_x p^n y = m'$, then $m' = 0$ and like $y \in M$, then $y = \sum_{i \in I} a_i x_i$, where $a_i \in R$, then $0 = r_x p^n y = \sum_{i \in I} r_x p^n a_i x_i$, then $r_x p^n a_i = 0\ \forall i \in I$, then $r_x a_i = 0\ \forall i \in I$, since $y \neq 0$ so $\exists i \in I$ such that $a_i \neq 0$, and by integrity of R, we have $r_x = 0$ hence $r_x x = 0$, i.e., $a = 0$ then $\ker(\lambda) = 0$ which implies that λ is a monomorphism.

Using the fact That α checks the extension property then there exists $\widetilde{\alpha} \in Aut(N)$ such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & N \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \widetilde{\alpha}} \\ M & \xrightarrow[\lambda]{} & N \end{array}$$

i.e., $\forall x \in M$,

$$\widetilde{\alpha}\lambda(x) = \lambda\alpha(x), \tag{1}$$

we have:

$$\begin{aligned}\tilde{\alpha}\lambda(x) &= \tilde{\alpha}(p^n y)\\ &= p^n\tilde{\alpha}(y)\\ &= p^n(r'_x y + c'_{i_1} + \dots + c'_{i_n})\\ &= r'_x p^n y + p^n c'_{i_1} + \dots + p^n c'_{i_n}\\ &= r'_x p^n y.\end{aligned}$$

And we have:

$$\begin{aligned}\lambda\alpha(x) &= \lambda(r_x x + m')\\ &= r_x p^n y + c_{i_1} + \dots + c_{i_n},\end{aligned}$$

where all the $c_{i_k} \in C_{i_k}$.

(1) Show that:

$$r'_x p^n y = r_x p^n y + c_{i_1} + \dots + c_{i_n},$$

i.e., $0 = c_{i_n}$ which is absurd, hence $m' \in (M')^1$, by Lemma 2.9 $\alpha(x) = rx + m'$, where $r \in R$ and $m' \in (M')^1$. But since $(M')^1 \subseteq M^1$ then $\alpha(x) - rx \in M^1$.

□

Theorem 2.11. *Let R be a Dedekind Domain not a field, M be a free R-module and* $\alpha \in Aut_R(M)$. *Then the following assertions are equivalent*:

(i) α *has the extension property*.

(ii) *There exists* $r \in R$ *invertible such that* $\alpha = r.id_M$.

Proof. (i) $\Rightarrow$ (ii) Since M is a free R-module, then $M = \bigoplus_{i\in I} < x_i >$, then $\forall i \in I, < x_i >$ is a direct summand of M then by Lemma 2.10: $\exists\, r \in R$ such that $\alpha(x_i) = rx_i + m^1$, where $m^1 \in M^1$.

Then $\forall\, m \in M$, we have:

$$\begin{aligned}\alpha(m) &= \alpha\left(\sum_{i\in I} \beta_i x_i\right)\\ &= \sum_{i\in I} \beta_i \alpha(x_i)\\ &= \sum_{i\in I} \beta_i (rx_i + m^1)\\ &= r\sum_{i\in I} \beta_i x_i + \sum_{i\in I} \beta_i m^1\\ &= rm + m_1^1.\end{aligned}$$

where $m_1^1 = \sum_{i\in I} \beta_i m^1 \in M^1$ and $r \in R$.

Since M is a free R-module, hence M is a torsion-free, so M^1 is a maximum divisible submodule of M, since a divisible submodule of a free R-module M is zero (R not a field), then $M^1 = 0$. Then there exists $r \in R$ invertible such that $\alpha = r.id_M$.

(ii) $\Rightarrow$ (i) Suppose that $\alpha(m) = rm + d$, where $r \in R$ invertible and d contained in the maximum divisible submodule of M, and let $\lambda : M \to N$ be a monomorphism, if we take $\tilde{\alpha}(n) = rn + \lambda(d)$ then we have:

- $\tilde{\alpha}(n) = 0 \Rightarrow rn + \lambda(d) = 0 \Rightarrow rn = \lambda(d) = 0 \Rightarrow n = 0.$

• Let $y \in N$, we take $x = r'y - r'\lambda(d)$ with $rr' = r'r = 1_R$, we have:

$$\begin{aligned}\tilde{\alpha}(x) &= \tilde{\alpha}(r'y - r'\lambda(d)) \\ &= r(r'y - r'\lambda(d)) + \lambda(d) \\ &= rr'y - rr'\lambda(d) + \lambda(d) \\ &= y - \lambda(d) + \lambda(d) \\ &= y.\end{aligned}$$

Then $\tilde{\alpha} \in Aut_R(N)$ such that $\forall\, m \in M$, $\tilde{\alpha}\lambda(m) = r\lambda(m) + \lambda(d) = \lambda(rm + d)$ $= \lambda\alpha(m)$.

Then α satisfies the extension property.

□

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