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FRACTIONAL FLOW EQUATIONS: A MODEL FOR PRESSURE DEFICIT IN AN OIL WELL

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Abstract

This article presents a novel system of flow equations that models the pressure deficit of a reservoir considered as a triple continuous medium formed by the rock matrix, vugular medium and fracture. In non- conventional reservoirs, the velocity of the fluid particles is altered due to physical and chemical phenomena caused by the interaction of the fluid with the medium, this behaviour is defined as anomalous. A more exact model can be obtained with the inclusion of the memory formalism concept that can be expressed through the use of fractional derivatives. Using Laplace transform of the Caputo fractional derivative and Bessel functions, a semi-analytical solution is reached in the Laplace space.

1. Introduction

The modelling of an oil reservoir is of paramount importance, since it allows to take decisions that can improve the extraction of hydrocarbons. Over the years, various approaches have been used to obtain a more comprehensive model of the fluid behaviour within the reservoir through the interpretation of pressure deficit. Here is presents a new semi-analytical solution of a system of partial fractional differential equations of flows coupled in a system with triple porosity and triple permeability using Caputo type temporary fractional derivatives. This system is reduced by Laplace transforms to one where each equation can be see as a Bessel type of the second type. This system is solved in the Laplace space using algebraic method, the necessary definitions and concepts are given previously such that fractional calculus, Bessel equations and Laplace transforms.

2. Previous Works

As mentioned above, many approximations have been developed to model correctly the fluid behaviour in an oil reservoir. Warren & Root in [1] proposed equations where they considered that matrix and the fractures systems had an Euclidean structure.

From this approach, Chang and Yortsos in [2] presented a formulation where a fractal fracture in an Euclidean matrix is considered. Camacho-Velazquez et al. in [3] take up this issue again, and proposed a model of double porosity in naturally fractured vugular sites. In their model they make use of a fractional order Caputo-type derivative, which has already been proposed in flow models by Metzler et al. as can be seen in [4]. Camacho-Velazquez et al. in their article [5] generalized a classical flow equation to one that considers the medium as the union of two or three porous media (fractured, vugular and matrix media for the latter case).

Classical models are constructed from the principle of conservation of the mass of each of the fluids involved in the same media and the Darcy's law for fluid in porous media as illustrated by Ertekin in [6]. Martínez-Salgado et al. [7] developed a model for a triadic medium with triple porosity and triple permeability using Caputo fractional derivatives for time and Weyl fractional derivatives for space. In [8-12], fractional derivatives were used in the search of complex roots of non-linear equations by the fractional iterative methods. Martínez et al. solved in [13] and [14] numerically diffusion equations with Riesz fractional derivative in space by radial-based functions. The equation with fractional derivative uses a fractional Darcy's law deduced by Le Mehaute as seen in [15] in the same way that appears in Raghavan's article, [16], where the order of fractional derivatives is expressed in terms of the Hausdorff dimension of the medium. Furthermore, taking the fractional Darcy's law, the order can be obtained from data [17] or through inverse problems [18].

3. Methods

The classical model assumes that the properties of rock and fluids are stable, the hydrodynamics of the flow of fluids in the porous medium is adequately described by Darcy's law, the geometry of the reservoir is of the Euclidean type.

The basis of the model is found in the continuity equation and Darcy's law for a flow through a porous medium, as illustrated in [19] and [20], these equations can be expressed as

$$\frac{\partial(\rho\theta)}{\partial t} + \nabla \cdot p(\rho q) = \rho \Upsilon,$$

$$q = -\frac{1}{\mu} k(p)(\nabla p - \rho g \nabla D), \tag{1}$$

where θ is the volumetric content of the fluid; $q = (q_1, q_2, q_3)$ is the flow of Darcy; with its spatial components (x, y, z), t is the time; ρ is the density of the fluid; μ is the dynamic viscosity of the fluid; g is the gravitational acceleration, Υ is a source term and represents a volume contributed by fluid per unit volume of porous medium in the unit of time; p is the pressure; D is the depth as a function of spatial coordinates, generally assimilated to the vertical coordinate z; k is the permeability tensor of the porous medium, $\theta(p)$ and $k(p)$ are characteristics of the fluid dynamics of the medium.

The general equation of fluid transfer is obtained by combining the equations as in [21]:

$$\frac{\partial(\rho\theta)}{\partial t} = \nabla \cdot p[\frac{\rho}{\mu} k(p)(\nabla p - \rho g \nabla D)] + \rho \Upsilon.$$

This differential equation contains two dependent variables, namely, the moisture content θ and the fluid pressure p, which are related. For this reason, the saturation $S(p)$ is defined as

$$\theta(p) = \phi(p) S(p), \tag{2}$$

where ϕ is the total porosity of the medium. The specific capacity is defined by

$$C(p) = \frac{d(\rho\phi S)}{dp} = \phi S \frac{d\rho}{dp} + \rho S \frac{d\phi}{dp} + \rho\phi \frac{dS}{dp}, \tag{3}$$

in consequence

$$\frac{\partial(\rho\theta)}{\partial t} = C(p)\frac{\partial p}{\partial t}. \tag{4}$$

In the classical model, the particles move smoothly in the medium, giving a relationship mean square displacement linear with time, this movement is said to be Markovian because it does not depend on previous positions and is represented by a Gaussian diffusion equation. This relationship does not exist in the anomalous diffusion that is altered due to particle entrapment, pore throat closure, fractures, which gives a non-linear relationship, it is said that this is non-Markovian because the movement depends on the previous time or previous position Equation (6).

$$\langle x^2 \rangle \propto t \tag{5}$$

$$\langle x^2 \rangle \propto t^{\alpha};\ \alpha \neq 1 \tag{6}$$

in the case that $0 < \alpha < 1$ it is said to be a subdiffusion, and if $\alpha > 1$ is a superdiffusion [4]. The memory formalism can be expressed as a modification of Darcy's law, through the fractional derivative Caputo, see (14), this equation can explain the diffusion in a case where the effects of past events exist [22]:

$$v(x, t) = -\frac{K_\alpha}{\mu}\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}\frac{\partial p(x, t)}{\partial x}, \tag{7}$$

α is a parameter, Raghavan and Chen [22] has directly addressed estimates for α : $0.56 < \alpha < 0.91$ and $0.77 < \alpha < 0.94$ for fractures and matrix of rock, respectively. Considering cylindrical coordinates and assuming the well axis passes through the origin, the application of the conservation of mass principle to a control volume:

$$\frac{1}{r}\frac{\partial}{\partial r}v(r, t) = \phi c\frac{\partial}{\partial t}p(r, t), \tag{8}$$

where ϕ is the porosity of the medium, and r is the distance from the well center. On substituting the left-hand side of Equation (8) for $v(r, t)$; we obtain the partial differential equation for transient diffusion under subdiffusive flow to be

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\widetilde{k}}{\mu}\frac{1}{r}\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}\frac{\partial p(r, t)}{\partial r}\right) = \phi c\frac{\partial}{\partial t}p(r, t). \tag{9}$$

Since $\frac{\widetilde{k}}{\mu}$ it does not depend on r, the above equation becomes

$$\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\widetilde{k}}{\mu}\frac{1}{r}\frac{\partial p(r, t)}{\partial r}\right)\right) = \phi c\frac{\partial}{\partial t}p(r, t), \tag{10}$$

then applying the Riemann-Liouville integral: ${}_tJ^{\alpha-1}$ to both sides, applying the Equation (16) is obtained:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\widetilde{k}}{\mu}\frac{1}{r}\frac{\partial p(r, t)}{\partial r}\right) = \phi c\frac{\partial^{\alpha}}{\partial t^{\alpha}}p(r, t). \tag{11}$$

3.1. Fractional calculus

There are several definitions of fractional derivative: the most widespread is that of Riemann-Liouville, we will only provide the definition of the derivative Caputo because it is the one we will use, a very complete reference in the area can be consulted in the book by Baleanu et al. [23]. The left-sided fractional integral of Riemann-Liouville of order α is defined as

$${}_tJ^{\alpha}f(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau,\ \alpha > 0, \tag{12}$$

where the convention ${}_tJ^0 = I$ (identity operator) and the semigroup property:

$$ {}_tJ^{\alpha}{}_tJ^{\beta} = {}_tJ^{\beta}{}_tJ^{\alpha} = {}_tJ^{\alpha+\beta}, \ \alpha, \beta \geq 0. \tag{13}$$

We define the left-sided Caputo fractional derivative of order $\alpha > 0$ as the operator ${}_tD_*^{\mu}$ such that ${}_tD_*^{\mu}f(t) := {}_tJ^{m-\mu}{}_tD^m f(t)$, with $D = \frac{d}{dx}$ hence

$$ {}_tD_*^{\mu}f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\mu)}\displaystyle\int_0^t \frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\mu+1-m}} \ m-1 < \mu < m, \\ \dfrac{d^m}{dt^m} f(t), \ \mu = m, \end{cases} \tag{14}$$

with $m = \lfloor \alpha \rfloor + 1$ if $\alpha \notin \mathbb{N}$, $m = \alpha$ if $\alpha \in \mathbb{N}$.

In particular, when $0 < \alpha < 1$, then

$$ {}_tD_*^{\alpha} = {}_tJ^{1-\alpha}D. \tag{15}$$

The above equation implies

$$ {}_tJ^{\alpha}{}_tD_*^{\alpha}f(t) = {}_tJ^{\alpha}{}_tJ^{1-\alpha}Df(t) = JDf(t) = f(t) - f(0). \tag{16}$$

The fractional derivative Caputo satisfies the property that it is zero when is applied to a constant. Another important property is that you can apply a Laplace transform:

$$ \mathcal{L}\{{}_tD_*^{\mu}f(t); \, s\} = s^{\mu}\tilde{f}(s) - \sum_{k=0}^{m-1} s^{\mu-1-k}f^{(k)}(0^+), $$

$$ m - 1 < \mu < k, \tag{17}$$

where

$$ \tilde{f}(s) = \mathcal{L}\{f(t); \, s\} = \int_0^{\infty} e^{-st}f(t)dt, \ s \in \mathbb{C}, $$

and

$$f^{(k)}(0^+) := \lim_{t \to 0^+} f(t).$$

3.2. Bessel functions

The following differential equation of second order

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} - (z^2 + \nu^2) y = 0, \tag{18}$$

where ν is a real constant is called the modified Bessel equation, the solutions to the above equation are called modified Bessel functions which take the following form:

$$K_\nu(z) = \left(\frac{\pi}{2}\right) \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \tag{19}$$

where $I_\nu(z)$ are the modified Bessel functions of the first type, it is noted that I_ν and $I_{-\nu}$- form a set of solutions for the Equation (18) and the Equation (19) it is known as the modified Bessel function of the second type. Some properties of the modified Bessel function of the second type are:

$$\frac{d}{dz} K_\nu(\alpha z) = -\alpha K_{\nu-1}(\alpha z) - \frac{\nu}{z} K_\nu(\alpha z), \tag{20}$$

$$\frac{d}{dz} K_\nu(\alpha z) = -\alpha K_{\nu+1}(\alpha z) + \frac{\nu}{z} K_\nu(\alpha z). \tag{21}$$

3.3. Flow equation (fractional time derivative)

Let's assume the Equation (11) that represents the fluid, where the medium is a whole, so we have

$$\phi c_\alpha \frac{\partial^\alpha p}{\partial t^\alpha} = \frac{k}{\mu} \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial p}{\partial r}\right), \tag{22}$$

where the derivative expression on the left denotes the fractional derivative Caputo of order $\alpha \in \mathbb{R}$, with dimensionless variables, the Equation (22) is

$$\phi c_{D\alpha} \frac{\partial^{\alpha} p_D}{\partial t_D^{\alpha}} = \kappa_D \frac{1}{r} \frac{\partial}{\partial r_D} \left(r_D \frac{\partial p_D}{\partial r_D} \right), \tag{23}$$

where

$$p_D = \frac{2\pi hk(p_i - p)}{Q_0 B_0 \mu}, \; t_D = t \frac{k}{\phi c r_w^2 \mu}, \; r_D = \frac{r}{r_w}, \tag{24}$$

where ϕ represents the porosity (dimensionless), c represents the compressibility of the medium in units of Pa^{-1}, k represents the permeability of the medium with units of m^2, p represents the fluid pressure in the middle with units of Pa, μ is the viscosity of the fluid with units of $\mathrm{Pa} \cdot \mathrm{s}$, t represents the time in units of s, r represents the distance of the well in units of m, r_w is a reference parameter: well radius with units of m, h is the well thickness measured in m, p_i is the initial reservoir pressure, the value of Q_0 is the flow rate with units of $\mathrm{m}^3\mathrm{s}^{-1}$ and B_0 is the fluid factor (dimensionless).

3.4. Laplace transform

The Laplace transform applied to the Equation (23) gives the following result using the Equation (17)

$$u^{\alpha} \overline{p}_D = \frac{1}{r_D} \frac{\partial}{\partial r_D} \left(r_D \frac{\partial \overline{p}_D}{\partial r_D} \right), \; u > 0, \tag{25}$$

where $\overline{p}_D(t_0) = 0$, because $p = p_i$, in $t = t_0$.

3.4.1. Bessel functions

The spatial derivatives to be developed in the Equation (25) present the following form:

$$r_D^2 \frac{\partial^2 \overline{p}_D}{\partial r_D^2} + r_D \frac{\partial \overline{p}_D}{\partial r_D} - r_D^2 u^{\alpha} \overline{p}_D = 0, \tag{26}$$

which is a Bessel equation, so the solution is

$$\overline{p}_D = AK_0(\beta r_D). \tag{27}$$

By substituting the Equation (27) into the Equation (25) and considering the Equations (20) and (21) to find the value of β, we have

$$\beta = \pm\sqrt{u^{\alpha}},\ u > 0. \tag{28}$$

The Equation (27) when considering the value of β, Equation (28) is

$$\overline{p}_D = AK_0(r_D\sqrt{u^{\alpha}}). \tag{29}$$

In the Equation (29) $\beta = -\sqrt{u^{\alpha}}$ is discarded because the modified Bessel function of second type is not defined for negative values.

3.4.2. Border conditions

To find the solution to the Equation (23), the following boundary condition is considered:

$$r_D \frac{\partial \overline{p}_D}{\partial r_D}\bigg|_{r_D=1} = -\frac{1}{u}. \tag{30}$$

Substituting the Equation (29) in Equation (30) generates the following:

$$A = \frac{1}{u}[\sqrt{u^{\alpha}}K_1(\sqrt{u^{\alpha}})]^{-1}, \tag{31}$$

$$\overline{p}_D = \frac{1}{u}[\sqrt{u^{\alpha}}K_1(\sqrt{u^{\alpha}})]^{-1}K_0(r_D\sqrt{u^{\alpha}}). \tag{32}$$

Therefore, the value of the pressure at the boundary of the well $(r_D = 1)$ is in the space of Laplace:

$$\bar{p}_D\big|_{r_D=1} = \frac{1}{u}\left[\sqrt{u^\alpha}K_1(\sqrt{u^\alpha})\right]^{-1}K_0(\sqrt{u^\alpha}). \tag{33}$$

4. Flow Equation with Triple Porosity and Triple Permeability with Fractional Time Derivative

From the classical transfer equations, Martínez-Salgado et al. in [7] proposes a system of coupled flow equations with triple porosity and triple permeability, which have the following form.

We consider the following notation:

$$c_{s_1} := \frac{1}{\phi_{s_1}}\frac{\partial \phi_{s_1}}{\partial p_{s_1}}, \text{ with } s_1 = m, f, v,$$

$$\Delta_{s_1 s_2}(p) := p_{s_1} - p_{s_2},$$

$$\phi_m c_m \frac{\partial p_m}{\partial t} = \frac{k_m}{\mu}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_m}{\partial r}\right) + a_{mf}\Delta_{fm}(p) + a_{mv}\Delta_{vm}(p), \tag{34}$$

$$\phi_f c_f \frac{\partial p_f}{\partial t} = \frac{k_f}{\mu}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_f}{\partial r}\right) - a_{mf}\Delta_{fm}(p) + a_{fv}\Delta_{vf}(p), \tag{35}$$

$$\phi_v c_v \frac{\partial p_v}{\partial t} = \frac{k_v}{\mu}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_v}{\partial r}\right) - a_{mv}\Delta_{vm}(p) - a_{fv}\Delta_{vf}(p), \tag{36}$$

where ϕ_m, ϕ_f, ϕ_v represent the porosities of the soil matrix, the fractured medium, and the vugular medium respectively in units of $\mathrm{m}^3/\mathrm{m}^3$; c_m, c_f, c_v represent the compressibility in each porous medium in units of pa^{-1}; k_m, k_f, k_v represent the permeability of each porous medium with units of m^2; p_m, p_f, p_v represent the fluid pressure in each medium porous with units of Pa; μ is the viscosity of the fluid with units of $\text{Pa} \cdot \text{s}$; a_{mf}, a_{mv}, a_{fv} are the transfer terms in the matrix-fracture, matrix-void, and fracture-vortex interfaces respectively with units of $\text{Pa}^{-1}\text{s}^{-1}$, t represents the time in units of s and r represents the distance to the well in units of m.

4.1. A dimensionalization of the flow equations

In order to handle the Equations (34), (35) and (36) in an easier way, the dimensionlessness of variables is applied. The dimensionlessness is a technique commonly used to make the parameters or variables in an equation have no units, to rank the possible values of a variable or a constant in order that its value is known and thus more manipulable. The system of Equations (34), (35) and (36) takes, after applying the dimensionlessness, the following form:

$$\omega_m \frac{\partial p_{Dm}}{\partial t_D} = \kappa_m \frac{1}{r_d} \frac{\partial}{\partial r_D}\left(r_D \frac{\partial p_{Dm}}{\partial r_D} \right) + \lambda_{mf} \Delta_{fm}(p_D) + \lambda_{mv} \Delta_{vm}(p_D), \tag{37}$$

$$\omega_f \frac{\partial p_{Df}}{\partial t_D} = \kappa_f \frac{1}{r_d} \frac{\partial}{\partial r_D}\left(r_D \frac{\partial p_{Df}}{\partial r_D} \right) - \lambda_{mf} \Delta_{fm}(p_D) + \lambda_{fv} \Delta_{vf}(p_D), \tag{38}$$

$$\omega_v \frac{\partial p_{Dv}}{\partial t_D} = \kappa_v \frac{1}{r_d} \frac{\partial}{\partial r_D}\left(r_D \frac{\partial p_{Dv}}{\partial r_D} \right) - \lambda_{mv} \Delta_{vm}(p_D) - \lambda_{fv} \Delta_{vf}(p_D), \tag{39}$$

with

$$\omega_m = 1 - \omega_f - \omega_v, \tag{40}$$

$$\kappa_m = 1 - \kappa_f - \kappa_v, \tag{41}$$

where

$$\omega_{s_1} = \frac{\phi_{s_1} c_{s_1}}{\phi_m c_m + \phi_f c_f + \phi_v c_v}, \; s_1 = f, v, \tag{42}$$

$$r_D = \frac{r}{r_w}, \; \kappa_{s_1} = \frac{\kappa_{s_1}}{\kappa_m + \kappa_f + \kappa_v}, \; s_1 = f, v, \tag{43}$$

$$\lambda_{s_1 s_2} = \frac{a_{s_1 s_2} \mu r_w^2}{k_m + k_f + k_v}, \; s_1 s_2 = mf, \; mv, \; fv, \tag{44}$$

$$p_{Dj} = \frac{2\pi h (k_m + k_f + k_v)(p_i - p_j)}{Q_0 B_0 \mu}, \tag{45a}$$

$$t_D = \frac{t(k_m + k_f + k_v)}{\mu r_w^2 (\phi_m c_m + \phi_f c_f + \phi_v c_v)}. \tag{45b}$$

The Equations (42)-(45) represent the dimensionless variables, it can be verified that these variables do not have units; in the Equation (44) the value of r_w is a reference parameter, in this case the radius of the well, in order that the variable r_D has the minimum value equal to 1, with units of m. In the Equation (45), the value of h represents the thickness of the oil field with units of m; p_j is the pressures in the different porous media, where $j = m, f, v$; p_i is the initial pressure in the field; the value of Q_0 is the flow rate with units of $\mathrm{m}^3\mathrm{s}^{-1}$ and B_0 is the fluid formation factor (dimensionless).

4.2. The system with fractional derivative

From the system of equations with dimensionless variables (37)-(39), using the equation of flow with fractional time derivative (23), we express a system with a fractional time derivative:

$$\omega_m \frac{\partial^{\beta_m} p_{Dm}}{\partial t_D^{\beta_m}} = \kappa_m \frac{1}{r_D} \frac{\partial}{\partial r_D} \left(r_D \frac{\partial p_{Dm}}{\partial r_D} \right) + \lambda_{mf} \Delta_{fm}(p_D) + \lambda_{mv} \Delta_{vm}(p_D), \tag{46}$$

$$\omega_f \frac{\partial^{\beta_f} p_{Df}}{\partial t_D^{\beta_f}} = \kappa_f \frac{1}{r_d} \frac{\partial}{\partial r_D} \left(r_D \frac{\partial p_{Df}}{\partial r_D} \right) - \lambda_{mf} \Delta_{fm}(p_D) + \lambda_{fv} \Delta_{vf}(p_D), \tag{47}$$

$$\omega_v \frac{\partial^{\beta_v} p_{Dv}}{\partial t_D^{\beta_v}} = \kappa_v \frac{1}{r_d} \frac{\partial}{\partial r_D} \left(r_D \frac{\partial p_{Dv}}{\partial r_D} \right) - \lambda_{mv} \Delta_{vm}(p_D) - \lambda_{fv} \Delta_{vf}(p_D), \tag{48}$$

where the variables shown in the Equations (46)-(48) have the same meaning as the Equations (42)-(44). By means of the Laplace transform and with the use of the Equation (17), the following system is reached:

$$\omega_m u^{\beta_m} \overline{p}_{Dm} = \kappa_m \frac{1}{r_D} \frac{\partial}{\partial r_D} \left(r_D \frac{\partial \overline{p}_{Dm}}{\partial r_d} \right) + \lambda_{mf} \Delta_{fm}(\overline{p}_D) + \lambda_{mv} \Delta_{vm}(\overline{p}_D), \tag{49}$$

$$\omega_f u^{\beta_f} \overline{p}_{Df} = \kappa_f \frac{1}{r_D} \frac{\partial}{\partial r_D} \left(r_D \frac{\partial \overline{p}_{Df}}{\partial r_d} \right) - \lambda_{mf} \Delta_{fm}(\overline{p}_D) + \lambda_{fv} \Delta_{vf}(\overline{p}_D), \tag{50}$$

$$\omega_v u^{\beta_v} \overline{p}_{Dv} = \kappa_v \frac{1}{r_D} \frac{\partial}{\partial r_D} \left(r_D \frac{\partial \overline{p}_{Dv}}{\partial r_d} \right) - \lambda_{mv} \Delta_{vm}(\overline{p}_D) - \lambda_{fv} \Delta_{vf}(\overline{p}_D), \tag{51}$$

where the variables shown in the Equations (46)-(48) have the same meaning as those where $\overline{p}_{Dm}$, $\overline{p}_{Df}$ and $\overline{p}_{Dv}$ represent the Laplace transforms of the variables p_{Dm}, p_{Df} and p_{Dv}. After developing Equations (49)-(51), it is easy to see that they comply with the form of a Bessel equation and therefore their solutions, as in the case $\beta_m = \beta_f = \beta_v = 1$, they are

$$\overline{p}_{Dm} = AK_0(\alpha r_D), \tag{52}$$

$$\overline{p}_{Df} = BK_0(\alpha r_D), \tag{53}$$

$$\overline{p}_{Dv} = CK_0(\alpha r_D). \tag{54}$$

In order to simplify the successive equations, the following terms are defined:

$$m_1(u) = u^{\beta_m}\omega_m + \lambda_{mf} + \lambda_{mv}, \tag{55a}$$

$$m_2 = \lambda_{mf}, \tag{55b}$$

$$m_3 = \lambda_{mv}, \tag{55c}$$

$$m_4(u) = u^{\beta_f}\omega_f + \lambda_{mf} + \lambda_{fv}, \tag{56a}$$

$$m_5 = \lambda_{fv}, \tag{56b}$$

$$m_6(u) = u^{\beta_v}\omega_v + \lambda_{mv} + \lambda_{fv}. \tag{56c}$$

As a result of replacing the Equations (52)-(54) in the system shown in (49)-(51) and making use of the definitions shown by (55)-(56), we have the following:

$$K_0(\alpha r_D)\{A[\kappa_m\alpha^2 - m_1] + B_{m_2} + C_{m_3}\} = 0, \tag{57}$$

$$K_0(\alpha r_D)\{Am_2 + B[\kappa_f\alpha^2 - m_4] + Cm_5\} = 0, \tag{58}$$

$$K_0(\alpha r_D)\{Am_3 + Bm_5 + C[\kappa_v\alpha^2 - m_6]\} = 0. \tag{59}$$

Since the modified second-species Bessel functions have an asymptotic behaviour, that is, they never take the value of zero, then the system shown in the Equations (57)-(59) can be expressed as follows:

$$\begin{bmatrix} \kappa_m\alpha^2 - m_1 & m_2 & m_3 \\ m_2 & \kappa_f\alpha^2 - m_4 & m_5 \\ m_3 & m_5 & \kappa_v\alpha^2 - m_6 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{60}$$

The Equation (60) is used to find the values of A, B, and C. Keeping this in mind, we note two principal cases: the determinant of the matrix 3×3 or it is different from zero. The first case gives us the trivial solution $A = B = C = 0$. Linear algebra is known to have a solution to the equation if and only if the determinant equals zero. The second case, where the determinant is equal to zero, we obtain the equation of degree six that follows:

$$\kappa_m \kappa_f \kappa_v \alpha^6 - [\kappa_m(\kappa_f m_6 + \kappa_v m_4) + \kappa_f \kappa_v m_1]\alpha^4$$
$$+ [\kappa_m m_4 m_6 - \kappa_m m_5^2 + (\kappa_f m_6 + \kappa_v m_4) m_1 - \kappa_v m_2^2 - \kappa_f m_2^3]\alpha^2$$
$$- m_1 m_4 m_6 + m_1 m_5^2 + m_2^2 m_6 + 2 m_2 m_3 m_5 + m_3^2 m_4 = 0. \tag{61}$$

In the previous equation the powers of α are even, therefore it can be solved as an equation of degree three. This equation has three real roots. The general solutions of Equations (49)-(51), when incorporating the three real roots, are:

$$\bar{p}_{Dm} = A_1 D_1 K_0(\alpha_1 r_D) + A_2 D_2 K_0(\alpha_2 r_D) + A_3 D_3 K_0(\alpha_3 r_D), \tag{62}$$

$$\bar{p}_{Df} = B_1 D_1 K_0(\alpha_1 r_D) + B_2 D_2 K_0(\alpha_2 r_D) + B_3 D_3 K_0(\alpha_3 r_D), \tag{63}$$

$$\bar{p}_{Dv} = D_1 K_0(\alpha_1 r_D) + D_2 K_0(\alpha_2 r_D) + D_3 K_0(\alpha_3 r_D), \tag{64}$$

where the terms A_i, B_i with $i = 1, 2, 3,$ have the form

$$A_i = \frac{m_3(\kappa_f \alpha_i^2 - m_4) - m_2 m_5}{m_2^5 - [\kappa_m \alpha_i^2 - m_1][\kappa_f \alpha_i^2 - m_4]}, \tag{65}$$

$$B_i = \frac{-m_3 - A_i[\kappa_m \alpha_i^2 - m_1]}{m_2}, \tag{66}$$

where the terms D_1, D_2, and D_3 are obtained from the boundary conditions and $\alpha_1, \alpha_2, \alpha_3$ are the positive roots of $\alpha_1^2, \alpha_2^2, \alpha_3^2$. The values of D_1, D_2, and D_3 are obtained from the boundary conditions and they are equal to

$$\begin{aligned} D_1 = \frac{1}{u}\Bigg\{ & \alpha_1 E_1 K_1(\alpha_1) + \alpha_3 E_3 K_1(\alpha_3)\frac{(B_1 - 1)K_0(\alpha_1)}{(1 - B_3)K_0(\alpha_3)} \\ & + \frac{\left[(1 - A_1)K_0(\alpha_1) + (1 - A_3)\dfrac{(B_1 - 1)}{(1 - B_3)}K_0(\alpha_1)\right]}{\left[(A_2 - 1)K_0(\alpha_2) + (A_3 - 1)\dfrac{(B_2 - 1)}{(1 - B_3)}K_0(\alpha_2)\right]} \\ & \times \alpha_2 E_2 K_1(\alpha_2) + \alpha_3 E_3 K_1(\alpha_3)\frac{(B_2 - 1)K_0(\alpha_2)}{(1 - B_3)K_0(\alpha_3)}\Bigg\}^{-1}, \end{aligned} \tag{67a}$$

$$\begin{aligned} D_2 = \frac{1}{u}\Bigg\{ & \frac{\left[(A_2 - 1)K_0(\alpha_2) + (A_3 - 1)\dfrac{(B_2 - 1)}{(1 - B_3)}K_0(\alpha_2)\right]}{\left[(1 - A_1)K_0(\alpha_1) + (1 - A_3)\dfrac{(B_1 - 1)}{(1 - B_3)}K_0(\alpha_1)\right]} \\ & + \left[\alpha_1 E_1 K_1(\alpha_1) + \alpha_3 E_3 K_1(\alpha_3)\frac{(B_1 - 1)K_0(\alpha_1)}{(1 - B_3)K_0(\alpha_3)}\right] \\ & + \left[\alpha_2 E_2 K_1(\alpha_2) + \alpha_3 E_3 K_1(\alpha_3)\frac{(B_2 - 1)K_0(\alpha_2)}{(1 - B_3)K_0(\alpha_3)}\right]\Bigg\}^{-1}, \end{aligned} \tag{67b}$$

and

$$D_3 = \frac{1}{u}\left\{\alpha_1 E_1 K_1(\alpha_1)\frac{(1-B_3)K(\alpha_3)}{(B_1-1)K_0(\alpha_1)} + \alpha_3 E_3 K_1(\alpha_3)\right.$$

$$+\left[\frac{(1-A_1)(1-B_3)+(1-A_3)(B_1-1)}{(A_2-1)(1-B_3)+(A_3-1)(B_2-1)}\right]$$

$$\left.\times\left[\frac{\alpha_2 E_2(1-B_3)K_1(\alpha_2)K_0(\alpha_3)+\alpha_3 E_3(B_2-1)K_1(\alpha_3)K_0(\alpha_2)}{(B_1-1)K_0(\alpha_2)}\right]\right\}^{-1}$$

$$+\frac{1}{u}\left\{\alpha_2 E_2 K_1(\alpha_2)\frac{(1-B_3)K_0(\alpha_3)}{(B_2-1)K_0(\alpha_2)} + \alpha_3 E_3 K_1(\alpha_3)\right.$$

$$+\left[\frac{(1-A_2)(1-B_3)+(1-A_3)(B_2-1)}{(A_1-1)(1-B_3)+(A_3-1)(B_1-1)}\right]$$

$$\left.\times\left[\frac{\alpha_1 E_1(1-B_3)K_1(\alpha_1)K_0(\alpha_3)+\alpha_3 E_3(B_1-1)K_1(\alpha_3)K_0(\alpha_1)}{(B_2-1)K_0(\alpha_1)}\right]\right\}^{-1}, \tag{67c}$$

where

$$E_i = [\kappa_m A_i + \kappa_f B_i + \kappa_v], \quad i = 1, 2, 3.$$

The following equations are obtained as the result of substituting the Equations (62)-(64) in boundary conditions.

$$\alpha_1 K_1(\alpha_1) D_1[\kappa_m A_1 + \kappa_f B_1 + \kappa_v]$$

$$+ \alpha_2 K_1(\alpha_2) D_2[\kappa_m A_2 + \kappa_f B_2 + \kappa_v]$$

$$+ \alpha_3 K_1(\alpha_3) D_3[\kappa_m A_3 + \kappa_f B_3 + \kappa_v] = \frac{1}{u}, \tag{68}$$

$$(A_1-1)D_1K_0(\alpha_1) + (A_2-1)D_2K_0(\alpha_2)$$

$$+ (A_3-1)D_3K_0(\alpha_3) = 0, \tag{69}$$

$$(B_1-1)D_1K_0(\alpha_1) + (B_2-1)D_2K_0(\alpha_2)$$

$$+ (B_3-1)D_3K_0(\alpha_3) = 0. \tag{70}$$

In order to simplify the Equations (68)-(70) the following terms are defined:

$$P_i = \alpha_i K_1(\alpha_i)[\kappa_m A_i + \kappa_f B_i + \kappa_v], \tag{71}$$

$$Q_i = (A_i - 1)K_0(\alpha_i),\ R_i = (B_i - 1)K_0(\alpha_i), \tag{72}$$

where $i = 1, 2, 3$. The matrix equation associated with the system of Equations (68)-(70) has the form

$$\begin{bmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 1/u \\ 0 \\ 0 \end{bmatrix}. \tag{73}$$

It defines

$$m = Q_1 R_2 P_3 - Q_1 P_2 R_3 - R_1 Q_2 P_3 - R_2 P_1 Q_3 + P_2 R_1 Q_3 + P_1 Q_2 R_3. \tag{74}$$

The solution of the matrix equation is

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \frac{1}{m} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_2 \end{bmatrix} \times \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}. \tag{75}$$

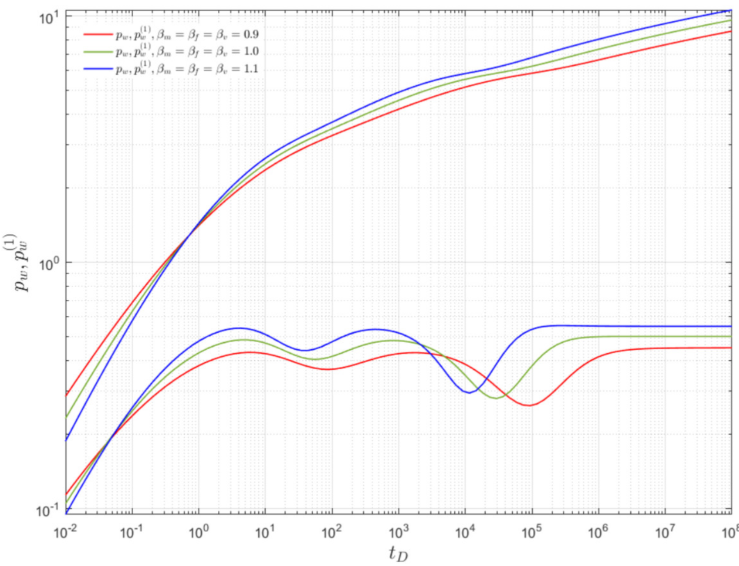
Now that we have obtained all the terms that define the solution of the system of Equations (49)-(51) have been obtained, it is necessary to show the value of the pressure at the boundary of the well, this value is:

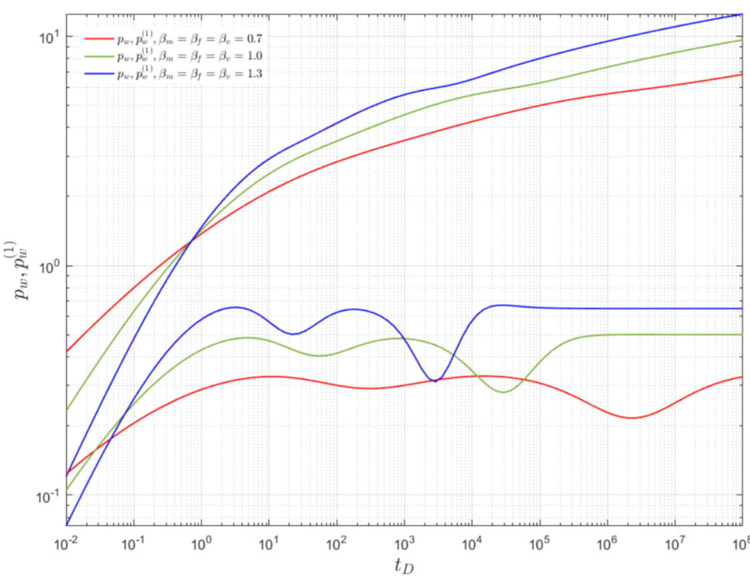
$$\bar{p}_w|_{r_d=1} = \sum_{i=1}^{3} D_i K_0(\alpha_i) = \sum_{i=1}^{3} A_i D_i K_0(\alpha_i) = \sum_{i=1}^{3} B_i D_i K_0(\alpha_i). \tag{76}$$

It should be mentioned that the numerical computation of an inverse Laplace transform is not easily done. In general, it is called a badly conditioned or ill-posed problem [24]. Some methods work quite well for certain functions, while for other functions they give poor results. As a general rule, numerical inversion methods work best for problems where the essential behaviour of the original function is concentrated in a finite interval $[0, T]$, i.e., the original function should be decaying for increasing values of its independent variable [24]. Considering the aforementioned and the fact that the pressure p_w is expected to satisfy the following condition:

$$\lim_{r_d \to \infty} p_w \to 0, \tag{77}$$

then, it is possible to implement the Stehfest algorithm [25] in the Equation (76). Finally, by obtaining the pressure p_w in real space, it is possible to numerically approximate its derivative $p_w^{(1)}$ to obtain the graphs presented in Figure 1.


$p_w, p_w^{(1)}, \beta_m = \beta_f = \beta_v = 0.9$
$p_w, p_w^{(1)}, \beta_m = \beta_f = \beta_v = 1.0$
$p_w, p_w^{(1)}, \beta_m = \beta_f = \beta_v = 1.1$
$p_w, p_w^{(1)}$
t_D
10^{1}
10^{0}
10^{-1}
10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3} 10^{4} 10^{5} 10^{6} 10^{7} 10^{8}


$p_w, p_w^{(1)}, \beta_m = \beta_f = \beta_v = 0.7$
$p_w, p_w^{(1)}, \beta_m = \beta_f = \beta_v = 1.0$
$p_w, p_w^{(1)}, \beta_m = \beta_f = \beta_v = 1.3$
$p_w, p_w^{(1)}$
t_D
10^{1}
10^{0}
10^{-1}
10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3} 10^{4} 10^{5} 10^{6} 10^{7} 10^{8}

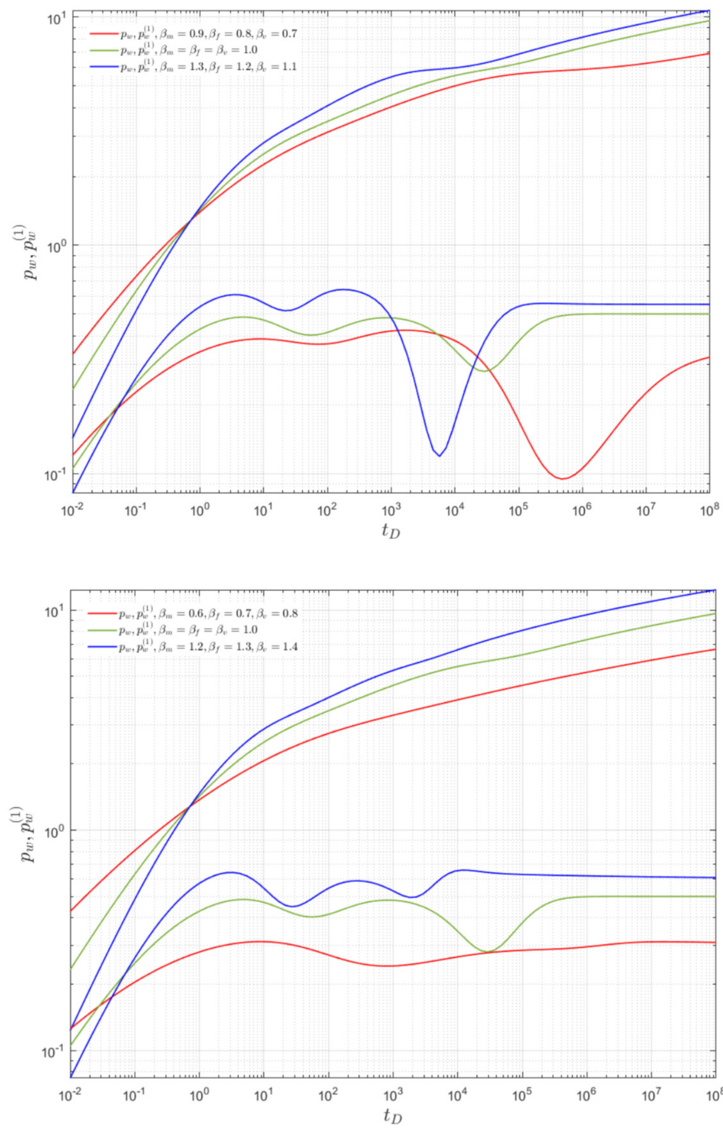


Figure 1. Behaviour of pressure p_w and pressure derivative $p_w^{(1)}$ for different values of β_m, β_f and β_v. All solutions were obtained with the particular values $\kappa_f = 0.75$, $\kappa_v = \omega_f = 0.02$, $\omega_v = 0.8$, $\lambda_{mf} = 10^{-3}$, $\lambda_{vf} = 10^{-8}$ and $\lambda_{fv} = 10^{-5}$. The solution in green in all the graphs corresponds to the classic case.

5. Conclusion

The memory formalism expressed through a generalized Darcy's law with Caputo derivative can facilitate reservoir modelling without the need to know their geometric structure to approach models that reflect anomalous behaviour of fluids in porous media. The property of symmetry is important, because it allows reducing the problem to one dimension. The simplification of the equations to an algebraic problem facilitate the solution.

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