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MULTIPLICITY OF SOLUTIONS FOR GENERALIZED QUASILINEAR SCHRÖDINGER EQUATIONS

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Abstract

In this paper, we study the following quasilinear Schrödinger equations:

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = K(x)h(u) + \lambda W(x)|u|^{p-2}u,\ x \in \mathbb{R}^N, \quad \text{(P)}$$

where $N \geq 3$, $V, K \in C(\mathbb{R}^N, \mathbb{R}^+)$ are given potentials, $1 < p < 2$, $\lambda > 0$ is a small parameter, g is a C^1 even function with $g(0) = 1$, $\lim_{t\to+\infty} g(t) = a \in (0, 1)$ and $g'(t) \leq 0$ for all $t \geq 0$, and $h \in C(\mathbb{R}, \mathbb{R})$ satisfies superlinear growth at infinity. We get the existence results of multiplicity of nontrivial solutions for problem (P).

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1. Introduction

In the last few years, there are many papers have studied the following quasilinear Schrödinger equations:

$$iz_t = -\Delta z + W(x)z - \eta(x, |z|^2)z + \kappa[\Delta(\varphi(|z|^2)\varphi'(|z|^2)]z, \quad x \in \mathbb{R}^N, \quad (1.1)$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, κ is a parameter and $\eta : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$, $\varphi : \mathbb{R}^+ \to \mathbb{R}$ are real functions. Quasilinear equations of the form (1.1) have been derived as mathematical models of several physical phenomena corresponding to various types of the nonlinear terms φ and η. For example, if $\varphi(s) = s$ and $\eta(s) = -\alpha - \dfrac{\beta}{(a+s)^3}$ for some positive constants α and β, then (1.1) become the basic equation to study the oscillations of a superfluid film, for more details on the physical backgrounds, we refer the readers to [9, 11, 21] and references therein. The nonlinear Schrödinger equations describe the propagation of optical pulses with higher-dispersion. These solitons play an important and key role for information transfer via optical fibers [37]. Our interests in the present paper are existence of standing wave solutions for (1.1), that is, solutions of the form $z(t, x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and u is a real function. Submitting $z(t, x) = \exp(-iEt)u(x)$ into (1.1), we get

$$-\Delta u + V(x)u + \kappa[\Delta(\varphi(|u|^2))\varphi'(|u|^2)]u = \eta(x, u^2)u, \quad x \in \mathbb{R}^N, \quad (1.2)$$

where $V(x) := W(x) - E$. In particular, setting $l(x, t) := \eta(x, t^2)t$, then if $\varphi(t) = t$ and $\varphi(t) = \sqrt{1+t}$, (1.2) turns into

$$-\Delta u + V(x)u + \kappa[\Delta|u|^2]u = l(x, u), \quad x \in \mathbb{R}^N, \quad (1.3)$$

and

$$-\Delta u + V(x)u + \kappa[\Delta(1+u^2)^{1/2}]\frac{u}{2(1+u^2)^{1/2}} = l(x, u), \quad x \in \mathbb{R}^N. \quad (1.4)$$

If $\kappa = 0$, problems (1.3) and (1.4) are semilinear Schrödinger equations, which have been studied extensively via variational methods in past thirty years, see, e.g., [8, 23, 32] and references therein.

If $\kappa < 0$, a lot of existence and multiplicity of weak solutions for (1.3) have been obtained by many papers via variational methods, see, e.g., [1, 12, 13, 17, 27, 38, 40, 42, 51, 53, 54], where there are three main arguments used in the above references, one is the constrained minimization argument, namely consider a minimization on a convenient constraint to get existence results, see [27, 38, 40, 51]. The second is perturbed argument, namely add a high order perturbation term such that the perturbed function is smooth, see [33, 34, 35, 49, 50]. The third is to make a change of variable (dual approach $v = f^{-1}(u)$ with f satisfies $f'(t) = 1/\sqrt{1-\kappa f^2(t)}$ for $t \in [0, +\infty)$ and $f(t) = -f(-t)$ for $t \in (-\infty, 0]$) to reduce (1.3) into a semilinear equation like the case of $\kappa = 0$, see [12, 17, 42]. Moreover, a lot of existence and multiplicity of weak solutions results for (1.4) have been obtained by many authors via variational methods with the change of variable $v = f^{-1}(u)$, see, e.g., [14, 15, 16, 28, 44, 45, 46, 47].

More recently, there are many papers have studied existence results of nontrivial solutions for the following generalized quasilinear Schrödinger equations, see [30, 31, 45, 47, 48]

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = l(x, u), \; x \in \mathbb{R}^N, \quad (1.5)$$

where $g \in C^1(\mathbb{R}, \mathbb{R}^+)$ satisfies

(G) g is a even function with $g(0) = 1$ and $g'(t) \geq 0$ for all $t \geq 0$.

In these papers, a change of variable was used to reduced the problem (1.5) to semilinear problem. In fact, their results focused on the general function $\varphi(|u|^2)$ in (1.2). Obviously, the change of variable $v = f^{-1}(u)$ satisfies condition (G).

If $\kappa > 0$ is small enough, by using a change of variables, namely, $s = G^{-1}(t)$ for $t \in [0, +\infty)$ and $G^{-1}(t) = -G^{-1}(-t)$ for $t \in (-\infty, 0)$, where the inverse of $G^{-1}(t)$ is

$$G(s) = \int_0^s g(t)dt, \tag{1.6}$$

$$g(t) = \begin{cases} \sqrt{1-\kappa t^2}, & 0 \le t < \sqrt{\dfrac{1}{3\kappa}}, \\ \dfrac{1}{3\sqrt{2\kappa}t} + \sqrt{\dfrac{1}{6}}, & t \ge \sqrt{\dfrac{1}{3\kappa}}, \\ g(-t), & t < 0, \end{cases} \tag{1.7}$$

the authors in [2, 4, 41, 43, 53] reduced (1.3) to a semilinear one, and an Orlicz space framework was used. The same change of variable was also used in [52]. Soon after, Huang and Jia get the existence of positive solutions for (1.3) with $\kappa = 1$ by using same change of variable, see [22]. Moreover, if $0 < \kappa < 2$, Shen and Wang have studied the existence of nontrivial solutions for (1.4) by using the change of variable (1.6), where $g(t) = \sqrt{1 - (\kappa t^2)/[2(1+t^2)]}$, see [44].

In order to study problem (1.2) with the general functions $\varphi(|u|^2)$, we concerned the existence of positive solutions for (1.5) under the conditions (GD) and (V), see [25, 26].

(GD) g is a even function with $g(0) = 1$, $\lim_{t\to+\infty} g(t) = a \in (0, 1)$ and $g'(t) \le 0$ for all $t \ge 0$.

(V) $0 < V_0 = \inf_{x \in \mathbb{R}^N} V(x)$, $V(x) \le V_\infty := \lim_{|x| \to \infty} V(x) < \infty$.

In fact, we say the potential $V(x)$ is the case of well potential if $V(x)$ satisfies condition (V). Obviously, there are many functions satisfy condition (GD). For example, $g(t)$ is defined in (1.7) with $a = \sqrt{1/6}$ and $g(t) = \sqrt{1 - (\kappa t^2)/[2(1+t^2)]}$ with $a = \sqrt{1 - \kappa/2}$, where $0 < \kappa < 2$. However, as far as we know in the literature there is no result of existence of nontrivial solutions for (1.5). In the present paper, we try to fill this gap and give an affirmative answer to this question. We consider problem (1.5) with $l(x, u) = K(x)h(u) + \lambda W(x)|u|^{p-2}u$, that is the following generalized quasilinear Schrödinger equations:

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = K(x)h(u) + \lambda W(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.8}$$

where $N \ge 3$, $1 < p < 2$, $\lambda > 0$ is a small parameter, $W \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$, $h \in C(\mathbb{R}, \mathbb{R})$ satisfies superlinear growth at infinity, g is of $C^1(\mathbb{R}, \mathbb{R}^+)$ satisfies (GD), and $V(x)$, $K(x)$ are positive continuous functions and $(V, K) \in \mathcal{K}$, that is, V and K satisfy the following conditions:

(V_0) $V(x) > 0$, $\forall x \in \mathbb{R}^N$ and $\lim_{|x| \to \infty} V(x) = 0$.

(K_1) (i) $K(x) > 0$, $\forall x \in \mathbb{R}^N$, $K \in L^\infty(\mathbb{R}^N)$.

(ii) if $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets such that $|A_n| \le R$ for all n and some $R > 0$, then

$$\lim_{r \to +\infty} \int_{A_n \cap B_r^c(0)} K(x)dx = 0.$$

(K_2) $\frac{K}{V} \in L^{\infty}(\mathbb{R}^N)$.

(K_3) There is $\sigma \in (2, 2^*)$ such that $\lim_{|x|\to\infty} \frac{K(x)}{[V(x)]^{\frac{2^*-\sigma}{2^*-2}}} = 0$.

For the case of potential $V(x)$ vanishes at infinity, there are many papers have shown existence of solutions for Schrödinger equations (see [5, 6, 10, 24, 29, 55, 56]) and quasilinear Schrödinger equations (see [1, 2, 19, 20, 36, 48] and references therein). In [5], Ambrosetti et al. considered the following Schrödinger equation:

$$-\varepsilon^2 \Delta u + V(x)u = K(x)|u|^p,\ x \in \mathbb{R}^N,$$

where $N \geq 3$, V, $K : \mathbb{R}^N \to \mathbb{R}$ are smooth functions and there exist constants c_1, c_2, c_3, α_1, $\alpha_2 > 0$ such that

$$(VK)\ \frac{c_3}{1+|x|^{c_1}} \leq V(x) \leq \alpha_1 \text{ and } 0 < K(x) \leq \frac{\alpha_2}{1+|x|^{c_2}},\ \forall x \in \mathbb{R}^N.$$

Moreover, c_1, c_2 satisfy

$$\frac{N+2}{N-2} - \frac{4c_2}{c_1(N-2)} < p \text{ if } 0 < c_2 < c_1 \text{ or } p > 1, \text{ where } c_2 > c_1.$$

Later, in [7], Ambrosetti and Wang considered the condition (VK), but the condition on V was assumed only outside of a ball centered at origin. Moreover, in [3], Alves and Souto considered a more general condition on V and K. Precisely, they assumed that $(V, K) \in \mathcal{K}$, they got existence of positive ground state solution for the following equation:

$$-\Delta u + V(x)u = K(x)h(u),\ x \in \mathbb{R}^N.$$

In the present paper, we consider the case $(V, K) \in \mathcal{K}$. To state our main result, we assume that h satisfies

(h1) $\lim_{|t|\to 0} \frac{h(t)}{|t|} = 0$ if (K_2) holds; $\lim_{|t|\to 0} \frac{h(t)}{|t|^{q-1}} < +\infty$ if (K_3) holds,

where $2 < q < 2^*$.

(h2) There exists $C > 0$ such that $|h(t)| \le C(1 + |t|^{q-1})$ for all $t \in \mathbb{R}$.

(h3) There exists $\theta > 2$ such that $0 \le \theta H(t) \le h(t)t$ for all $t \in \mathbb{R}$, where $H(t) = \int_0^t h(s)ds$.

(h4) $h(t) = -h(-t)$ for all $t \in \mathbb{R}$.

Now, we state our main results as follows:

Theorem 1.1. *Assume that* (h1)-(h3) *hold*, $(V, K) \in \mathcal{K}$ *and* $W \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$, $1 < p < 2$, *then there exists a constant* $\lambda_0 > 0$ *such that for all* $\lambda \in (0, \lambda_0)$, *problem* (1.8) *has at least two nontrivial solutions.*

Theorem 1.2. *Assume that* (h1)-(h4) *hold*, $(V, K) \in \mathcal{K}$ *and* $W \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$, $1 < p < 2$, *then problem* (1.8) *has infinity many nontrivial solutions.*

This paper is organized as follows: in Section 2, we will provide a useful lemma and present some embedding results. In Section 3, we will prove Theorems 1.1 and 1.2. Throughout this paper, we will denote the following notations: $\|u\|_p (1 < p \le \infty)$ is the norm in $L^p(\mathbb{R}^N)$; $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively; $\langle \cdot, \cdot \rangle$ denotes the duality pairing between a Banach space and its dual space; $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$ and $B_R(0)$ denotes a ball centered at the origin with radius $R > 0$.

2. Preliminaries

We introduce the subspace

$$H = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty \right\},$$

which is a Hilbert space equipped with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv) dx, \; u, v \in H,$$

and the norm

$$\|u\| = \left[\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx \right]^{1/2}.$$

Denote by $L_K^r(\mathbb{R}^N)$ the weighted Lebesgue function space

$$L_K^r(\mathbb{R}^N) = \left\{ u \in H^1(\mathbb{R}^N) : u \text{ is measurable and } \int_{\mathbb{R}^N} K(x)|u|^r dx < +\infty \right\},$$

and owe with the norm

$$\|u\|_{r,K} = \left(\int_{\mathbb{R}^N} K(x)|u|^r dx \right)^{1/r}, \; u \in H.$$

The natural energy functional corresponding to problem (1.8) is given by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u|^2 dx - \int_{\mathbb{R}^N} K(x)H(u) dx$$

$$- \frac{\lambda}{p} \int_{\mathbb{R}^N} W(x)|u|^p dx,$$

where $H(t) = \int_0^t h(s)ds$. By making the following change of variable

$$v = G(u) = \int_0^u g(s)ds,$$

the functional $I(u)$ can be written by

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx - \int_{\mathbb{R}^N} K(x)H(G^{-1}(v))dx$$
$$- \frac{\lambda}{p}\int_{\mathbb{R}^N} W(x)|G^{-1}(v)|^p dx.$$

$J(v)$ is the variational functional of the following equation:

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = K(x)\frac{h(G^{-1}(v))}{g(G^{-1}(v))} + \lambda W(x)\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))},$$
$$x \in \mathbb{R}^N, \qquad (2.1)$$

where $G^{-1}(v)$ is the inverse function of $G(u)$. Next, we will establish the smoothness of J. First, we give some properties of $g(t)$ and $G(t)$.

Lemma 2.1 ([25], Lemma 2.1)**.** *The functions* $g(t)$, $G(t) = \int_0^t g(s)ds$ *enjoy the following properties under the condition* (GD)

(1) $G(t)$ *is invertible,* $G(t)$ *and the inverse* $G^{-1}(t)$ *are odd.*

(2) $\frac{t}{g(t)}g'(t) \le 0$ *for all* $t \ge 0$.

(3) $|t| \le |G^{-1}(t)| \le \frac{|t|}{a}$ *for all* $t \in \mathbb{R}$.

(4) $\dfrac{G^{-1}(t)}{t}$ *is nondecreasing for all* $t \in \mathbb{R}$ *and* $\lim\limits_{t\to 0}\dfrac{G^{-1}(t)}{t} = 1$, $\lim\limits_{t\to\infty}\dfrac{G^{-1}(t)}{t} = \dfrac{1}{a}$.

(5) $t^2 \le \dfrac{t}{g(t)}G(t) \le \dfrac{t^2}{a}$ *for all* $t \in \mathbb{R}$.

Lemma 2.2. *J is well defined in H and $J \in C^1(H, \mathbb{R})$. Moreover,*

$$\langle J'(v), \psi\rangle = \int_{\mathbb{R}^N} \nabla v \nabla \psi dx + \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\psi dx - \int_{\mathbb{R}^N} K(x)\frac{h(G^{-1}(v))}{g(G^{-1}(v))}\psi dx - \lambda\int_{\mathbb{R}^N} W(x)\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))}\psi dx, \quad \forall \psi \in H.$$

Proof. It deduces from Lemma 2.1(3) that

$$\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx \le \frac{1}{a^2}\int_{\mathbb{R}^N} V(x)|v|^2 dx, \ \forall v \in H.$$

By (h1) and (h2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|h(t)| \le \varepsilon|t| + C_\varepsilon|t|^{q-1}, \ \forall t \in \mathbb{R},$$

where $2 < q < 2^*$. Then

$$|H(t)| \le \frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{q}|t|^q, \ \forall t \in \mathbb{R}. \tag{2.2}$$

Hence,

$$\int_{\mathbb{R}^N} K(x)|H(G^{-1}(v))|dx \le \int_{\mathbb{R}^N} K(x)\left[\frac{\varepsilon}{2a^2}|v|^2 + \frac{C_\varepsilon}{qa^q}|v|^q\right]dx < +\infty, \ \forall v \in H.$$

Since $W \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$, $1 < p < 2$, by the Hölder inequality, we have

$$\int_{\mathbb{R}^N} W(x)|G^{-1}(v)|^p dx \le \frac{1}{a^p} \int_{\mathbb{R}^N} W(x)|v|^p dx$$

$$\le \frac{1}{a^p}\left[\int_{\mathbb{R}^N} |W(x)|^{\frac{2}{2-p}} dx\right]^{\frac{2-p}{2}}\left[\int_{\mathbb{R}^N} |v|^2 dx\right]^{\frac{p}{2}} < +\infty, \ \forall v \in H.$$

Thus, J is well defined.

Next, we verify that $J \in C^1(H, \mathbb{R})$. Note that for any $v, \psi \in H$ is fixed, given $0 < t < 1$, by the mean value theorem, there exists $\epsilon \in (0, 1)$ such that

$$\left|\frac{|G^{-1}(v + t\psi)|^2 - |G^{-1}(v)|^2}{t}\right| = 2\left|\frac{G^{-1}(v + \epsilon t\psi)}{g(G^{-1}(v + \epsilon t\psi))}\psi\right|$$

$$\le \frac{2}{a^2}|(v + \epsilon t\psi)\psi|$$

$$\le \frac{2}{a^2}(|v| + |\psi|)|\psi| \in L^1(\mathbb{R}^N).$$

Then by the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{t\to 0} \frac{1}{2} \int_{\mathbb{R}^N} V(x) \frac{|G^{-1}(v + t\psi)|^2 - |G^{-1}(v)|^2}{t} dx = \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi dx.$$

Similarly, by (2.2), we have

$$\lim_{t\to 0} \int_{\mathbb{R}^N} K(x) \frac{H(G^{-1}(v + t\psi)) - H(G^{-1}(v))}{t} dx = \int_{\mathbb{R}^N} K(x) \frac{h(G^{-1}(v))}{g(G^{-1}(v))} \psi dx.$$

Moreover, for any $v, \psi \in H$ are fixed, given $0 < t < 1$, by the mean value theorem, there exists $\mu \in (0, 1)$ such that

$$\left|\frac{|G^{-1}(v + t\psi)|^p - |G^{-1}(v)|^p}{t}\right| = p\frac{|G^{-1}(v + \mu t\psi)|^{p-1}}{g(G^{-1}(v + \mu t\psi))}|\psi|$$

$$\leq \frac{p}{a^p}|v + \mu t\psi|^{p-1}|\psi|.$$

Since $\frac{2-p}{2} + \frac{p-1}{2} + \frac{1}{2} = 1$, $W \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$, $1 < p < 2$, by the Hölder inequality and Minkowski inequality, we get

$$\int_{\mathbb{R}^N} W(x)|v + \mu t\psi|^{p-1}|\psi|dx$$

$$\leq \left[\int_{\mathbb{R}^N} |W(x)|^{\frac{2}{2-p}}dx\right]^{\frac{2-p}{2}}\left[\int_{\mathbb{R}^N} |v + \mu t\psi|^2 dx\right]^{\frac{p-1}{2}}\left[\int_{\mathbb{R}^N} |\psi|^2 dx\right]^{\frac{1}{2}}$$

$$\leq \left[\int_{\mathbb{R}^N} |W(x)|^{\frac{2}{2-p}}dx\right]^{\frac{2-p}{2}}\left[\left(\int_{\mathbb{R}^N} |v|^2 dx\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^N} |\psi|^2 dx\right)^{\frac{1}{2}}\right]^{p-1}\left[\int_{\mathbb{R}^N} |\psi|^2 dx\right]^{\frac{1}{2}}$$

$$< +\infty.$$

Hence, by the Lebesgue dominated convergence theorem, we have

$$\lim_{t\to 0}\frac{\lambda}{p}\int_{\mathbb{R}^N} W(x)\frac{|G^{-1}(v + t\psi)|^p - |G^{-1}(v)|^p}{t}dx = \lambda\int_{\mathbb{R}^N} W(x)\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))}\psi dx.$$

This means that $J \in C^1(H, \mathbb{R})$ and

$$\langle J'(v), \psi \rangle = \int_{\mathbb{R}^N} \nabla v \nabla \psi dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi dx$$

$$- \int_{\mathbb{R}^N} K(x) \frac{h(G^{-1}(v))}{g(G^{-1}(v))} \psi dx - \lambda \int_{\mathbb{R}^N} W(x) \frac{|G^{-1}(v)|^{p-2} G^{-1}(v)}{g(G^{-1}(v))} \psi dx,$$

$$\forall \psi \in H. \qquad \square$$

Second, we show relations with critical point of J and weak solution of problem (1.8).

Lemma 2.3. *Assume that* (V_0) *and* (h1)-(h2) *hold. If* $v \in H$ *is a critical point of* J, *then* $u = G^{-1}(v)$ *is a weak solution of problem* (1.8).

Proof. Since v is a critical point of J, we have

$$\int_{\mathbb{R}^N} \nabla v \nabla \psi dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi dx$$

$$= \int_{\mathbb{R}^N} K(x) \frac{h(G^{-1}(v))}{g(G^{-1}(v))} \psi dx + \lambda \int_{\mathbb{R}^N} W(x) \frac{|G^{-1}(v)|^{p-2} G^{-1}(v)}{g(G^{-1}(v))} \psi dx, \quad (2.3)$$

for all $\psi \in H$.

Since $v \in H$, it deduces from (V_0) and Lemma 2.1 (3) that $u := G^{-1}(v) \in H$. For each $\varphi \in C_0^\infty(\mathbb{R}^N)$, taking $\psi := g(u)\varphi \in H$ in (2.3), we have

$$\int_{\mathbb{R}^N} \nabla v (g'(u)\varphi \nabla u + g(u)\nabla \varphi) dx + \int_{\mathbb{R}^N} V(x) \frac{u}{g(u)} g(u)\varphi dx$$

$$= \int_{\mathbb{R}^N} \frac{h(u)}{g(u)} g(u)\varphi dx + \lambda \int_{\mathbb{R}^N} W(x) \frac{|u|^{p-2} u}{g(u)} g(u)\varphi dx.$$

Note that $v = G(u)$ and $\nabla v = g(u)\nabla u$, then

$$\int_{\mathbb{R}^N} [g^2(u)\nabla u\nabla\varphi + g(u)g'(u)|\nabla u|^2\varphi]dx + \int_{\mathbb{R}^N} V(x)u\varphi dx$$

$$= \int_{\mathbb{R}^N} h(u)\varphi dx + \lambda \int_{\mathbb{R}^N} W(x)|u|^{p-2}u\varphi dx,$$

for $\varphi \in C_0^\infty(\mathbb{R}^N)$. Therefore, u is a weak solution of problem (1.8). □

By (V_0), we have the following embedding theorems.

Lemma 2.4 ([18], Lemma 2.1)**.** *Assume that* $(V, K) \in \mathcal{K}$, *then* $H \hookrightarrow L_K^r(\mathbb{R}^N)$ *is continuous for all* $r \in [2, 2^*]$ *if* (K_2) *holds*; $H \hookrightarrow L_K^\sigma(\mathbb{R}^N)$ *is continuous for all* $\sigma \in (2, 2^*)$ *if* (K_3) *holds.*

It deduces from Lemma 2.4 that there exists a constant $\gamma_r > 0$ which is dependent of r such that

$$\|u\|_{r,K} \le \gamma_r \|u\| \quad \text{for all} \quad r \in [2, 2^*]. \tag{2.4}$$

Lemma 2.5 ([3], Proposition 2.1)**.** *Assume that* $(V, K) \in \mathcal{K}$, *then* $H \hookrightarrow L_K^q(\mathbb{R}^N)$ *is compact for all* $q \in (2, 2^*)$ *if* (K_2) *holds*; $H \hookrightarrow L_K^\sigma(\mathbb{R}^N)$ *is compact for all* $\sigma \in (2, 2^*)$ *if* (K_3) *holds.*

3. Proof of Theorems 1.1 and 1.2

First, we discuss J satisfies the Mountain Pass geometry.

Lemma 3.1. *Assume that h satisfies* (h1) *and* (h2), *then*

(1) *there exist* $\rho, \alpha > 0$ *such that* $J(v) \ge \alpha$, $\|v\| = \rho$.

(2) *there is* $v \in H \setminus \{0\}$ *such that* $J(v) < 0$.

Proof. (1) Set $S_\rho = \{v \in H : \|v\| = \rho\}$, combining Lemma 2.1(3), (2.4) and (2.2), let $\varepsilon = \dfrac{a^2}{2\gamma_2^2}$, we have

$$J(v) \geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|v|^2 dx - \frac{\lambda}{pa^p}\int_{\mathbb{R}^N} W(x)|v|^p dx$$

$$- \frac{\varepsilon}{2a^2}\int_{\mathbb{R}^N} K(x)|v|^2 dx - \frac{C_\varepsilon}{qa^q}\int_{\mathbb{R}^N} K(x)|v|^q dx$$

$$\geq \frac{1}{2}\|v\|^2 - \frac{\lambda}{pa^p}\|W(x)\|_{\frac{2}{2-p}}\|v\|^p - \frac{\varepsilon\gamma_2^2}{2a^2}\|v\|^2 - \frac{C_\varepsilon}{qa^q}\gamma_q^q\|v\|^q$$

$$= \|v\|^p\left[\frac{1}{4}\|v\|^{2-p} - \frac{C_\varepsilon}{qa^q}\gamma_q^q\|v\|^{q-p} - \frac{\lambda}{pa^p}\|W(x)\|_{\frac{2}{2-p}}\right].$$

Denote

$$\eta(t) = \frac{1}{4}t^{2-p} - \frac{C_\varepsilon}{qa^q}\gamma_q^q t^{q-p} - \frac{\lambda}{pa^p}\|W(x)\|_{\frac{2}{2-p}},\ t \geq 0,$$

hence,

$$\eta'(t) = \frac{2-p}{4}t^{1-p} - \frac{C_\varepsilon}{qa^q}\gamma_q^q(q-p)t^{q-p-1},\ t \geq 0.$$

Then $\eta(t)$ arrives its maximum at $t = \left[\dfrac{(2-p)q}{4C_\varepsilon a^q\gamma_q^q(q-p)}\right]^{\frac{1}{q-2}}$. That is, there is $\rho > 0$ such that

$$\max_{t\geq 0}\eta(t) = \eta(\rho) = \frac{q-2}{4(q-p)}\left[\frac{(2-p)q}{4C_\varepsilon a^q\gamma_q^q(q-p)}\right]^{\frac{2-p}{q-2}}.$$

Therefore, there exist constants λ_0 and α such that $J(v)|_{S_\rho} \geq \alpha > 0$ for all $\lambda \in (0, \lambda_0)$.

(2) It deduces from (h3) that

$$\lim_{t\to+\infty} \frac{H(t)}{t^2} = +\infty. \tag{3.1}$$

By Lemma 2.1(3), we have

$$\begin{aligned}\frac{J(t\varphi)}{t^2} &\leq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\varphi|^2 dx + \frac{1}{2a^2}\int_{\mathbb{R}^N} V(x)|\varphi|^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} W(x)\frac{|t\varphi|^p}{t^2} dx \\ &\quad - \int_{\mathbb{R}^N} K(x)\frac{H(G^{-1}(t\varphi))}{(t\varphi)^2}\varphi^2 dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\varphi|^2 dx + \frac{1}{2a^2}\int_{\mathbb{R}^N} V(x)|\varphi|^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} W(x)\frac{|t\varphi|^p}{t^2} dx \\ &\quad - \int_{\mathbb{R}^N} K(x)\frac{H(G^{-1}(t\varphi))}{(G^{-1}(t\varphi))^2}\varphi^2 dx \\ &\to -\infty,\ t \to +\infty,\end{aligned}$$

therefore, there exists large enough $t_0 > 0$, let $v = t_0\varphi$ with $\|v\| > \rho$ such that $J(v) < 0$. □

By Lemma 3.1, J satisfies the Mountain Pass geometry. Then there is a (PS) sequence $\{v_n\} \in H$ at level c such that

$$J(v_n) \to c,\ J'(v_n) \to 0 \text{ as } n \to \infty, \tag{3.2}$$

where

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} J(\gamma),\ \Gamma = \{\gamma \in C([0, 1], H) : \gamma(0) = 0,\ \gamma(1) \neq 0,\ J(\gamma(1)) < 0\}.$$

Second, we establish the boundedness of $(PS)_c$ sequence $\{v_n\}$ which is defined in (3.2).

Lemma 3.2. *Suppose that h satisfies* (h1)-(h3) *and* $(V, K) \in \mathcal{K}$, *then any* $(PS)_c$ *sequence* $\{v_n\}$ *is bounded in* H.

Proof. By (3.2), (2) and (3) of Lemma 2.1, we know that

$$\begin{aligned} c + 1 + o_n(1)\|v_n\| &\geq J(v_n) - \frac{1}{\theta}\langle J'(v_n),\, G^{-1}(v_n)g(G^{-1}(v_n))\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{1}{\theta}\int_{\mathbb{R}^N} \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\, g'(G^{-1}(v_n))|\nabla v_n|^2 dx \\ &\quad + \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx \\ &\quad + \int_{\mathbb{R}^N} K(x)\left[\frac{1}{\theta}h(G^{-1}(v_n))G^{-1}(v_n) - H(G^{-1}(v_n))\right] dx \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{p}\right)\lambda\int_{\mathbb{R}^N} W(x)|G^{-1}(v_n)|^p dx. \end{aligned}$$

Hence, by (h3), we have

$$\begin{aligned} c + 1 + o_n(1)\|v_n\| + \left(\frac{1}{p} - \frac{1}{\theta}\right)\lambda\int_{\mathbb{R}^N} W(x)|G^{-1}(v_n)|^p dx \\ \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v|^2 dx + \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx \\ + \int_{\mathbb{R}^N} K(x)\left[\frac{1}{\theta}h(G^{-1}(v_n)G^{-1}(v_n - H(G^{-1}(v_n))\right] dx \\ \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v|^2 dx + \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} V(x)|v_n|^2 dx. \end{aligned}$$

By the Hölder inequality, we have

$$\int_{\mathbb{R}^N} W(x)|G^{-1}(v_n)|^p\,dx \le \frac{1}{a^p}\int_{\mathbb{R}^N} W(x)|v_n|^p\,dx \le \|W(x)\|_{\frac{2}{2-p}}\|v\|^p.$$

Therefore,

$$c+1+o_n(1)\|v_n\| + \left(\frac{1}{p}-\frac{1}{\theta}\right)\lambda\|W(x)\|_{\frac{2}{2-p}}\|v\|^p \ge \left(\frac{1}{2}-\frac{1}{\theta}\right)\|v_n\|^2.$$

Then $\{v_n\}$ is bounded in H. □

Denote

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|v|^2dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} W(x)|G^{-1}(v_n)|^p\,dx - \int_{\mathbb{R}^N} K(x)F(x,v)dx,$$

where

$$F(x,v) = \int_0^v f(x,s)ds = \frac{1}{2}\frac{V(x)}{K(x)}v^2 - \frac{1}{2}\frac{V(x)}{K(x)}|G^{-1}(v)|^2 + H(G^{-1}(v)).$$

Taking the derivative, we have

$$\frac{d}{dt}F(x,t) = f(x,t) = \frac{V(x)}{K(x)}t - \frac{V(x)}{K(x)}\frac{G^{-1}(t)}{g(G^{-1}(t))} + \frac{h(G^{-1}(t))}{g(G^{-1}(t))}.$$

Moreover, we have the following properties:

Lemma 3.3. *Suppose that h satisfies* (h2) *and* $(V,K)\in\mathcal{K}$, *then*

(1) $\lim_{t\to 0}\dfrac{f(x,t)}{t} = 0.$

(2) $\lim_{t\to+\infty}\dfrac{f(x,t)}{t^{2^*-1}} = 0.$

Proof. Since h satisfies (h2), combining (4) of Lemma 2.1, by a direct calculation, we get

(1)

$$\lim_{t\to 0}\frac{f(x,t)}{t}=\lim_{t\to 0}\left[\frac{V(x)}{K(x)}-\frac{V(x)}{K(x)}\frac{G^{-1}(t)}{tg(G^{-1}(t))}+\frac{h(G^{-1}(t))}{tg(G^{-1}(t))}\right]$$

$$=\frac{V(x)}{K(x)}-\lim_{t\to 0}\frac{V(x)}{K(x)}\frac{G^{-1}(t)}{tg(G^{-1}(t))}+\lim_{t\to 0}\frac{h(G^{-1}(t))}{G^{-1}(t)}\frac{G^{-1}(t)}{tg(G^{-1}(t))}$$

$$=0.$$

(2)

$$\lim_{t\to+\infty}\frac{f(x,t)}{t^{2^*-1}}$$

$$=\lim_{t\to+\infty}\left[\frac{V(x)}{K(x)t^{2^*-2}}-\frac{V(x)}{K(x)}\frac{G^{-1}(t)}{t^{2^*-1}g(G^{-1}(t))}+\frac{h(G^{-1}(t))}{t^{2^*-1}g(G^{-1}(t))}\right]$$

$$=\lim_{t\to+\infty}\left[\frac{V(x)}{K(x)t^{2^*-2}}-\frac{V(x)}{K(x)}\frac{G^{-1}(t)}{t}\frac{1}{t^{2^*-2}g(G^{-1}(t))}+\frac{h(G^{-1}(t))}{(G^{-1}(t))^{2^*-1}}\frac{(G^{-1}(t))^{2^*-1}}{t^{2^*-1}g(G^{-1}(t))}\right]$$

$$=0.$$

This completes the proof. □

Lemma 3.4. *Assume that h satisfies* (h2) *and $(V,K)\in\mathcal{K}$, let $\{v_n\}\subset H$ be a bounded $(PS)_c$ sequence and $v_n\rightharpoonup v$, $v\in H$, then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}K(x)f(x,v_n)v_n dx=\int_{\mathbb{R}^N}K(x)f(x,v)v dx,$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}K(x)f(x,v_n)v dx=\int_{\mathbb{R}^N}K(x)f(x,v)v dx.$$

Proof. It deduces from Lemma 3.3 and (h2), f satisfies all conditions of Lemma 1.1 in [3], we obtain the conclusion. □

Lemma 3.5. *Assume that* $W \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$, $1 < p < 2$, *let* $\{v_n\} \subset H$ *be a bounded* $(PS)_c$ *sequence and* $v_n \rightharpoonup v$, $v \in H$, *then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} W(x) \frac{|G^{-1}(v_n)|^{p-2}G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n dx = \int_{\mathbb{R}^N} W(x) \frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))} v dx,$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} W(x) \frac{|G^{-1}(v_n)|^{p-2}G^{-1}(v_n)}{g(G^{-1}(v_n))} v dx = \int_{\mathbb{R}^N} W(x) \frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))} v dx.$$

Proof. Let $\xi(t) = \dfrac{|G^{-1}(t)|^{p-2}G^{-1}(t)}{g(G^{-1}(t))} t$, since G^{-1} is odd and $G^{-1}(t) \geq 0$ when $t \geq 0$, by (3) and (5) of Lemma 2.1, we know that

$$0 \leq \xi(t) \leq \frac{1}{a^p} |t|^p.$$

By Lemma 2.7 of [56], we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} W(x) |v_n - v|^p dx = 0.$$

Hence,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} W(x) \xi(v_n - v) \leq \frac{1}{a^p} \lim_{n\to\infty} \int_{\mathbb{R}^N} W(x) |v_n - v|^p dx = 0.$$

This completes the proof. □

Third, we prove J satisfies $(PS)_c$ condition.

Lemma 3.6. *Suppose that h satisfies* (h2) *and* $(V, K) \in \mathcal{K}$, *then any* $(PS)_c$ *sequence* $\{v_n\} \subset H$ *is obtained in* (3.2) *has a strong convergence subsequence.*

Proof. By Lemma 3.2, $\{v_n\}$ in bounded in H, up to a subsequence, denote as $\{v_n\}$. Assume that $v_n \rightharpoonup v$, $v \in H$, since $\langle J'_{\kappa,\lambda}(v_n), v_n \rangle = o_n(1)$, together with Lemma 3.4, we have

$$\lim_{n\to\infty} \|v_n\|^2 = \lim_{n\to\infty} \int_{\mathbb{R}^N} K(x) f(x, v_n) v_n dx + \lim_{n\to\infty} \int_{\mathbb{R}^N} W(x) \frac{|G^{-1}(v_n)|^{p-2} G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n dx$$

$$= \int_{\mathbb{R}^N} K(x) f(x, v) v dx + \int_{\mathbb{R}^N} W(x) \frac{|G^{-1}(v)|^{p-2} G^{-1}(v)}{g(G^{-1}(v))} v dx.$$

Moreover, it deduces from $\langle J'(v_n), v \rangle = o_n(1)$ that

$$\langle v_n, v \rangle = \int_{\mathbb{R}^N} K(x) f(x, v_n) v dx + \int_{\mathbb{R}^N} W(x) \frac{|G^{-1}(v_n)|^{p-2} G^{-1}(v_n)}{g(G^{-1}(v_n))} v dx + o_n(1).$$

Then

$$\lim_{n\to\infty} \langle v_n, v \rangle = \lim_{n\to\infty} \int_{\mathbb{R}^N} K(x) f(x, v_n) v dx + \int_{\mathbb{R}^N} W(x) \frac{|G^{-1}(v_n)|^{p-2} G^{-1}(v_n)}{g(G^{-1}(v_n))} v dx$$

$$= \int_{\mathbb{R}^N} K(x) f(x, v) v dx + \int_{\mathbb{R}^N} W(x) \frac{|G^{-1}(v)|^{p-2} G^{-1}(v)}{g(G^{-1}(v))} v dx$$

$$= \langle v, v \rangle,$$

that is,

$$\lim_{n\to\infty} \|v_n\|^2 = \|v\|^2.$$

Therefore, $v_n \to v$ in H. □

Proof of Theorem 1.1. By Lemma 3.1 and the Mountain Pass Lemma, for each $\lambda \in (0, \lambda_0)$, there exists a $(PS)_c$ sequence $\{v_n\} \subset H$ for J. By Lemma 3.2, $\{v_n\}$ is bounded in H. Moreover, by Lemma 3.6, there are a subsequence of $\{v_n\}$ and $v_1 \in H$ such that $v_n \to v_1$ with $J(v_1) = c \geq \rho > 0$.

Next, we state the second nontrivial solution of problem (2.1) through the local minimization. Since $W \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$, we choose a function $\phi \in H$ such that

$$\int_{\mathbb{R}^N} W(x)|\phi|^p dx > 0.$$

Thus, by (h3) and Lemma 2.1(3), we have

$$\begin{aligned} J(t\phi) &= \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla\phi|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(t\phi)|^2 dx - \int_{\mathbb{R}^N} K(x)H(G^{-1}(t\phi))dx \\ &\quad - \frac{\lambda}{p}\int_{\mathbb{R}^N} W(x)|G^{-1}(t\phi)|^p dx \\ &\leq \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla\phi|^2 dx + \frac{t^2}{2a^2}\int_{\mathbb{R}^N} V(x)|\phi|^2 dx - \frac{\lambda}{p}t^p \int_{\mathbb{R}^N} W(x)|\phi|^p dx \\ &\leq \frac{t^2}{2a^2}\|\phi\|^2 - \frac{\lambda}{p}t^p \int_{\mathbb{R}^N} W(x)|\phi|^p dx \\ &< 0, \end{aligned}$$

for $t > 0$ is small enough. Then, there exists $\rho > 0$ is given in Lemma 3.1 such that $-\infty < \inf_{v\in\overline{B}_\rho} J(v) < 0$. By the Ekeland's variational principle, there exists a minimizing sequence $\{v_n\} \subset \overline{B}_\rho$ such that $J(v_n) \to \inf_{v\in\overline{B}_\rho} J(v)$,

$J'(v_n) \to 0$ as $n \to \infty$. Hence, by Lemma 3.6, there exists a nontrivial solution v_2 of problem (2.1) satisfies

$$J(v_2) < 0 \quad \text{and} \quad \|v_2\| \le \rho.$$

Therefore, we can conclude that

$$J(v_2) < 0 < J(v_1),$$

for all $\lambda \in (0, \lambda_0)$. □

Proof of Theorem 1.2. Since H is a separable Hilbert space, H has a countable orthogonal basis $\{e_i\}$, $i = 1, 2, \ldots$. Let $V_k = \text{span}\{e_1, e_2, \cdots, e_k\}$ and $E_k = V_k^{\perp}$, then $H = V_k \oplus E_k$, $k \in \mathbb{N}$. Hence, by Lemma 3.1, we have

$$J|_{S_\rho \cap E_k} \ge \beta > 0.$$

Indeed, for any finite-dimensional subspace $\widetilde{H} \subset H$, there is a positive integral number k such that $\widetilde{H} \subset V_k$. Suppose to the contrary that there exists a sequence $\{v_n\}$ such that $\{v_n\} \subset \widetilde{H} \subset V_k$ and $\|v_n\| \to \infty$ as $n \to \infty$, but

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx > \int_{\mathbb{R}^N} K(x)H(G^{-1}(v_n))dx + \frac{\lambda}{p}\int_{\mathbb{R}^N} W(x)|G^{-1}(v_n)|^p dx.$$

By Lemma 2.1(3), we have

$$\left[\int_{\mathbb{R}^N} K(x)H(G^{-1}(v_n))dx + \frac{\lambda}{p}\int_{\mathbb{R}^N} W(x)|G^{-1}(v_n)|^p dx\right] / \|v_n\|^2 < \frac{1}{2a^2}. \quad (3.3)$$

Setting $\eta_n = v_n / \|v_n\|$, then, $\{\eta_n\}$ is bounded in H. Up to a subsequence, we assume that $\eta_n \rightharpoonup \eta$ in H, $\eta_n \to \eta$ in $L^r(\mathbb{R}^N)$, $r \in [2, 2^*)$ and $\eta_n(x) \to \eta(x)$ a.e. $x \in \mathbb{R}^N$. Let $\Omega_1 = \{x \in \mathbb{R}^N : \eta(x) \neq 0\}$ and $\Omega_2 = \{x \in \mathbb{R}^N : \eta(x) = 0\}$. If $\text{meas}(\Omega_1) > 0$, by (3.1) and the Fatou's Lemma, we have

$$\frac{\int_{\mathbb{R}^N} K(x)H(G^{-1}(v_n))dx}{\|v_n\|^2} = \int_{\mathbb{R}^N} \frac{K(x)H(G^{-1}(v_n))}{|G^{-1}(v_n)|^2} \frac{|G^{-1}(v_n)|^2}{|v_n|^2} \eta_n^2 dx \to +\infty$$

as $n \to \infty$.

Note that $1 < p < 2$, we know that

$$\frac{\frac{\lambda}{p}\int_{\mathbb{R}^N} W(x)|G^{-1}(v_n)|^p dx}{\|v_n\|^2} \geq \frac{\lambda}{p}\int_{\mathbb{R}^N} W(x)\frac{|v_n|^p}{\|v_n\|^2}dx = \frac{\lambda}{p}\int_{\mathbb{R}^N} W(x)\frac{|\eta_n|^p}{\|v_n\|^{2-p}}dx \to 0$$

as $n \to \infty$.

Which is a contradiction with (3.3). Hence, $\eta_n \to 0$ in $L^r(\mathbb{R}^N)$. Since all norms are equivalent on the finite dimensional space, there exists a constant $C > 0$ such that

$$\|u\|_r \geq C\|u\|, \ \forall u \in \widetilde{H}.$$

Hence,

$$0 = \lim_{n\to\infty} \|\eta_n\|_r \geq C \lim_{n\to\infty} \|\eta_n\| = C > 0.$$

This is a contradiction and the desired conclusion is obtained. This means that there is a constant $R = R(\widetilde{H}) > 0$ such that $J < 0$ on $\widetilde{H} \setminus B_{R(\widetilde{H})}$. Moreover, by conditions (h1) and (h4), we know that $J(0) = 0$ and J is even. Thereby, it follows from Theorem 9.12 in [39], we complete the proof. □

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