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TOPOLOGIES ON SEQUENCE SPACES IN NON-ARCHIMEDEAN ANALYSIS

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Abstract

In this paper, we deal with polar and solid topologies in separated dual pairs $\langle E(X), E(Y)\rangle$ of vector-valued sequence spaces over a non-archimedean valued field K, especially we study the natural topology Na. We give a relation between Schauder basis of $(E(X), \tau)$ and Schauder basis of (X, τ_j) $j \in \mathbb{N}$, where τ_j are topologies, which we define in the paper. Furthermore, we investigate some properties of the $\sigma_\Lambda\left(E(X), E(X)^\Lambda\right)$ and $\beta_\Lambda\left(E(X), E(X)^\Lambda\right)$ topologies, Λ is a scalar sequence space over K.

1. Introduction

The study of non-archimedean $(n.a)$ scalar sequence spaces was first attempted by Monna [10] and Dorleyn [1]. They obtained the basic set-

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theoretical results about perfect sequence spaces as well as the properties in the weak topology. In [6], De Grande De-Kimpe gave a characterization of the natural topology Na over an *n.a* perfect sequence space; this topology was also studied by Garling in the classical case and noted σ_γ [2]. In the present paper, some properties of this topology are generalized on a sequence space $E(X)$ over a topological vector space X in paragraph 2, we give also some results concerning polar and solid topology τ on $E(X)$. Relations between Schauder basis of $(E(X), \tau)$ and Schauder basis of (X, τ_j) $j \in \mathbb{N}$, where τ_j are topologies, which we define in the paper and investigate.

In [13], Ruckle constructed and studied the $\sigma S\left(T, T^S\right)$, $\beta S\left(T, T^S\right)$, and $\beta\varphi$-topologies, S and T are two scalar sequence spaces over a field $K(K = \mathbb{R}$ or $K = \mathbb{C})$ and $T^S = \{(\lambda_i)_i \in \omega(K) : (\lambda_i\mu_i)_i \in S$ for all $(\mu_i)_i \in T\}$. These classes of topologies include the topologies studied by Köthe and Toeplitz [9] and Garling [2]. In [4], Hassani and Babahmed gave a generalization of Köthe-Toeplitz duals based of a separated duality of *n.a* Banach spaces. In paragraph 3 of this work, we give a generalization of some result of [4] and [6] for the locally K-convex sequence spaces over an *n.a* topological vector spaces.

Throughout this paper, K is an *n.a* non trivially valued complete field with valuation $|.|$, X and Y are two *n.a* topological vector spaces over K (or K vector spaces) that are in separated duality $\langle X, Y\rangle$. The duality theory for locally K-convex spaces can be found more extensively in [3], [16], and [18].

A nonempty subset A of a K-vector space X is called *K-convex*, if $\lambda x + \mu y + \gamma z \in A$, whenever $x, y, z \in A$, $\lambda, \mu, \gamma \in K$, $|\lambda| \le 1$, $|\mu| \le 1$, $|\gamma| \le 1$, and $\lambda + \mu + \gamma = 1$. A is said to be *absolutely K-convex*, if $\lambda x + \mu y \in A$, whenever $x, y \in A$, $\lambda, \mu \in K$, $|\lambda| \le 1$, $|\mu| \le 1$. For a nonempty set $A \subset X$, its absolutely K-convex hull $c_0(A)$ is the smallest absolutely K-convex set that contains A. If A is a finite set $\{x_1, \ldots, x_n\}$, we sometimes write $c_0(x_1, \ldots, x_n)$ instead of $c_0(A)$.

A topological vector space X over K is called *locally K-convex space*, if X has a neighbourhood base of zero consisting of locally K-convex sets.

A subset A of a locally K-convex space X is compactoid, if for each neighbourhood U of zero, there exist $x_1, \ldots, x_n \in X$ such that $A \subset U + c_0(x_1, \ldots, x_n)$. An absolutely K-convex subset A of X is said to be *c-compact*, if every convex filter on A has a clusterpoint on A.

An absolutely K-convex subset of a locally K-convex space X is called *K-closed*, if for every $x \in X$, the set $\{|\lambda| : \lambda \in K, \lambda x \in A\}$ is closed in $|K|$. If the valuation on K is discrete, every absolutely K-convex set A is K-closed. If K has a dense valuation, an absolutely K-convex set A is K-closed, if and only if from $x \in E$, $\lambda x \in A$ for all $\lambda \in K$, $|\lambda| \prec 1$, it follows that $x \in A$. Intersections of K-closed sets are K-closed. For an absolutely K-convex set A, the K-closed hull of A is the smallest subset of X, that is, K-closed and contains A, it is denoted by $K_c(A)$. If K is discrete, we have $K_c(A) = A$ and if K is dense, $K_c(A) = \bigcap\{\lambda A : \lambda \in K$ and $|\lambda| \succ 1\}$ ([3] p. 220). For each absolutely K-convex subset A of Y, $K_c\left(\overline{A}^{\sigma(Y, X)}\right) = A^{\circ\circ}$ ([3], Corollary 4.3, p. 233). A subset A of Y is said to be *X-closed*, if for every $y \in Y \setminus A$, there exits $x \in X$ such that $|\langle x, y\rangle| \succ 1$ and $|\langle x, A\rangle| \leq 1$. Intersections of X-closed sets are X-closed. For a subset A of Y, the X-closed hull $X_c(A)$ of A is the smallest X-closed subset of Y that contains A. For each subset A of Y, $X_c(A) = A^{\circ\circ}$ ([3], Proposition 2.5, p. 224). Using these two results and by [3], Theorem 4.2, p. 233, we have: For all absolutely K-convex subset A of Y, A is X-closed, if and only if, A is K-closed and $\sigma(Y, X)$- closed.

A sequence $(e_i)_i$ is a Schauder basis for X, if every $x \in X$ can be written uniquely as $x = \sum_{i=1}^{\infty} \lambda_i x_i$, where the coefficient functionals $f_j : x \mapsto \lambda_j$ are continuous.

Let $\mathcal{A}$ be a family of $\sigma(Y, X)$- bounded subsets of Y such that

(a) $\mathcal{A}$ is directed by inclusion,

(b) $Y = \bigcup_{A\in\mathcal{A}} A$,

(c) there exists $\lambda_0 \in K$, $|\lambda_0| > 1$ such that $\lambda_0 A \in \mathcal{A}$, for all $A \in \mathcal{A}$.

A topology τ on X is called *polar topology of* $\mathcal{A}$*- convergence*, if τ has a fundamental system of zero-neighbourhood consisting of $\{A^\circ / A \in \mathcal{A}\}$.

For further information about polar topology of $\mathcal{A}$- convergence and general properties of locally K-convex spaces, we refer to [3], [16], and [18].

$(\omega(X), \tau_\omega(X)) =$ the linear space of all sequences in X endowed with the product topology $\tau_\omega(X)$, which is generated by the family of $n.a$ seminorms $(p_n)_{n\in\mathbb{N}, p\in(\mathcal{P})}$, $p_n(\overline{x}) = p(x_n)$ for all $\overline{x} = (x_n)_n \in \omega(X)$ and all $p \in (\mathcal{P})$, if X is a locally K-convex space and $(\mathcal{P})$ is a family of $n.a$ seminorms, which define its topology; this space is noted $\omega(K)$ (or ω, for short) in case when $X = K$. A sequence space over X is a subspace of $\omega(X)$.

We define the following sequence spaces over X:

$c_0(X) = \{(x_k)_k \in \omega(X) : (x_k)_k \text{converges to zero}\}$,

$c(X) = \{(x_k)_k \in \omega(X) : (x_k)_k \text{ converges in } X\}$,

$\varphi(X) = \{(x_k)_k \in \omega(X) : \text{there exists } k_0 \in \mathbb{N} : x_k = 0 \text{ for all } k \geq k_0\}$,

$m(X) = \{(x_k)_k \in \omega(X) : (x_k)_k \text{ is bounded in } X\}$.

Over $m(X)$, we define the sequence of $n.a$ seminorms $(\overline{p})_{p\in(\mathcal{P})}$ by:

$$\overline{p}(\overline{x}) = \sup_k p(x_k) \text{ for all } \overline{x} = (x_k)_k \in m(X).$$

Let $\tau_\infty(X)$ be the topology on $m(X)$ defined with the sequence of $n.a$ seminorms $(\overline{p})_{p\in(\mathcal{P})}$.

If $A \subset \omega(X)$, the β-dual of A is the subspace of $\omega(Y)$, which is define by $A^{\beta} = \{(y_n)_n \in \omega(Y) : \lim_n \langle x_n, y_n \rangle = 0 \text{ for all } (x_n)_n \in A\}$. A is called *perfect*, if $A^{\beta\beta} = A$. If A is perfect, then $\varphi(X) \subset A$. For all $A \subset \omega(X)$, A^{β} is perfect. We define B^{β}, if $B \subset \omega(Y)$ in the same way.

Let $E(X)$ and $E(Y)$ be two sequence spaces on X and Y, respectively, such that $E(Y) \subset E(X)^{\beta}$, we define on the pair $(E(X), E(Y))$, the following duality $\langle (x_n)_n, (y_n)_n \rangle = \sum_{n=1}^{\infty} \langle x_n, y_n \rangle$ for all $(x_n)_n \in E(X)$ and all $(y_n)_n \in E(Y)$. In this paragraph, we consider this duality such that $\varphi(X) \subset E(X)$ and $\varphi(Y) \subset E(Y)$, this duality is separated.

For every $j \geq 1$, we consider the following linear maps:

$$\prod_j^X : E(X) \to X, \qquad \delta_j^X : X \to E(X),$$

$$(x_n)_n \mapsto x_j, \qquad x \mapsto \delta_j(x),$$

where $\delta_j(x)$ is the sequence with x in the j-th place and 0's elsewhere.

We define also $\prod_j^Y$ and δ_j^Y.

The net $(x^i)_{i \in I}$ in $E(X)$ is called *TK-convergent*, if for every $j \geq 1$, $\left(x_j^i\right)_{i \in I}$ is convergent in (X, τ_j), where τ_j is the initial topology on X associated with the map δ_j^X.

For every $y = (y_n)_n \in E(X)^{\beta}$, we consider $w_y(x) = \sup_n |\langle x_n, y_n \rangle|$ and $v_y(x) = |\sum_{n=1}^{\infty} \langle x_n, y_n \rangle|$ for every $x \in E(X)$, then w_y and v_y are two *n.a* seminorms over $E(X)$, and we have for every $y \in E(X)^{\beta}$, $v_y \leq w_y$. Let *Na* and τ_0 be the topologies over $E(X)$ generated by the family of *n.a* seminorms $(w_y)_{y \in E(X)^{\beta}}$ and $(v_y)_{y \in E(X)^{\beta}}$, respectively. *Na* is the

natural topology studied by De Grande De-Kimpe in the particular case of *n.a* scalar sequences [6], it is also the $\sigma_\gamma(T, T^\gamma)$-topology studied by Garling in the classical case ($K = \mathbb{R}$ or $K = \mathbb{C}$) [2].

A subset D of $\omega(X)$ is said to be *solid*, if for every $\bar{x} = (x_k)_k \in D$ and $\alpha = (\alpha_k)_k \in \omega$ such that $|\alpha_k| \leq 1$ for all k, we have $\alpha\bar{x} = (\alpha_k x_k)_k \in D$. The solid hull $S(D)$ of D is the smallest solid set of sequence containing D.

A topology on $E(X)$, with respect to the duality $\langle E(X), E(X)^\beta \rangle$, will be called *solid*, if the elements of the determining family of weakly bounded subsets of $E(X)^\beta$ are solid sets.

A sequence space (E, τ_E) on X, is called an *X-space*, if the inclusion map: $(E, \tau_E) \to (\omega(X), \tau_\omega(X))$ is continuous; it is called *AX-space*, if for every $x = (x_i)_i \in E$, $\lim_{n\to\infty} x^{[n]} = x$, $x^{[n]} = (x_1, x_2, \ldots, x_n, 0, \ldots)$; it is called *PX-space*, if for all net $(x^i)_i$ in E, $(x^i)_i$ converges to 0 in (E, τ_E), if and only if for all $j \geq 1$, the net $\left(\prod_j^X (x^i)\right)_i$ converges to 0 in X. And it is called *SPX-space*, if for all sequence $(x^n)_n$ in E, $(x^n)_n$ converges to 0 in (E, τ_E), if and only if for all $j \geq 1$, the sequence $\left(\prod_j^X (x^n)\right)_n$ converges to 0 in X. It is obvious that $(\omega(X), \tau_\omega(X))$ is a *PX*-space and every *PX*-space is an *SPX*-space.

If p is a non-archimedean seminorm over X, $B_p(0, 1)$ is the set $\{x \in X : p(x) \leq 1\}$.

2. The τ_0 and Na-Topologies on $E(X)$

Proposition 1. $\tau_0 = \sigma\left(E(X), E(X)^\beta\right)$; *where* σ *is the weak topology on* $E(X)$.

Proof. Obvious. □

Lemma 1. *If* $E(X)$ *is perfect, then it is weakly sequentially complete.*

Proof. By Proposition 1 and ([4], Theorem 2.1, p. 117). □

Proposition 2. *If* $E(X)$ *is perfect, then every weakly bounded subset of* $E(X)$ *is strongly bounded.*

Proof. The weakly closed subsets of $E(X)$ are weakly sequentially complete by Lemma 1. Then apply the non-archimedean analogue of Banach-Mackey ([11], p. 124). □

Corollary 1. *If* $E(X)$ *is perfect, then all the polar topologies on* $E(X)$ *yield the same bounded subsets.*

Lemma 2. *If* $E(X)$ *is perfect, then the solid hull of a finite subset of* $E(X)^{\beta}$ *is* $\tau_0\left(E(X)^{\beta}, E(X)\right)$*-bounded.*

Proof. It suffices to prove that $S(y)$ is σ-bounded for every $y \in E(X)^{\beta}$. Let $y = (y_k)_k \in E(X)^{\beta}$, then for every $x = (x_k)_k \in E(X)$ and every $\alpha = (\alpha_k)_k \in \omega$ such that $|\alpha_k| \leq 1$ for all k, we have $v_x(\alpha y) \leq w_x(y) \prec +\infty$. □

Lemma 3. *If* $E(X)$ *is perfect, then for every* σ*-bounded and solid subset* A *of* $E(X)^{\beta}$, *the polar* A° *of* A *in the duality* $\left\langle E(X), E(X)^{\beta}\right\rangle$ *is given by*:

$$A^{\circ} = \{x = (x_n) \in E(X) / w_y(x) \leq 1 \text{ for all } y \in A\} \left(= \bigcap_{y \in A} B_{w_y}(0, 1)\right).$$

Proof. Obvious. □

Proposition 3. *The Na topology on* $E(X)$ *is a polar topology.*

Proof. Let τ be the polar topology of $\mathcal{M}$-convergence on $E(X)$, $\mathcal{M}$ standing for the family of the solid hull of a finite subsets of $E(X)^{\beta}$, then $\tau = Na$, since for every $y \in E(X)^{\beta}$, $S(y)^{\circ} = B_{w_y}(0, 1)$ (Lemma 3). □

Remark 1. The topology Na on $E(X)$ is a polar and solid topology. It is also the coarsest solid and polar topology on $E(X)$.

Proposition 4. *If $E(X)$ is perfect, then it is Na-sequentially complete.*

Proof. It suffices to use Lemma 1, Proposition 3, and ([3], Theorem 3.2, p. 230). □

Proposition 5. *If $E(X)$ is perfect, then the topology Na on $E(X)$ is compatible with the duality $\langle E(X),\ E(X)^{\beta}\rangle$.*

Proof. Na is a polar topology.

For every $y = (y_n)_n \in E(X)^{\beta}$, $S(y)^{\circ} = \left[K_c\left(\overline{S(y)}^{\sigma\left(E(X)^{\beta},\ E(X)\right)}\right)\right]^{\circ}$ ([3], Corollary 4.3, p. 233) and we have $K_c\left(\overline{S(y)}^{\sigma\left(E(X)^{\beta},\ E(X)\right)}\right)$ is $\sigma\left(E(X)^{\beta},\ E(X)\right)$-bounded, $E(X)^{\beta} = \bigcup_{y\in E(X)^{\beta}} K_c\left(\overline{S(y)}^{\sigma\left(E(X)^{\beta},\ E(X)\right)}\right)$ and $K_c\left(\overline{S(y)}^{\sigma\left(E(X)^{\beta},\ E(X)\right)}\right)$ is closed in $E(X)^{\beta}$ equipped with the topology $\sigma\left(E(X)^{\beta},\ E(X)\right)$, so it is $E(X)$-closed.

Furthermore, for every y and z in $E(X)^{\beta}$, we have:

$$K_c\left(\overline{S(y)}^{\sigma\left(E(X)^{\beta},\ E(X)\right)}\right) \cup K_c\left(\overline{S(z)}^{\sigma\left(E(X)^{\beta},\ E(X)\right)}\right)$$

$$\subset K_c\left(\overline{S(\{y,\ z\})}^{\sigma\left(E(X)^{\beta},\ E(X)\right)}\right),$$

and for every $\lambda \in K$:

$$|\lambda| \succ 1\ \lambda \cdot K_c\left(\overline{S(y)}^{\sigma\left(E(X)^{\beta},\ E(X)\right)}\right) = K_c\left(\overline{S(\lambda y)}^{\sigma\left(E(X)^{\beta},\ E(X)\right)}\right).$$

Therefore, if we take

$$\mathcal{A} = \left\{ K_c\left(\overline{S(A)}^{\sigma\left(E(X)^{\beta},\, E(X)\right)} \right) : A \subset E(X)^{\beta} \text{ and } A \text{ is finished} \right\},$$

so Na is a polar topology of $\mathcal{A}$-convergence, where $\mathcal{A}$ is formed by a $\sigma\left(E(X)^{\beta},\, E(X)\right)$-bounded and $E(X)$-closed subsets of $E(X)^{\beta}$. Then by Theorem 4.3, p. 233 of [3], Na is compatible with the duality $\left\langle E(X),\, E(X)^{\beta} \right\rangle$. □

Proposition 6. *Let $A \subset E(X)$, then if A is Na-bounded, then $S(A)$ is Na-bounded.*

Proof. Obvious. □

Proposition 7. *Let τ be a polar topology in $E(X)$ such that $E(X)$ is perfect, then for every $A \subset E(X)$, if A is τ-bounded, then $S(A)$ is τ-bounded.*

Proof. A is τ-bounded $\Rightarrow$ A is Na-bounded (Corollary 1 and Proposition 3) $\Rightarrow S(A)$ is Na-bounded (Proposition 6) $\Rightarrow S(A)$ is τ-bounded (Corollary 1 and Proposition 3). □

Corollary 2. *If $E(X)$ is perfect, then $\beta\left(E(X),\, E(X)^{\beta}\right)$, the strong topology on $E(X)$, is a polar and solid topology.*

Proof. Obvious. □

Lemma 4. *Let (X, τ) be a topological vector space, $\langle X, Y \rangle$ be a separated duality, and A be a compactoid subset of X.*

If $(y_n)_n$ is a τ-equicontinuous sequence of Y, which converges pointwise to y, then the sequence $(y_n)_n$ converges to y uniformly on A.

Proof. ([16], Lemma 10.6, p. 220). □

The following proposition is an obvious adaptation of the one of Proposition 14 of [6].

Proposition 8. *Let A be a compactoid subset of $(E(X), Na)$, then for all $y = (y_k)_k \in E(X)^\beta$, we have $\lim_k \sup_{x=(x_k)\in A} |\langle x_k, y_k \rangle| = 0$.*

Remark 2. Assume that K is spherically complete and let τ be a locally K-convex and compatible topology with the duality $\left\langle E(X), E(X)^\beta \right\rangle$ on $E(X)$. Then, for every absolutely K-convex, bounded and c-compact subset of $(E(X), \tau)$ and for every $y = (y_n) \in E(X)^\beta$,

$$\lim_n \sup_{x=(x_k)\in A} |\langle x_n, y_n \rangle| = 0.$$

Proof. By Proposition 8 and ([15], Theorem 9, $\alpha \Leftrightarrow \beta$, p. 139) and ([5], Proposition 3, p. 177). □

Lemma 5. *For every $j \geq 1, \prod_j^X$ is $(\sigma(E(X), E(Y)), \sigma(X, Y))$-continuous and δ_j^X is $(\sigma(X, Y), \sigma(E(X), E(Y)))$-continuous.*

Proof. For every $j \geq 1$, $\left(\prod_j^X\right)^*(Y) \subset E(Y)$ and $\left(\delta_j^X\right)^*(E(Y)) \subset Y$, then the lemma holds by ([14], p. 128), $\left(\prod_j^X\right)^*$ and $\left(\delta_j^X\right)^*$ are the adjoint maps of $\prod_j^X$ and δ_j^X, respectively. □

Theorem 1. *If $E(X)$ is perfect and $(e_n)_n$ is a Schauder basis of $X_\sigma = (X, \sigma(X, Y))$, then for every polar, solid and compatible topology τ with the duality $\langle E(X), E(Y)\rangle, (\delta_j(e_n))_{n,j}$ is a Schauder basis of $(E(X), \tau)$.*

Proof. It suffices to prove the result for the Na topology. So take $x = (x_k)_k \in E(X)$. Then, for all $j \geq 1, x_j = \prod_j^X(x) \in X$, so $x_j = \sum_{n=1}^\infty \lambda_n^j e_n$ in X_σ since $(e_n)_n$ is a Schauder basis of X_σ, therefore, for all $j \geq 1$, $X_j = \delta_j^X(x_j) = \sum_{n=1}^\infty \lambda_n^j \delta_j^X(e_n)$ in $(E(X), \sigma(E(X), E(Y)))$, since δ_j^X is $(\sigma(X, Y), \sigma(E(X), E(Y)))$-continuous (lemma before). Let $y = (y_n)_n \in$

$E(Y)$, then $w_y(\sum_{n=1}^{\infty}\lambda_n^j\delta_j(e_n)) = |\langle x_j, y_j\rangle| \to 0$. Hence, $\sum_{j\geq 1}(\sum_{n=1}^{\infty}\lambda_n^j\delta_j(e_n))$ is convergent in $(E(X), Na)$. Consequently, every element $x \in E(X)$ can be written uniquely as $x = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\lambda_n^j\delta_j(e_n)$.

On the other hand, for every $n, j \geq 1$, let $f_{nj} : (E(X), Na) \to K$

$$x = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\lambda_n^j\delta_j(e_n) \mapsto \lambda_n^j.$$

Then, for all $n, j \geq 1$, $f_{nj} = f_n \circ \prod_j^X$, $\prod_j^X$ is $(\sigma(E(X), E(Y)), \sigma(X, Y))$-continuous by lemma before and $(f_n)_n$ is the weak Schauder basis associated with the basis $(e_n)_n$ ([7], Lemma 3, p. 402), then f_{nj} is continuous. □

Proposition 9. *If $E(X)$ is perfect, then for every $y \in E(X)^{\beta}$, $S(y)$ is compactoid in $\left(E(X)^{\beta}, Na\right)$.*

Proof. Let $y = (y_k)_k \in E(X)^{\beta}$, then for every $x = (x_k)_k \in E(X)$, there exists $n_0 \in \mathbb{N}$ such that $\sup_{i\geq n_0}|\langle x_i, y_i\rangle| \leq 1$. For every $i \geq 1$, put $y^i = \delta_i(y_i)$ and let $z = (\lambda_k y_k)_k$ with $(\lambda_k)_k \in \omega(K)$ such that for all $k \geq 1$, $|\lambda_k| \leq 1$. Then, for all $i \geq n_0$, $z - z^{[i]} \in B_{w_x}(0, 1)$, where $z^{[i]} = (z_1, z_2, \ldots, z_i, 0, \ldots)$ and so $z \in z^{[n_0]} + B_{w_x}(0, 1)$.

Then, $S(y) \subset c_0(y^1, \ldots, y^{n_0}) + B_{w_x}(0,1)$, which proves the proposition. □

Corollary 3. *If $E(X)$ is perfect, then the Na topology is weaker than the polar topology of $\mathcal{M}$-convergence, $\mathcal{M}$ standing for the family of all compactoid subsets of $\left(E(X)^{\beta}, Na\right)$.*

This inequality is not strict (see [17], Proposition 3.2.9, p. 212).

Proposition 10. *If* $(x^n)_n$ *is a Schauder basis of* $(E(X), \tau)$, *then for every* $j \geq 1$, $\left(\prod_j^X (x^n)\right)_n$ *is a Schauder basis of* $\left(X, \tau_{\prod_j}\right)$, *where* $\tau_{\prod_j}$ *is the final topology on* X *associated with the map* $\prod_j^X$.

Proof. Let $j \geq 1$, then for every $x \in X$, $\delta_j^X(x) \in E(X)$. Hence, there exists $(\lambda_n^j)_n \in \omega(K)$ such that $\delta_j^X(x) = \sum_{n=1}^{\infty} \lambda_n^j x^n$ in $(E(X), \tau)$. Then

$$x = \prod_j^X \left(\delta_j^X(x)\right) = \sum_{n=1}^{\infty} \lambda_n^j \prod_j^X (x^n), \tag{1}$$

since $\prod_j^X : (E(X), \tau) \to \left(X, \tau_{\prod_j}\right)$ is continuous.

Further, it is easy to see that the representation of x is unique and that the coefficient-functionals $f_n : \left(X, \tau_{\prod_j}\right) \to K$, $x = \sum_{n=1}^{\infty} \lambda_n^j \prod_j^X (x^n) \mapsto \lambda_n^j$ are continuous because $f_n \circ \prod_j^X$ is continuous

$$((E(X), \tau) \xrightarrow{\prod_j^X} \left(X, \tau_{\prod_j}\right) \xrightarrow{f_n} K,$$

$$\sum_{n=1}^{\infty} \lambda_n x^n \mapsto \sum_{n=1}^{\infty} \lambda_n \prod_j^X (x^n) \mapsto \lambda_n).$$

Thus $\left(\prod_j^X (x^n)\right)_n$ is a Schauder basis of $\left(X, \tau_{\prod_j}\right)$. □

Remark 3. For all $j \geq 1$, if $\prod_j^X$ is (τ, τ_j)-continuous, then $\left(\prod_j^X (x^n)\right)_n$ is a Schauder basis of (X, τ_j), where τ_j is the initial topology on X associated with the map δ_j^X. This is true in particular case when τ is solid and polar topology of $\mathcal{A}$-convergence, since $\prod_j^X (A^\circ) \subset \left(\prod_j^Y (A)\right)^\circ$ for all $A \in \mathcal{A}$.

Lemma 6. *Let* τ *be a locally K-convex and polar topology on* $E(X)$ *and* $j \geq 1$, *such that for all subset A of* $E(Y)$, *which is* $E(X)$*-closed,* $\prod_j^Y(A)$ *is X-closed. Then if* τ *is compatible with the duality* $\langle E(X), E(Y)\rangle$, *then* τ_j *is compatible with the duality* $\langle X, Y\rangle$.

Proof. τ is a polar topology of $\mathcal{A}$-convergence, $\mathcal{A}$ standing for the family of absolutely K-convex, $\sigma(E(Y), E(X))$-bounded and $E(X)$-closed subsets of $E(Y)$ ([3], Theorem 4.3, p. 233). $\prod_j^Y$ is $(\sigma(E(Y), E(X)), \sigma(Y, X))$-continuous, then for all $A \in \mathcal{A}$, $\prod_j^Y(A)$ is absolutely K-convex, $\sigma(Y, X)$-bounded and X-closed, hence τ_j is compatible with the duality $\langle X, Y\rangle$ ([3], Theorem 4.3, p. 233). □

Theorem 2. *Let* τ *be a locally K-convex and polar topology on* $E(X)$, *which is compatible with the duality* $\langle E(X), E(Y)\rangle$ *and there is* $j \geq 1$, *such that* $\prod_j^X$ *is* (τ, τ_j)*-continuous and for all subset A of* $E(Y)$, *which is* $E(X)$*-closed,* $\prod_j^Y(A)$ *is X-closed, then if* $(x^n)_n$ *is a Schauder basis of* $(E(X), \tau)$, *then* $\left(\prod_j^X(x^n)\right)_n$ *is a Schauder basis of* $X_\sigma = (X, \sigma(X, Y))$.

Proof. By Remark 3, $\left(\prod_j^X(x^n)\right)_n$ is a Schauder basis of (X, τ_j). But, since τ_j is compatible with the duality $\langle X, Y\rangle$ (lemma before), we have $\sigma \leq \tau_j$. Therefore, $\left(\prod_j^X(x^n)\right)_n$ is a Schauder basis of $X_\sigma = (X, \sigma(X, Y))$.

□

Remark 4. Theorem before is true, if we replace the condition $\prod_j^X$ is (τ, τ_j)-continuous, by τ is solid.

3. Topologies on Sequence Spaces

Let Λ be a sequence space over a field K, the Λ dual of $E(X)$ (= E, for short) is the subset of $\omega(Y)$ defined as:

$$E^{\Lambda} = \{(y_n)_n \in \omega(Y) : (\langle x_n, y_n \rangle)_n \in \Lambda \text{ for all } (x_n)_n \in E\},$$

E is called Λ *perfect*, if $E^{\Lambda\Lambda} = E$.

We define also F^{Λ} for F a sequence space on Y.

The proof of the following proposition is elementary:

Proposition 11. (1) *For every scalar sequence space* Λ *containing* φ, $\varphi(X)^{\Lambda} = \omega(Y)$.

(2) *For every sequence space* E *on* X, *and every scalar sequence space* Λ,

(i) $E \subset E^{\Lambda\Lambda}$;

(ii) E^{Λ} *is* Λ *perfect*.

(3) *For every sequence space* E *on* X, *and every sequence spaces* Λ *and* Δ *on* K, *if* $\Lambda \subset \Delta$, *then* $E^{\Lambda} \subset E^{\Delta}$.

(4) *For every sequence spaces* E *and* F *on* X, *and every sequence space* Λ *on* K, *if* $E \subset F$, *then* $F^{\Lambda} \subset E^{\Lambda}$.

Let $(\mathcal{P})$ be a family of *n.a* seminorms, which defines the locally K-convex topology of scalar sequence space Λ, for all $y = (y_n)_n \in E^{\Lambda}$ and all $p \in (\mathcal{P})$, we define on E, the family of *n.a* seminorm p_y as follows: For all $x = (x_n)_n \in E$, $p_y(x) = p((\langle x_n, y_n \rangle)_n)$.

Then, for every non empty subset U of E^{Λ}, the family of *n.a* seminorms $(p_y)_{y \in U,\, p \in (\mathcal{P})}$ define an *n.a* locally K-convex topology on E denoted by $\sigma_{\Lambda}(E, U)$.

Example 1. (a) The space $\Lambda = cs$ (the space of all scalar sequences $x = (x_n)_n$, for which $\sum_{n \geq 1} x_n$ is convergent) coincides with the space $c_0(K)$, then $E^{\Lambda} = E^{\beta}$ and if we equip Λ with the *n.a* seminorm

$p((x_n)_n) = \left|\sum_{n=1}^{+\infty} x_n\right|$, the $\sigma_{cs}(E, U)$-topology is defined by the $n.a$ seminorms $(v_y)_{y\in U}$, where $U \subset E^{\beta}$ and $\sigma_{cs}\left(E, E^{\beta}\right) = \sigma\left(E, E^{\beta}\right)$. If we equip Λ with the usual maximum norm, we obtain that $\sigma_{\Lambda}\left(E, E^{\Lambda}\right) = Na$. So all the study carried out in the paper about the spaces $\left(E, \sigma_{\Lambda}\left(E, E^{\Lambda}\right)\right)$ can be applied to the space $(E(X), Na)$.

(b) If $\Lambda = bs$ (the space of all scalar sequences $x = (x_i)_i$, for which $\sup_n|\sum_{k=1}^{n} x_k|$ is finite) equipped with the $n.a$ seminorm $p((x_k)_k) = \sup_n|\sum_{k=1}^{n} x_k|$, then E^{Λ} is the γ-dual of E, and the $\sigma_{bs}(E, U)$-topology is defined by the family of $n.a$ seminorms $(w_y)_{y\in U}$, where $U \subset E^{\gamma}$ and we have $\sigma_{bs}(E, U) = \sigma_{\gamma}(E, U)$ topology studied by Garling in the scalar and classical case [2], we have also $\Lambda = m(K)$ and the norm p coincides with the usual supremum norm considered in $m(K)$ and $\sigma_{bs}(E, U) = Na$, if $U \subset E^{\beta}$.

For every $y \in E(X)^{\Lambda}$, we define the linear map:

$$f_y : E \to \Lambda,$$

$$x = (x_n)_n \mapsto (\langle x_n, y_n \rangle)_n.$$

Proposition 12. *For every $y \in U$, the linear map f_y is $(\sigma_{\Lambda}(E,U), \tau)$-continuous*; *where τ is the topology on Λ.*

Proof. Obvious. □

Proposition 13. *If Λ is a K-space, then $(E, \sigma_{\Lambda}(E, U))$ is a K-space.*

Proof. Obvious. □

Remarks 5. (a) A net $(x^i)_{i\in I}$ is a Cauchy net in $(E, \sigma_{\Lambda}(E, U))$, if and only if for every $y = (y_n)_n \in U$, $(x^i y)_{i\in I}$ is a Cauchy net in Λ.

(b) A net $(x^i)_{i\in I}$ converges to x in $(E, \sigma_\Lambda(E, U))$, if and only if for every $y = (y_n)_n \in U$, the net $(x^i y)_{i\in I}$ converges to xy in Λ; where $zt = f_t(z) = (\langle z_n, t_n \rangle)_n$ for every $t = (t_n)_n \in U$ and every $z = (z_n)_n \in E$.

We find the definitions given by Hassani and Babahmed for sequences in ([4], Definition 2.1, p. 114).

Proposition 14. *If* $\varphi \subset \Lambda$ *and* Λ *is an AK-space, then* $(E, \sigma_\Lambda(E, U))$ *is an AX-space.*

Proof. Obvious. □

Proposition 15. *If* Λ *is an AK-space such that* $\varphi \subset \Lambda$ *and* $(E, \sigma_\Lambda(E, U))$ *is sequentially complete, then E is* Λ*-perfect and* $E = U^\Lambda$.

Proof. $U \subset E^\Lambda \Rightarrow E \subset E^{\Lambda\Lambda} \subset U^\Lambda$.

Conversely, let $x = (x_n)_n \in U^\Lambda$, then for all $n \geq 1$, $x^{[n]} \in E$ because $\varphi(X) \subset E$.

For every $y \in U$, the sequence $\left((xy)^{[n]}\right)_n$ converges to xy in Λ, then $\left(x^{[n]}\right)_n$ is a Cauchy sequence in $(E, \sigma_\Lambda(E, U))$, which is sequentially complete, therefore the sequence $\left(x^{[n]}\right)_n$ converges to x in $(E, \sigma_\Lambda(E, U))$ and so $x \in E$.

Hence, $E = U^\Lambda = E^{\Lambda\Lambda}$. □

Theorem 3. $E(X)$ *is perfect, if and only if* $\varphi(x) \subset E(X)$ *and* $E(X)$ *is weakly sequentially complete.*

Proof. The necessary condition is Lemma 1.

Sufficiency, let $\Lambda = c_0(K)$ endowed with the usual maximum norm, then Λ is an *AK*-space, $\varphi \subset \Lambda$ and $\sigma_\Lambda\left(E, E^\beta\right) = Na$ (Example 1(a) of p. 206). E is σ-sequentially complete, then (E, Na) is sequentially complete ([3], Theorem 3.2, p. 230). Therefore, the conclusion holds by proposition before. □

Theorem 4. *Let* Λ *be an SPK-space and contains the space* φ. *Then*:

(i) *If E is* Λ*- perfect, then E is* $\sigma_\Lambda\left(E, E^\Lambda\right)$*- sequentially complete.*

(ii) E^Λ *is* $\sigma_\Lambda\left(E^\Lambda, E\right)$*- sequentially complete.*

Proof. (i) Let $(x^n)_n$ be a Cauchy-sequence in $\left(E, \sigma_\Lambda\left(E, E^\Lambda\right)\right)$, then for all $y \in E^\Lambda$ and all $j \geq 1$, $(\prod_j \circ f_y(x^n))_n$ is a Cauchy-sequence in K and since $\varphi(Y) \subset E^\Lambda$, we have for all $y \in Y$ and all $j \geq 1$, $\left(\left\langle x_j^n, y\right\rangle\right)_n$ is a Cauchy-sequence in K. Therefore, for every $j \geq 1$, there exists $x_j \in X$ such that the sequence $\left(x_j^n\right)_n$ converges to x_j in $X_\sigma = (X, \sigma(X, Y))$ ([4], Lemma 2.3, p. 115).

Let $x = (x_j)_j$, then for every $y \in E^\Lambda$, we have $(x^n y)_n \subset \Lambda$ and for all $j \geq 1$, the sequence $(\prod_j((x^n - x)y))_n$ converges to 0 in K and so $(x^n y)_n$ converges to xy in Λ. Hence, $xy \in \Lambda$ so $x \in E^{\Lambda\Lambda} = E$, and $(x^n)_n$ converges to x in $\left(E, \sigma_\Lambda\left(E, E^\Lambda\right)\right)$.

(ii) E^Λ is Λ- perfect, then it suffices to use (i). □

Remark 6. If $\Lambda = cs$ or bs or l^1, we find Theorem 2.1 of ([4], Theorem 2.1, p. 117).

Theorem 5. *Let* Λ *be a PK-space, which contains the space* φ *and* X_σ *be complete, then*:

(i) *If E is* Λ*- perfect, then* $\left(E, \sigma_\Lambda\left(E, E^\Lambda\right)\right)$ *is complete.*

(ii) $\left(E^\Lambda, \sigma_\Lambda\left(E^\Lambda, E\right)\right)$ *is complete.*

Proof. (i) Let $(x^\alpha)_{\alpha \in I}$ be a Cauchy-net in $\left(E, \sigma_\Lambda\left(E, E^\Lambda\right)\right)$, then for all $y \in Y$ and all $j \geq 1$, $\left(\left\langle x_j^\alpha, y\right\rangle\right)_\alpha$ is a Cauchy-net in K. Therefore, for

every $j \geq 1$, there exists $x_j \in X$ such that the sequence $\left(x_j^{\alpha}\right)_{\alpha}$ converges to x_j in X_σ (X_σ is complete). Then $(x^{\alpha})_{\alpha \in I}$ converge to $x = (x_j)_j$ in $\left(E, \sigma_\Lambda\left(E, E^\Lambda\right)\right)$.

(ii) E^Λ is Λ-perfect, then it suffices to use (i). □

Let A be a subset of E, then A is $\sigma_\Lambda(E, U)$-bounded, if and only if for every $y \in U$, yA is bounded in Λ. The α-boundedness, β-boundedness, and γ-boundedness of ([4], p. 118) are a particular case of $\sigma_\Lambda(E, U)$-boundedness.

A subset A of E is called *strongly bounded*, if for every $\sigma_\Lambda\left(E^\Lambda, E\right)$-bounded subset B of E^Λ, AB is bounded in Λ, where

$$AB = \{xy = (\langle x_n, y_n \rangle)_n / x = (x_n)_n \in A \text{ and } y = (y_n)_n \in B\}.$$

$$AB = \bigcup_{y \in B} f_y(A).$$

It is obvious that strongly bounded sets are $\sigma_\Lambda(E, U)$-bounded.

Proposition 16. *If Λ is an SPK-space, then all $\sigma_\Lambda\left(E, E^\Lambda\right)$-bounded subset in E is strongly bounded in E.*

Proof. Suppose A was a $\sigma_\Lambda\left(E, E^\Lambda\right)$-bounded subset of E, which is not strongly bounded. Then, there would be a $\sigma_\Lambda\left(E^\Lambda, E\right)$-bounded subset B of E^Λ and an *n.a* seminorm $p \in (\mathcal{P})$ such that $p(AB)$ is not bounded. Let $(a^n)_n$ be a sequence in A and $(b^n)_n$ be a sequence in B such that for every $n \geq 1$,

$$p(a^n b^n) \succ \varrho^{2n}, \tag{2}$$

where ϱ is the number defined in ([18], p. 251).

An increasing subsequence of integers $N_1, N_2, N_3, \ldots$ can be defined by induction such that:

For every $i \geq 2$ and every $n \in \mathbb{N} : 1 \leq n \leq N_{i-1}$,

$$\varrho^{-N_i} \sup_{b \in B} p(a^n b) \prec \varrho^{-i}, \tag{3}$$

$$\varrho^{-N_i} \sup_{a \in A} p(ab^n) \prec \varrho^{-i}. \tag{4}$$

Let $(\lambda_n)_{n \in \mathbb{Z}} \subset K$ such that for all $n \in \mathbb{Z}$, $|\lambda_n| = \varrho^n$ ([18], p. 251), then the series $\sum_{i \geq 1} \lambda_{-N_i} b^{N_i}$ converges to a $b \in E^\Lambda$ with respect to the $\sigma_\Lambda\left(E^\Lambda, E\right)$ topology. This is because E^Λ is $\sigma_\Lambda\left(E^\Lambda, E\right)$- sequentially complete by Theorem 4 (ii) and for each seminorm $q \in (\mathcal{P})$ and each $x \in E$, we have:

For all $m \geq 1$, $\sum_{i=1}^{m} q\left(\lambda_{-N_i} x b^{N_i}\right)$

$$= \sum_{i=1}^{m} \varrho^{-N_i} q\left(x b^{N_i}\right)$$

$$\leq \sum_{i=1}^{m} \varrho^{-N_i} \sup_{b \in B} q(xb)$$

$$\leq \left(\sum_{i=1}^{+\infty} \varrho^{-i}\right) \sup_{b \in B} q(xb)$$

$$\prec +\infty.$$

For all $n \geq 1$, let $a_n = a^{N_n}$, then

$$p(a_n b) = p\left(\sum_{i=1}^{+\infty} \lambda_{-N_i} a^{N_n} b^{N_i}\right)$$

$$\geq p\left(\lambda_{-N_n} a^{N_n} b^{N_n}\right) - \sum_{i=1}^{n-1} \varrho^{-N_i} p\left(a^{N_n} b^{N_i}\right)$$

$$- \sum_{i=n+1}^{+\infty} \varrho^{-N_i} p\left(a^{N_n} b^{N_i}\right).$$

But $\sum_{i=1}^{n-1} \varrho^{-N_i} p\left(a^{N_n} b^{N_i}\right) \prec \sum_{i=1}^{n-1} \varrho^{-i}$ by (4) and $\sum_{i=n+1}^{+\infty} \varrho^{-N_i} p\left(a^{N_n} b^{N_i}\right) \prec \sum_{i=n+1}^{+\infty} \varrho^{-i}$ by (3).

Therefore, $p(a_n b) \geq \varrho^{-N_n}\varrho^{2N_n} - \sum_{i=1}^{n-1}\varrho^{-i} - \sum_{i=n+1}^{+\infty}\varrho^{-i}$

$$\geq \varrho^{N_n} - \sum_{i=1}^{+\infty}\varrho^{-i} + \varrho^{-n}$$

$$\geq \varrho^{N_n} - \frac{\varrho^2}{\varrho - 1} + \varrho^{-n}.$$

But, this leads to a contradiction of the $\sigma_\Lambda\left(E, E^\Lambda\right)$- boundedness of A in E. □

For U a subset of E^Λ, the $\beta_\Lambda(E, U)$ topology on E is determined by the family of $n.a$ seminorms $(p_A)_{p\in(\mathcal{P}),\, A\in\mathcal{A}}$, where $\mathcal{A}$ standing for the family of the $\sigma_\Lambda\left(E^\Lambda, E\right)$- bounded and absolutely K-convex subsets of E^Λ and for every $A \in \mathcal{A}$ and every $x \in E$, $p_A(x) = \sup_{a\in A} p(xa)$. We define in the same way, the $\beta_\Lambda\left(E^\Lambda, U\right)$- topology for $U \subset E$. If $\Lambda = l^1$ (resp., *cs*; resp., *bs*), we find the seminorms α_B (resp., β_B; resp., γ_B) studied by Hassani and Babahmed ([4], p. 118).

Proposition 17. *Suppose that* $\varphi \subset \Lambda, \varphi(X) \subset E$, *and* $\left(E^\Lambda, \beta_\Lambda\left(E^\Lambda, E\right)\right)$ *is an AK-space. A sequence* $(x^n)_n$ *in E is* $\sigma_\Lambda\left(E, E^\Lambda\right)$- *Cauchy, if and only if it is* $\sigma_\Lambda\left(E, E^\Lambda\right)$- *bounded and for every* $j \geq 1$, *the sequence* $\left(x_j^n\right)_n$ *is* $\sigma(X, Y)$- *convergent in X.*

Proof. Let $(x^n)_n$ be a Cauchy sequence in $\left(E, \sigma_\Lambda\left(E, E^\Lambda\right)\right)$, then it is bounded and for every y in Y and every $j \in \mathbb{N}$, $\delta_j(y) \in E^\Lambda(\varphi \subset \Lambda)$ so $\left(\left\langle x_j^n, y\right\rangle\right)_n$ is a Cauchy sequence in K, which is complete, then $\left(x_j^n\right)_n$ is $\sigma(X, Y)$- convergent in X.

Conversely, let $(x^n)_n$ be a sequence in E, which is $\sigma_\Lambda\left(E, E^\Lambda\right)$- bounded and for every $j \geq 1$, the sequence $\left(x_j^n\right)_n$ is $\sigma(X, Y)$- convergent in X. Let $y = (y_n)_n \in E^\Lambda$, $\varepsilon \succ 0$ and $p \in (\mathcal{P})$. Since $\left(E^\Lambda, \beta_\Lambda\left(E^\Lambda, E\right)\right)$ is

an AX-space, there is n_0 such that $p_B\left(y - y^{[n]}\right) \prec \varepsilon$ for $n \geq n_0$, where $B = \{(x^n)_n\}$. On the other hand, $\left(x_j^n\right)_n$ is $\sigma(X, Y)$-convergent in X for all $j \geq 1$, then there exists $n_1 \in \mathbb{N}$ such that for all $m, n \geq n_1$,

$$p\left((x^n y)^{[n_0]} - (x^m y)^{[n_0]}\right) \prec \varepsilon.$$

Then for every $m, n \geq n_1$, we have:

$$\begin{aligned} p((x^n - x^m)y) &\leq \max\left(\begin{matrix} p\left(x^n y - (x^n y)^{[n_0]}\right), p\left((x^n y)^{[n_0]} - (x^m y)^{[n_0]}\right), \\ p\left((x^m y)^{[n_0]} - x^m y\right) \end{matrix}\right) \\ &\leq \max\left(p_B\left(y - y^{[n_0]}\right), \varepsilon\right) \\ &\leq \varepsilon. \end{aligned}$$

□

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