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WEYL-MECHANICAL SYSTEMS ON GENERALIZED (PARA)-KÄHLER SPACE FORM

ZEKI KASAP

Department of Elementary Mathematics
Faculty of Education
Pamukkale University
Denizli
Turkey
e-mail: zekikasap@hotmail.com

Abstract

Euler-Lagrange and Hamilton equations on Kähler-Weyl manifolds were presented and the para-complex mathematical aspects of Lagrangian and Hamilton operator, dynamic equation, the action functional, Lagrangian and Hamilton's principle and equations and so on were given. The most important result revealed by this study, how to find the Lagrangian and Hamiltonian equations of motion without using the dynamic equations. For this, theorems were used as an alternative method of finding equations. As a result of this study, Weyl-Euler-Lagrange and Weyl-Hamilton partial differential equations were obtained for movement of objects on Kähler-Weyl manifolds.

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1. Introduction

A Kähler manifold is a manifold with three mutually compatible structures, which are complex, Riemann and symplectic structure. It is well known that a conformal structure C on the manifold M is the collection of all conformally equivalent Riemannian metrics given on M. A Weyl structure ∇ on a conformal manifold (M, C) is a torsion-free linear connection ∇^J preserving the conformal structure C. Generalizing the usual Euler-Lagrange and Hamilton equations can be found with complex and to the para-complex settings to the context of a generalized (para)-Kählerian space. Also, using the mechanical principle, the theory of modern differential geometry and advanced calculus, Lagrangian and Hamiltonian mechanics for mechanical systems can be extended to Kähler-Weyl manifolds.

Gilkey and Nikcević [1] submitted in both complex and para-complex categories and examine (para)-Kähler Weyl structures in both geometric and in the algebraic settings. León and Rodrigues [2] offered a geometrical approach of Lagrangian and Hamiltonian formalisms involving higher order derivatives. Vries [3] shown that the Hamiltonian and Lagrangian motion equations have a very simple interpretation in relativistic quantum mechanics. Tekkoyun [4] presented that para-complex analogue of the Euler-Lagrange and Hamilton equations was obtained in the framework of para-Kählerian manifold. Tekkoyun and Sari [5] introduced bi-para-complex analogue of Lagrangian and Hamiltonian systems. Tekkoyun and Yayli [6] presented generalized-quaternionic Kählerian analogue of Lagrangian and Hamiltonian mechanical systems. Gilkey and Nikčević [7] gave an elementary proof of the fact that any 4-dimensional para-Hermitian manifold admits a unique para-Kähler-Weyl structure. Ganchev and Mihova [8] submitted warped product Kähler manifolds that they used as an underlying manifold an alpha-Sasakian manifold. Bejan and Chiriac [9] introduced a new covariant differential ${}^{A}\nabla$ adapted to an almost paracontact structure. Cherney et al. [10] presented a gauge invariant generalization of Maxwell's equations and p-form electromagnetism to a Kähler space-time. Sun and Wang [11] defined a class of geometric flows on a complete Kähler manifold to unify some physical and mechanical models. Freed [12] gave an intrinsic definition of the special geometry which arises in global $N = 2$ supersymmetry in four dimensions. Tyurin [13] showed a new approach to geometrical quantization procedure. Lyakhovich and Sharapov [14] examined that necessary and sufficient conditions were established for the second class constraint surface to be (an almost) Kähler manifold. Rongye [15] discussed Newtonian mechanics on a Kähler manifold and also gave the complex mathematical aspects of Newton's law. Kasap and Tekkoyun [16] introduced Lagrangian and Hamiltonian formalism for mechanical systems using para/pseudo-Kähler manifolds. Petrović and Velimirović [17] defined a new class of generalized para-Kähler spaces by using two different kinds of covariant derivatives. Petrović [18] defined generalized hyperbolic Kähler spaces as a particular case of Eisenhart's generalized Riemannian spaces.

2. Preliminaries

Definition 1 ([19])**.** A conformal map is a function which preserves angles. A conformal manifold is a differentiable manifold equipped with an equivalence class of (pseudo) Riemann metric tensors, in which two metrics g' and g are equivalent if and only if

$$g' = \Psi^2 g, \tag{1}$$

where $\Psi > 0$ is a smooth positive function. An equivalence class of such metrics is known as a conformal class. A manifold with a conformal structure is called a *conformal manifold*.

Definition 2 ([19])**.** Two Riemann metrics g_1 and g_2 on M are said to be conformally equivalent if there exists a smooth function $f : M \to \mathbb{R}$ with

$$e^f g_1 = g_2. \tag{2}$$

If the metrics g_1 and g_2 are conformally equivalent, we will write $g_1 \sim g_2$.

Definition 3 ([19])**.** Let M be an n-dimensional smooth manifold. A pair (M, C), is said to be a conformal structure on M is an equivalence class C of Riemann metrics on M, is called a conformal structure.

Theorem 1 (See [20])**.** *Let ∇ be a connection on M and $g \in C$ be a fixed metric. ∇ is compatible with (M, C) if and only if there exists a 1-form ω with $\nabla_X g + \omega(X) g = 0$.*

Definition 4 ([21])**.** A compatible torsion-free connection is called a Weyl connection. The triple (M, C, ∇) is a Weyl structure.

Theorem 2 ([22])**.** *To each metric $g \in C$ and 1-form ω, there corresponds a unique Weyl connection ∇ satisfying $\nabla_X g + \omega(X) g = 0$.*

Definition 5 ([22])**.** If $F : \{\text{1-forms on } M\} \times C \to \{\text{Weyl connections}\}$ is a function defined $F(g, \omega) = \nabla$, where ∇ is the connection guaranteed by Theorem 2 we say that ∇ corresponds to (g, ω).

A function F defined in Definition 5 by has the properties:

Proposition 1 ([22])**.** (1) *F is surjective.*

(2) $F(g, \omega) = F(e^f g, \eta)$ *iff* $\eta = \omega - df$. *So*

$$F(e^f g) = F(g) - df. \tag{3}$$

Note that a Riemann metric g *and a one-form* ω *determine a Weyl structure, namely* $F : C \to \wedge^1 M$, *where* C *is a conformal structure and equivalence class of g and* $F(e^f g) = \omega - df$.

Proof. We will show that $\nabla_X g$ is equal to the $\nabla_X (e^f g)$. Suppose $F(g, \omega) = F(e^f g, \eta) = \nabla$. We have

$$\nabla_X (e^f g) + \eta(X) e^f g = X(e^f) g + e^f \nabla_X g + \eta(X) e^f g$$

$$= \nabla f(X) e^f g + e^f \nabla_X g + \eta(X) e^f g = 0. \tag{4}$$

Therefore $\nabla_X g = -(df(X) + \eta(X))$. On the other hand $\nabla_X g + \omega(X) g = 0$. Therefore $\omega = \eta + df$. Set $\nabla = F(g, \omega)$ and show that $\nabla = F(e^f g, \eta)$ and $\nabla_X (e^f g) + \eta(X) e^f g = 0$. To calculate

$$\nabla_X (e^f g) + \eta(X) e^f g = e^f df(X) g + e^f \nabla_X g + (\omega(X) - df(X)) e^f g$$

$$= e^f (\nabla_X g + \omega(X) g) = 0. \tag{5}$$

Such a geometric structure was introduced by Weyl in 1922 in an attempt to unify gravity with electromagnetism [23]. Einstein believed till the end of his life in 1955 that the non-symmetric gravitational theory will be the right choice for the so-called theory of everything. But Eisenhart's generalized Riemannian spaces are important, because these spaces were fundamental in unifying the theory of gravitation and electromagnetism by Einstein who considered a manifold with a non-symmetric basic tensor [24].

3. Pseudo-Riemannian Weyl Geometry

Definition 6 ([25]). A pair (J, g) consisting of a complex structure J on M and a Hermitian metric g in the tangent bundle TM, a Riemannian metric g that is invariant under J, $g(JX, JY) = g(X, Y)$ for any vector fields X and Y on M. A Hermitian structure specifies in any tangent space T_pM the structure of a Hermitian vector space. A manifold with a Hermitian structure is called a Hermitian manifold.

Definition 7 ([25]). The most important class of Hermitian manifolds are Kähler manifolds. These are Hermitian manifolds for which the Hermitian form ω is closed: $d\omega = 0$. In this case the form ω is called a Kähler form. An almost Hermitian manifold whose associated can a form be of type (1, 1) is closed is naturally called an almost Kähler manifold. Any symplectic manifold admits a compatible almost complex structure making it into an almost Kähler manifold.

Let ∇ be a torsion-free connection on a pseudo-Riemannian manifold (M, g) of even dimension $n = 2m \geq 4$. The triple (M, g, ∇) is said to be a Weyl structure if there exists a smooth 1-form $\varnothing$ so that $\nabla g = -2\varnothing \otimes g$.

Let $n = 2m \geq 4$. A triple $(M, g, J_{\pm})$ is said to be an almost pseudo/para-Hermitian manifold with an almost pseudo/para-complex structure $J_{\pm}$. Here, J_{+} is called para and J_{-} is called pseudo, if g is a pseudo/para-Riemannian metric on M of neutral signature (m, m) and if $J_{\pm}$ is an endomorphism of the tangent bundle TM so that $J_{\pm}^2 = Id$ and so that $J_{\pm}^* g = \mp g$; $(M, g, J_{\pm})$ is said to be pseudo/para-Hermitian with an integrable complex structure $J_{\pm}$ if the pseudo/para-Nijenhuis tensor

$$N_{\pm}(x, y) = [x, y] \mp J_{\pm}[J_{\pm}x, y] \mp J_{\pm}[x, J_{\pm}y] \pm [J_{\pm}x, J_{\pm}y]. \tag{6}$$

Let $(M, g, J_{\pm}, \nabla)$ be an almost pseudo/para-Hermitian Weyl manifold. If $\nabla(J_{\pm}) = 0,$ then one says that this is a (para)-Kähler Weyl manifold. Note that necessarily $J_{\pm}$ is integrable in this setting. $J_{\pm}$ is integrable means there exist local coordinates $(x^1, \ldots, x^{\overline{m}}, y^1, \ldots, y^{\overline{m}})$ so $J_{\pm}(\partial x_i) = \partial y_i$ and $J_{\pm}(\partial y_i) = \pm\partial x_i,$ i.e., the Nijenhuis tensor $N_{\pm} = 0.$ Let $\Omega_{\pm}(x, y) := g(x, J_{\pm}y)$ be the (para)-Kähler form. The following conditions are equivalent and if any is satisfied $(M, g, \nabla, J_{\pm})$ is said to be Kähler. $\nabla^g J_{\pm} = 0, d\Omega_{\pm} = 0$ and $J_{\pm}$ is integrable and given $p \in M.$

Theorem 3 ([26])**.** (a) *If*

$$\widetilde{g} = e^{2f} g \tag{7}$$

is a conformally equivalent positive definite metric, then $(M, \widetilde{g}, \nabla)$ *is a Weyl manifold with associated* 1*-form* $\widetilde{\varnothing} = \varnothing - df.$ *If* ∇g *is the Levi-Civita connection, we may then express* $\nabla = \nabla^{\varnothing}$ *in the form*:

$$\nabla_x^{\varnothing} y = \nabla_x^g y + \varnothing(x)y + \varnothing(y)x - g(x, y)\varnothing^*, \tag{8}$$

where $\widetilde{\varnothing}^*$ *is the dual vector field. Thus* $\varnothing$ *determines* $\nabla.$ *Conversely, if* $\varnothing$ *is given and if we use Equation* (8) *to define* $\nabla,$ *then* ∇ *is a Weyl connection with associated* 1*-form* $\varnothing.$

(b) $(M, g, J_{\pm}, \nabla)$ *is Kähler-Weyl and* $m \geq 6,$ *then the Weyl structure is trivial, i.e., there is a conformally equivalent metric so that* $(M, \widetilde{g}, J_{\pm})$ *is Kähler.*

(c) *Let* $m = 4.$ If $(M, g, J_{\pm})$ *is pseudo/para-Hermitian, then there is a unique torsion free connection* ∇ *so* $(M, g, J_{\pm}, \nabla)$ *is Kähler Weyl*; *here* $\varnothing = \pm \frac{1}{2} J_{\pm}^* \delta\Omega_{\pm}.$

A symplectic form on a smooth manifold M is a smooth closed 2-form omega on M which is nondegenerate such that at every point p, the alternating bilinear form ω_p on the tangent space T_pM is non-degenerate.

4. Generalized (Para)-Kähler-Weyl Space Form

Definition 8 ([21, 28])**.** A Kähler form ω is a real closed non-degenerate symplectic two-form for which $\omega(X, JX) > 0$ for nonzero tangent vectors $X \in T_pM$. It must also satisfy $\omega(JX, JY) = \omega(X, Y)$, where J is the almost complex structure induced by multiplication by almost complex structure $(\mathbf{i}^2 = -1)$. That is,

$$J_{\pm}\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad J_{\pm}\left(\frac{\partial}{\partial y_i}\right) = \pm\frac{\partial}{\partial x_i},\ i = 1, 2. \tag{9}$$

The dual endomorphism $J_{\pm}^{*}$ of $T_P^{*}(M)$ at any point P of a manifold M satisfies $J_{\pm}^{*2} = \pm Id$ and it is defined by

$$J_{\pm}^{*}(dx_i) = dy_i, \quad J_{\pm}^{*}(dy_i) = \pm dx_i,\ i = 1, 2. \tag{10}$$

Proposition 2 ([29])**.** (a) *A manifold with a Weyl structure is known as a Weyl manifold. Let's we use* (7) *to get base structures. If we rewrite Equation* (9) *with conformal structure, we obtain the following equations*:

$$J_{\pm}\left(\frac{\partial}{\partial x_i}\right) = e^{2f}\frac{\partial}{\partial y_i}, \quad J_{\pm}\left(\frac{\partial}{\partial y_i}\right) = \pm e^{-2f}\frac{\partial}{\partial x_i}, \tag{11}$$

that are base structures for Euler-Lagrange equations, where $J_{\pm}$ is a conformal (*para*)*-complex structure to an integrable almost* (*para*)*-complex $J_{\pm}$ given as the above. f second structure was chosen the minus sign. Because the condition of the structure required to provide $J_{\pm}^2 = \pm Id$.*

(b) *Similarly* $J_{\pm}^{*}$ *the structures being the dual of* $J_{\pm}$ *structures are given with conformal structure as follows*:

$$J_{\pm}^{*}(dx_i) = e^{2f} dy_i, \quad J_{\pm}^{*}(dy_i) = \pm e^{-2f} dx_i, \tag{12}$$

are base structures for Hamilton equations. We continue our studies thinking of the $(M, g, \nabla, J_{\pm})$ *and* $(M, g, \nabla, J_{\pm}^{*})$ *instead of the almost para/pseudo-Kähler Weyl manifolds* $(M, g, \nabla, J_{\pm})$, *and also its dual* $(M, g, \nabla, J_{\pm}^{*})$, *respectively.*

Proof.

(a) Let's examine these structures holomorphic property

$$J_{\pm}^{2}\left(\frac{\partial}{\partial x_i}\right) = J_{\pm} \circ J_{\pm}\left(\frac{\partial}{\partial x_i}\right) = J_{\pm}\left(e^{2f}\frac{\partial}{\partial y_i}\right)$$

$$= e^{2f} J_{\pm}\left(\frac{\partial}{\partial y_i}\right) = \pm e^{2f} e^{-2f}\frac{\partial}{\partial x_i} = \pm\frac{\partial}{\partial x_i},$$

$$J_{\pm}^{2}\left(\frac{\partial}{\partial y_i}\right) = J_{\pm} \circ J_{\pm}\left(\frac{\partial}{\partial y_i}\right) = J_{\pm}\left(\pm e^{-2f}\frac{\partial}{\partial x_i}\right)$$

$$= \pm e^{-2f} J_{\pm}\left(\frac{\partial}{\partial x_i}\right) = \pm e^{-2f} e^{2f}\frac{\partial}{\partial y_i} = \pm\frac{\partial}{\partial y_i}. \tag{13}$$

(b) Similarly,

$$J_{\pm}^{*2}(dx_i) = J_{\pm}^{*} \circ J_{\pm}^{*}(dx_i) = J_{\pm}^{*}(e^{2f} dy_i)$$

$$= e^{2f} J_{\pm}^{*}(dy_i) = \pm e^{2f} e^{-2f} dx_i = \pm dx_i,$$

$$J_{\pm}^{*2}(dy_i) = J_{\pm}^{*} \circ J_{\pm}^{*}(dy_i) = J_{\pm}^{*}(\pm e^{-2f} dx_i)$$

$$= \pm e^{-2f} J_{\pm}^{*}(dx_i) = \pm e^{2f} e^{-2f} dy_i = \pm dy_i. \tag{14}$$

As we have seen, $J_{\pm}^{2} = \pm I$ are the para(pseudo) complex structures type.

4.1. (Euler)-Lagrange and Hamilton dynamics equations

Lagrange and Hamilton dynamic equations ([27]) were introduced by Klein in 1962 and they are shown as follows.

(a) Lagrange dynamic equation: Let M be an n-dimensional manifold and TM be its tangent bundle with canonical projection $\tau_M : TM \to M$. TM is called the phase space of velocities of the base manifold M. Let $L : TM \to \mathbb{R}$ be a differentiable function on TM called a *Lagrangian function*. We consider the closed 2-form on TM given by $\Phi_L = -d(d_J L)$. Lagrange dynamic equation is

$$i_\xi \Phi_L = dE_L. \tag{15}$$

Then ξ is a vector field, we shall see that (15) under a certain condition on ξ is the intrinsical expression of the Euler-Lagrange equations of motion. This equation is named as *Lagrange dynamical equation*. We shall see that for motion in a potential, $E_L = V(L) - L$ is an energy function and $V = J(\xi)$ a Liouville vector field. Here dE_L denotes the differential of E. The triple (TM, Φ_L, ξ) is known as *Lagrangian system* on the tangent bundle TM. If it is continued the operations on (15) for any coordinate system $(q^i(t), p_i(t))$, infinite dimension *Lagrange's equation* is obtained the form below:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i}, \frac{\partial q^i}{\partial t} = \dot{q}^i, \, i = 1, \ldots, n \qquad \text{(a proof see in [30])} \tag{16}$$

(b) Hamilton dynamic equation: Let M is a base manifold and its cotangent manifold T^*M. By a *symplectic form* we mean a 2-form Φ on T^*M. Let (T^*M, Φ) be a symplectic manifold. A vector field $X_H : T^*M \to TT^*M$ is called *Hamiltonian function* if there exists a C^1

function $H : T^*M \to \mathbb{R}$ such that *Hamilton dynamical equation* is determined by

$$i_{X_H}\Phi = dH. \tag{17}$$

We say that X_H is a locally Hamiltonian vector field. The 2-form Φ is closed and it is a canonical symplectic form given by $\Phi = -\, d\Omega$, where $\Omega = J^*(\omega)$, J^* is a dual of J, and ω is a 1-form on T^*M. The triple (T^*M, Φ, X_H) is called a *Hamiltonian system* which is defined on the cotangent bundle T^*M. In Darboux coordinates $(q^i(t), p_i(t))$ we have $\omega = dq^i \wedge dp_i$ and $X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$ so that the integral curves $(q^i(t), p_i(t))$ is an integral curve of X_H satisfy the Hamilton equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}. \qquad \text{(proof see [30]) (18)}$$

5. Weyl-Euler-Lagrangian Mechanics Systems

Theorem 4. *Let J be a para-complex or complex structure on the tangent space $T_p(M)$ of manifold M at the point $p \in M$. If the J base structure is known or can be found, the Euler-Lagrange equations of motion can be obtained without using the dynamic equation* (15) *for mechanical systems on the defined manifold of moving objects. The way to find the time-dependent Euler-Lagrange motion equations that can be obtained for mechanical systems without any calculations. In short, if the base structure is* $J\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_2}$, *the equation is* $\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_2}\right) = \frac{\partial L}{\partial x_1}$ *or* $\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_2}\right) - \frac{\partial L}{\partial x_1} = 0.$ *The transactions made here are as follows*: (a) *The Lagrangian function "L" is added to the base vectors of defined space*

$\frac{\partial}{\partial x_1} \to \frac{\partial L}{\partial x_1}$ *and their location changes* $J\left(\frac{\partial L}{\partial x_2}\right) = \frac{\partial L}{\partial x_1}$, (b) *Instead of the "J" holomorphic structure,* $\frac{\partial}{\partial t}$ *expressing the derivative by time is introduced* $\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_2}\right) = \frac{\partial L}{\partial x_1}$.

For example: Base structures are (1) $J\left(\frac{\partial}{\partial z^i}\right) = -\mathbf{j}\frac{\partial}{\partial z^i}$ and (2) $J\left(\frac{\partial}{\partial \bar{z}^i}\right) = \mathbf{j}\frac{\partial}{\partial \bar{z}^i}$. Equations are (1) $\mathbf{j}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial z^i}\right) + \frac{\partial L}{\partial z^i} = 0$ and (2) $\mathbf{j}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{z}^i}\right) + \frac{\partial L}{\partial \dot{z}^i} = 02$ [4]. □

Example. The motion equations for (11) are, according to Hamilton dynamical equation and Theorem 4 for a conformal (para)-complex structure to an integrable almost (para)-complex $J_{\pm}$,

$$(1) \qquad \frac{\partial}{\partial t}\left(e^{2f}\frac{\partial L}{\partial y_i}\right) - \frac{\partial L}{\partial x_i} = 0,$$

$$(2) \qquad \frac{\partial}{\partial t}\left(e^{-2f}\frac{\partial L}{\partial x_i}\right) \mp \frac{\partial L}{\partial y_i} = 0. \qquad (19)$$

The differential equations (19) are called conformal Weyl-Euler-Lagrange equations on a generalized (para)-Kählerian space form $(M, g, \nabla, J_{\pm})$. Finally, it may be said that the triple (M, Φ_L, ξ) is conformal Weyl-Euler-Lagrange mechanical system on a generalized (para)-Kählerian manifold of constant $J_{\pm}$-sectional curvature $(M, g, \nabla, J_{\pm})$.

Proof 1. We introduce the Weyl-Euler-Lagrange equations on a generalized (para)-Kählerian space form $(M, g, \nabla, J_{\pm})$ using dynamic equation (15).

First of all, let $J_{\pm}$ almost generalized (para) product structure and by (x_i, y_i) the coordinates of M. Also, on $(M, g, \nabla, J_{\pm})$, let semispray be a vector field given by

$$\xi = X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}, \; X_i = \dot{x} = y_i, \; Y_i = \dot{y}_i. \tag{20}$$

By Liouville vector field on a generalized (para)-Kählerian space form $(M, g, J_{\pm})$, we use the vector field determined by $V_{\pm} = J_{\pm}(\xi)$ and counted by

$$V_{\pm} = X_i e^{2f} \frac{\partial}{\partial y_i} \pm Y_i e^{-2f} \frac{\partial}{\partial x_i}. \tag{21}$$

Denote T by *the kinetic energy* and P by *the potential energy of mechanics system* on generalized (para)-Kählerian space of constant $J_{\pm}$-sectional curvature. So, we write by $L = T - P$ *a Lagrangian function* and by $E_L = V(L) - L$ *the energy function* associated with L. Operator $i_{J_{\pm}}$ defined by $i_{J_{\pm}} : \wedge^2 M \to \wedge^1 M$, is known as the interior product with $J_{\pm}$, or sometimes the insertion operator, or contraction by $J_{\pm}$. The exterior vertical derivation $d_{J_{\pm}}$ is determined by

$$\mathbf{d}_{J_{\pm}} = [i_{J_{\pm}}, d] = i_{J_{\pm}} d - d i_{J_{\pm}}, \tag{22}$$

where d is the usual exterior derivation. For an almost product structure $J_{\pm}$ determined by (11), the closed generalized (para)-Kählerian form is the 2-form given by $\Phi_L = -d(\mathbf{d}_{J_{\pm}} L)$ such that

$$d_{J_{\pm}} = e^{2f} \frac{\partial}{\partial y_i} dx_i \pm e^{-2f} \frac{\partial}{\partial x_i} dy_i, \; d_{J_{\pm}} : \mathcal{F}(M) \to \wedge^1 M. \tag{23}$$

Because of the closed generalized (para)-Kählerian form Φ_L on a generalized (para)-Kählerian space form $(M, g, \nabla, J_{\pm})$ is a $\pm$ symplectic structure,

$$\begin{aligned}\Phi_L &= 2e^{2f}\frac{\partial f}{\partial x_i}\frac{\partial L}{\partial y_i}dx_i \wedge dx_i + e^{2f}\frac{\partial^2 L}{\partial x_i \partial y_i}dx_i \wedge dx_i \\ &\mp 2e^{-2f}\frac{\partial f}{\partial x_i}\frac{\partial L}{\partial x_i}dy_i \wedge dx_i \pm e^{-2f}\frac{\partial^2 L}{\partial x_i \partial x_i}dy_i \wedge dx_i \\ &+ 2e^{2f}\frac{\partial f}{\partial y_i}\frac{\partial L}{\partial y_i}dx_i \wedge dy_i + e^{2f}\frac{\partial^2 L}{\partial y_i \partial y_i}dx_i \wedge dy_j \\ &\mp 2e^{-2f}\frac{\partial f}{\partial y_i}\frac{\partial L}{\partial x_i}dy_i \wedge dy_j \pm e^{-2f}\frac{\partial^2 L}{\partial y_i \partial x_i}dy_i \wedge dy_i. \end{aligned} \tag{24}$$

Therefore,

$$\Phi_L(\xi) = \begin{bmatrix} 2e^{2f}\frac{\partial f}{\partial x_i}\frac{\partial L}{\partial y_i}X_i dx_i - 2e^{2f}\frac{\partial f}{\partial x_i}\frac{\partial L}{\partial y_i}X_i dx_i \\ + e^{2f}\frac{\partial^2 L}{\partial x_i \partial y_i}X_i dx_i - e^{2f}\frac{\partial^2 L}{\partial x_i \partial y_i}X_i dx_i \\ \pm 2e^{-2f}\frac{\partial f}{\partial x_i}\frac{\partial L}{\partial x_i}X_i dy_i \mp e^{-2f}\frac{\partial^2 L}{\partial x_i \partial x_i}X_i dy_i \\ + 2e^{2f}\frac{\partial f}{\partial y_i}\frac{\partial L}{\partial y_i}X_i dy_i + e^{2f}\frac{\partial^2 L}{\partial y_i \partial y_i}X_i dy_i \\ + 2e^{-2f}\frac{\partial f}{\partial x_i}\frac{\partial L}{\partial x_i}Y_i dx_i + e^{-2f}\frac{\partial^2 L}{\partial x_i \partial x_i}Y_i dx_i \\ - 2e^{2f}\frac{\partial f}{\partial y_i}\frac{\partial L}{\partial y_i}Y_i dx_i - e^{2f}\frac{\partial^2 L}{\partial y_i \partial y_i}Y_i dx_i \\ \mp 2e^{-2f}\frac{\partial f}{\partial y_i}\frac{\partial L}{\partial x_i}Y_i dy_i \pm 2e^{-2f}\frac{\partial f}{\partial y_i}\frac{\partial L}{\partial x_i}Y_i dy_i \\ \pm e^{-2f}\frac{\partial^2 L}{\partial y_i \partial x_i}Y_i dy_i \mp e^{-2f}\frac{\partial^2 L}{\partial y_i \partial x_i}Y_i dy_i \end{bmatrix}. \tag{25}$$

Energy of system is calculated as follows:

$$E_L = V_{\pm}(L) - L = X_i e^{2f} \frac{\partial L}{\partial y_i} \pm Y_i e^{-2f} \frac{\partial L}{\partial x_i} - L, \tag{26}$$

and

$$\begin{aligned} dE_L &= X_i 2e^{2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial y_i} dx_i + X_i e^{2f} \frac{\partial^2 L}{\partial x_i \partial y_i} dx_i \\ &\mp 2Y_i e^{-2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_i} dx_i \pm Y_i e^{-2f} \frac{\partial^2 L}{\partial x_i \partial x_i} dx_i - \frac{\partial L}{\partial x_i} dx_i \\ &+ X_i 2e^{2f} \frac{\partial f}{\partial y_i} \frac{\partial L}{\partial y_i} dy_i + X_i e^{2f} \frac{\partial^2 L}{\partial y_i \partial y_i} dy_i \\ &\mp Y_i 2e^{-2f} \frac{\partial f}{\partial y_i} \frac{\partial L}{\partial x_i} dy_i \pm Y_i e^{-2f} \frac{\partial^2 L}{\partial y_i \partial x_i} dy_i - \frac{\partial L}{\partial y_i} dy_i. \end{aligned} \tag{27}$$

By taking into account $i_\xi \Phi_L = dE_L$ and the curve α, an integral curve of ξ, we obtain the equations determined by

$$\begin{aligned} &- 2e^{2f}\left[X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}\right]\left(\frac{\partial L}{\partial y_i}\right) - e^{2f}\left[X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}\right]\left(\frac{\partial L}{\partial y_i}\right) + \frac{\partial L}{\partial x_i} = 0, \\ &\pm 2e^{-2f}\left[X_i \frac{\partial f}{\partial x_i} + Y_i \frac{\partial f}{\partial y_i}\right]\left(\frac{\partial L}{\partial x_i}\right) \mp e^{-2f}\left[X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}\right]\left(\frac{\partial L}{\partial x_i}\right) + \frac{\partial L}{\partial y_i} = 0, \end{aligned} \tag{28}$$

or alternatively

$$\begin{aligned} &- 2e^{2f}\xi(f) \frac{\partial L}{\partial y_i} - e^{2f}\xi\left(\frac{\partial L}{\partial y_i}\right) + \frac{\partial L}{\partial x_i} = 0, \\ &\pm 2e^{-2f}\xi(f) \frac{\partial L}{\partial x_i} \mp e^{-2f}\xi\left(\frac{\partial L}{\partial x_i}\right) + \frac{\partial L}{\partial y_i} = 0. \end{aligned} \tag{29}$$

Also, by taking into account $\xi(\alpha(t)) = \dot{\alpha}(t) = \frac{\partial \alpha}{\partial t}$ we get equations as follows:

$$- 2e^{2f} \frac{\partial f}{\partial t} \frac{\partial L}{\partial y_i} - e^{2f} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial y_i}\right) + \frac{\partial L}{\partial x_i} = 0,$$

$$\pm 2e^{-2f} \frac{\partial f}{\partial t} \frac{\partial L}{\partial x_i} \mp e^{-2f} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right) + \frac{\partial L}{\partial y_i} = 0, \tag{30}$$

or shortly

$$(1) \qquad \frac{\partial}{\partial t}\left(e^{2f} \frac{\partial L}{\partial y_i}\right) - \frac{\partial L}{\partial x_i} = \mathbf{0},$$

$$(2) \qquad \frac{\partial}{\partial t}\left(e^{-2f} \frac{\partial L}{\partial x_i}\right) \mp \frac{\partial L}{\partial y_i} = \mathbf{0}, \tag{31}$$

such that the differential equations (19) are named *conformal Weyl-Euler-Lagrange equations* on a generalized (para)-Kählerian space form $(M, g, \nabla, J_\pm)$. Finally, it may be said that the triple (M, Φ_L, ξ) is conformal Weyl-Euler-Lagrange mechanical system on generalized (para)-Kählerian manifold of constant $J_\pm$ -sectional curvature $(M, g, \nabla, J_\pm)$.

Proof 2. Let's use the proposed Theorem 4 for the second proof.

(1) First, let's consider holomorphic structures (11).

$$J_\pm\left(\frac{\partial}{\partial x_i}\right) = e^{2f} \frac{\partial}{\partial y_i}, \quad J_\pm\left(\frac{\partial}{\partial y_i}\right) = \pm e^{-2f} \frac{\partial}{\partial x_i}. \tag{32}$$

(2) "L" of the Lagrangian function is added to derivative expressions,

$$J_\pm\left(\frac{\partial L}{\partial x_i}\right) = e^{2f} \frac{\partial L}{\partial y_i}, \quad J_\pm\left(\frac{\partial L}{\partial y_i}\right) = \pm e^{-2f} \frac{\partial L}{\partial x_i}. \tag{33}$$

(3) The places of the derivative expressions on both sides of the equality are completely displaced.

$$J_\pm\left(e^{2f} \frac{\partial L}{\partial y_i}\right) = \frac{\partial L}{\partial x_i}, \quad J_\pm\left(\pm\, e^{-2f} \frac{\partial L}{\partial x_i}\right) = \frac{\partial L}{\partial y_i}. \tag{34}$$

(4) In this step, $\left(\frac{\partial}{\partial t}\right)$ is written instead of $J_{\pm}$,

$$\frac{\partial}{\partial t}\left(e^{2f}\,\frac{\partial L}{\partial y_i}\right) = \frac{\partial L}{\partial x_i}, \quad \frac{\partial}{\partial t}\left(\pm\, e^{-2f}\,\frac{\partial L}{\partial x_i}\right) = \frac{\partial L}{\partial y_i}. \tag{35}$$

(5) If the distribution property is applied here, the equations are found as follows:

$$\frac{\partial}{\partial t}\left(e^{2f}\,\frac{\partial L}{\partial y_i}\right) - \frac{\partial L}{\partial x_i} = 0, \quad \frac{\partial}{\partial t}\left(e^{-2f}\,\frac{\partial L}{\partial x_i}\right) \mp \frac{\partial L}{\partial y_i} = 0. \tag{36}$$

So the proof is complete. □

6. Weyl-Hamiltonian Mechanics Systems

Theorem 5. *Let J be a para-complex or complex structures on the tangent space $T_p(M)$. Let J^* be which is dual to J. J^* be defined that a para-complex or complex structures of dual J, a Liouville form and a* 1*-form on definition manifolds are indicated by J^*, Ω and ω, respectively. If the dual endomorphism J^* of the cotangent space and Liouville form $\Omega = J^*(\omega)$, then Hamilton motion equations can be found as follows: Hamiltonian motion equations for $\lambda = \lambda(x, y) = \Omega(x, y)dx + \Psi(x, y)dy$ are*

$$\text{(1)} \qquad \frac{dx}{dt} = \frac{1}{\partial_y\Omega - \partial_x\Psi}\,\frac{\partial H}{\partial y},$$

$$\text{(2)} \qquad \frac{dy}{dt} = \frac{-1}{\partial_y\Omega - \partial_x\Psi}\,\frac{\partial H}{\partial x}. \tag{37}$$

For example: Base structures are (1) $J^*(z_i) = -\mathbf{j}dz_i$ and (2) $J^*(\bar{z}_i) = \mathbf{j}d\bar{z}_i$ for $\lambda = J^*(\omega) = \frac{1}{2}\mathbf{j}(z_i d\bar{z}_i - \bar{z}_i dz_i)$, $\Omega = \frac{1}{2}\mathbf{j}z_i$, $\Psi = -\frac{1}{2}\mathbf{j}\bar{z}_i$, $\partial_{z_i}\Omega = \frac{1}{2}\mathbf{j}$, $\partial_{\bar{z}_i}\Omega = -\frac{1}{2}\mathbf{j}$, $\partial_y\Omega - \partial_x\Psi = \mathbf{j}$, $\mathbf{j}^2 = 1$. Equations are (1) $\frac{dz_i}{dt} = -\mathbf{j}\frac{dH}{d\bar{z}_i}$ and (2) $\frac{d\bar{z}_i}{dt} = \mathbf{j}\frac{dH}{dz_i}$ [4]. □

Example. In this study, Weyl-Hamilton equations are on a generalized (para)-Kählerian space form $(M, g, \nabla, J_{\pm}^{*})$ using Theorem 5 for

$$\Omega_{\pm} = -\frac{1}{2}(-e^{2f}x_i dy_i \pm e^{-2f}y_i dx_i)$$

$$= \frac{1}{2}(\mp e^{-2f}y_i dx_i + e^{2f}x_i dy_i); \tag{38}$$

(3)
$$\frac{dx_i}{dt} = \frac{-2}{\left[e^{2f} + 2x_i\frac{\partial f}{\partial x_i} + e^{-2f} - 2x_i\frac{\partial f}{\partial y_i}\right]}\frac{\partial H}{\partial y_i},$$

(4)
$$\frac{dy_i}{dt} = \frac{2}{\left[e^{2f} + 2x_i\frac{\partial f}{\partial x_i} + e^{-2f} - 2x_i\frac{\partial f}{\partial y_i}\right]}\frac{\partial H}{\partial x_i}. \tag{39}$$

Hence, the differential equations presented in (39) are named *conformal Weyl-Hamilton equations* on the almost pseudo/para-Kähler Weyl manifolds $(M, g, \nabla, J_{\pm}^{*})$. After that the triple $(M, \Phi_{\pm}, X_H)$ is said to be a *conformal Weyl-Hamiltonian mechanical system* on $(M, g, \nabla, J_{\pm}^{*})$.

Proof 1. We provide using the dynamical equation (17) with information about Weyl-Hamilton equations on a generalized (para)-Kählerian space form $(M, g, \nabla, J_{\pm}^{*})$, of constant $J_{\pm}^{*}$-sectional curvature:

$$J_{\pm}^{*}(dx_i) = e^{2f}dy_i,$$

$$J_{\pm}^{*}(dy_i) = \pm e^{-2f}dx_i. \tag{40}$$

Let $J_{\pm}^{*}$ be an almost generalized (para)-product structure defined by (12) and $\Omega_{\pm}$ Liouville form determined by

$$\Omega_{\pm} = J_{\pm}^{*}(\omega_{\pm}) = -\frac{1}{2}(-e^{2f}x_i dy_i \pm e^{-2f}y_i dx_i) \tag{41}$$

such that

$$\omega_{\pm} = -\frac{1}{2}(-x_i dx_i \pm y_i dy_i) \tag{42}$$

is a 1-form on M [29], where $y_i = \dfrac{\partial L}{\partial \dot{x}_i}$. If $\Phi_{\pm} = -d\Omega_{\pm}$ is a generalized closed (para)-Kählerian form, then it is also a generalized (para)-symplectic structure on M.

Let $(M, g, \nabla, J_{\pm}^{*})$ be a generalized (para)-Kählerian space form, of constant $J_{\pm}^{*}$-sectional curvature, with a generalized closed (para)-Kählerian form $\Phi_{\pm}$.

Let us think that a Hamiltonian vector field X_H associated with Hamiltonian energy H is given by

$$X_H = X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}. \tag{43}$$

For the generalized closed (para)-Kählerian form $\Phi_{\pm}$ on M, we have

$$\begin{aligned}\Phi_{\pm} &= -\frac{1}{2}d(x_i e^{2f}dy_i \pm y_i e^{-2f}dx_i)\\ &= -\frac{1}{2}\left\{\begin{array}{c} e^{2f}\dfrac{\partial x_i}{\partial x_j}dx_j \wedge dy_i + 2x_i e^{2f}\dfrac{\partial f}{\partial x_j}dx_j \wedge dy_i \\ \pm e^{-2f}\dfrac{\partial y_i}{\partial x_j}dx_j \wedge dx_i \mp 2y_i e^{-2f}\dfrac{\partial f}{\partial x_j}dx_j \wedge dx_i \\ + e^{2f}\dfrac{\partial x_i}{\partial y_j}dy_j \wedge dy_i + 2x_i e^{2f}\dfrac{\partial f}{\partial y_j}dy_j \wedge dy_i \\ \pm e^{-2f}\dfrac{\partial y_i}{\partial y_j}dy_j \wedge dx_i \mp 2y_i e^{-2f}\dfrac{\partial f}{\partial y_j}dy_j \wedge dx_i \end{array}\right\}.\end{aligned} \tag{44}$$

Then it goes like the following:

$$i_{X_H}\Phi_{\pm} = \Phi_{\pm}(X_H)$$

$$= -\frac{1}{2}\left\{\begin{array}{c} e^{2f}X_i\left[dx_i \frac{\partial}{\partial x_i} dy_i - dy_i \frac{\partial}{\partial x_i} dx_i\right] \\ + 2x_i e^{2f} \frac{\partial f}{\partial x_i} X_i\left[dx_i \frac{\partial}{\partial x_i} dy_i - dy_i \frac{\partial}{\partial x_i} dx_i\right] \\ \pm e^{-2f}X_i\left[dy_i \frac{\partial}{\partial x_i} dx_i - dx_i \frac{\partial}{\partial x_i} dy_i\right] \\ \mp 2x_i e^{-2f} \frac{\partial f}{\partial y_i} X_i\left[dy_i \frac{\partial}{\partial x_i} dx_i - dx_i \frac{\partial}{\partial x_i} dy_i\right] \\ e^{2f}Y_i\left[dx_i \frac{\partial}{\partial y_i} dy_i - dy_i \frac{\partial}{\partial y_i} dx_i\right] \\ + 2x_i e^{2f} \frac{\partial f}{\partial x_i} Y_i\left[dx_i \frac{\partial}{\partial y_i} dy_i - dy_i \frac{\partial}{\partial y_i} dx_i\right] \\ \pm e^{-2f}Y_i\left[dy_i \frac{\partial}{\partial y_i} dx_i - dx_i \frac{\partial}{\partial y_i} dy_i\right] \\ \mp 2x_i e^{-2f} \frac{\partial f}{\partial y_i} Y_i\left[dy_i \frac{\partial}{\partial y_i} dx_i - dx_i \frac{\partial}{\partial y_i} dy_i\right] \end{array}\right\}. \tag{45}$$

From (45) with respect to $i_{X_H}\Phi_{\pm} = dH$, we obtain generalized (para)-Hamiltonian vector field on generalized (para)-Kählerian space of constant $J_{\pm}^{*}$-sectional curvature.

Let $(M, g, \nabla, J_{\pm}^{*})$ be a generalized (para)-Kählerian space form, of constant $J_{\pm}^{*}$-sectional curvature, with a generalized closed (para)-Kählerian form $\Phi_{\pm}$. Moreover, the differential of Hamiltonian energy H is gained by

$$dH = \frac{\partial H}{\partial x_i} dx_i + \frac{\partial H}{\partial y_i} dy_i. \tag{46}$$

Using the equation $i_{X_H}\Phi = dH$, the following statement is reached:

$$-\frac{1}{2}\left\{\begin{array}{c} e^{2f}X_i dy_i + 2x_i e^{2f}\dfrac{\partial f}{\partial x_i}X_i dy_i \\ \pm e^{-2f}X_i dy_i \mp 2x_i e^{-2f}\dfrac{\partial f}{\partial y_i}X_i dy_i \\ -e^{2f}Y_i dx_i - 2x_i e^{2f}\dfrac{\partial f}{\partial x_i}Y_i dx_i \\ \pm e^{-2f}Y_i dx_i \mp 2x_i e^{-2f}\dfrac{\partial f}{\partial y_i}Y_i dx_i \end{array}\right\} = \frac{\partial H}{\partial x_i}dx_i + \frac{\partial H}{\partial y_i}dy_i. \tag{47}$$

So, a generalized (para)-Hamiltonian vector field X_H associated with Hamiltonian energy H is decided by using (17), the Hamilton vector field is calculated

$$X_i = \frac{-2}{[e^{2f} + 2x_i e^{2f}\dfrac{\partial f}{\partial x_i} \pm e^{-2f} \mp 2x_i e^{-2f}\dfrac{\partial f}{\partial y_i}]}\frac{\partial H}{\partial y_i},$$

$$Y_i = \frac{2}{[e^{2f} + 2x_i e^{2f}\dfrac{\partial f}{\partial x_i} \pm e^{-2f} \mp 2x_i e^{-2f}\dfrac{\partial f}{\partial y_i}]}\frac{\partial H}{\partial x_i}.$$

Let $\alpha : I \subset \mathbb{R} \to M$ be an integral curve of a generalized (para)-Hamiltonian vector field X_H, i.e., $X_H(\alpha(t)) = \dot{\alpha}$, $t \in I$.

$$X_H = \frac{-2}{[e^{2f} + 2x_i e^{2f}\dfrac{\partial f}{\partial x_i} \pm e^{-2f} \mp 2x_i e^{-2f}\dfrac{\partial f}{\partial y_i}]}\frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i}$$

$$+ \frac{2}{[e^{2f} + 2x_i e^{2f}\dfrac{\partial f}{\partial x_i} \pm e^{-2f} \mp 2x_i e^{-2f}\dfrac{\partial f}{\partial y_i}]}\frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i}. \tag{48}$$

In the local coordinates we have $\alpha(t) = (x_i(t),\ y_i(t))$, and

$$\dot{\alpha}(t) = \frac{\partial \alpha}{\partial t} = \frac{dx_i}{dt}\frac{\partial}{\partial x_i} + \frac{dy_i}{dt}\frac{\partial}{\partial y_i}. \tag{49}$$

$$\frac{-2}{[e^{2f} + 2x_i e^{2f}\dfrac{\partial f}{\partial x_i} \pm e^{-2f} \mp 2x_i e^{-2f}\dfrac{\partial f}{\partial y_i}]}\frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i}$$

$$+\frac{2}{[e^{2f} + 2x_i e^{2f}\dfrac{\partial f}{\partial x_i} \pm e^{-2f} \mp 2x_i e^{-2f}\dfrac{\partial f}{\partial y_i}]}\frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i}$$

$$= \frac{dx_i}{dt}\frac{\partial}{\partial x_i} + \frac{dy_i}{dt}\frac{\partial}{\partial y_i}. \tag{50}$$

We infer the equations so-called conformal equations:

$$\frac{dx_i}{dt} = \frac{-2}{[e^{2f} + 2x_i e^{2f}\dfrac{\partial f}{\partial x_i} \pm e^{-2f} \mp 2x_i e^{-2f}\dfrac{\partial f}{\partial y_i}]}\frac{\partial H}{\partial y_i},$$

$$\frac{dy_i}{dt} = \frac{2}{[e^{2f} + 2x_i e^{2f}\dfrac{\partial f}{\partial x_i} \pm e^{-2f} \mp 2x_i e^{-2f}\dfrac{\partial f}{\partial y_i}]}\frac{\partial H}{\partial x_i}. \tag{51}$$

Proof 2. We use Theorem 5 for proof [$\lambda = \lambda(x, y) = \Omega(x, y)dx + \Psi(x, y)dy$, equations are (1) $\dfrac{dx}{dt} = \dfrac{1}{\partial_y\Omega - \partial_x\Psi}\dfrac{\partial H}{\partial y}$ and (2) $\dfrac{dy}{dt} = \dfrac{-1}{\partial_y\Omega - \partial_x\Psi}\dfrac{\partial H}{\partial x}$]. The following steps are followed for proof:

(1) $\lambda_{\pm} = -\dfrac{1}{2}(-e^{2f}x_i dy_i \pm e^{-2f}y_i dx_i)$

$= \dfrac{1}{2}(\mp e^{-2f}y_i dx_i + e^{2f}x_i dy_i)$,

(2) $\Omega = \mp\dfrac{1}{2}e^{-2f}y_i$ and $\Psi = \dfrac{1}{2}e^{2f}x_i$,

(3) $\partial_y \Omega = \frac{1}{2}\left(\mp e^{-2f} \pm 2y_i e^{-2f} \frac{\partial f}{\partial y_i}\right),$

$\partial_x \Psi = \frac{1}{2}(e^{2f} + 2e^{2f} x_i \frac{\partial f}{\partial x_i}),$

(4) $\partial_y \Omega - \partial_x \Psi = \frac{1}{2}(\mp e^{-2f} \pm 2e^{-2f} y_i \frac{\partial f}{\partial y_i})$

$-\frac{1}{2}\left(e^{2f} + 2e^{2f} x_i \frac{\partial f}{\partial x_i}\right)$

$$= \frac{1}{2}\left(\mp e^{-2f} \pm 2e^{-2f} y_i \frac{\partial f}{\partial y_i} - e^{2f} - 2e^{2f} x_i \frac{\partial f}{\partial x_i}\right), \tag{52}$$

(5) $\frac{dx}{dt} = \frac{1}{\partial_y \Omega - \partial_x \Psi}\frac{\partial H}{\partial y}$

$$= \frac{1}{\frac{1}{2}[\mp e^{-2f} \pm 2e^{-2f} y_i \frac{\partial f}{\partial y_i} - e^{2f} - 2e^{2f} x_i \frac{\partial f}{\partial x_i}]}\frac{\partial H}{\partial y},$$

(6) $\frac{dy}{dt} = \frac{-1}{\partial_y \Omega - \partial_x \Psi}\frac{\partial H}{\partial x}$

$$= \frac{-1}{\frac{1}{2}[\mp e^{-2f} \pm 2e^{-2f} y_i \frac{\partial f}{\partial y_i} - e^{2f} - 2e^{2f} x_i \frac{\partial f}{\partial x_i}]}\frac{\partial H}{\partial y},$$

Thus, the following equations are reached:

$$\frac{dx_i}{dt} = \frac{-2}{[e^{2f} + 2x_i e^{2f} \frac{\partial f}{\partial x_i} \pm e^{-2f} \mp 2x_i e^{-2f} \frac{\partial f}{\partial y_i}]}\frac{\partial H}{\partial y_i},$$

$$\frac{dy_i}{dt} = \frac{2}{[e^{2f} + 2x_i e^{2f} \frac{\partial f}{\partial x_i} \pm e^{-2f} \mp 2x_i e^{-2f} \frac{\partial f}{\partial y_i}]}\frac{\partial H}{\partial x_i}. \tag{53}$$

So the proof is complete. □

7. Conclusion

We derived the following conclusions:

(1) By using Weyl metric (7) the base structures given in (9) and (10) should be transferred to (11) and (12).

(2) Conformal Weyl-Euler-Lagrange equations or partial differential equations (19) found by using the Theorem 4 with a generalized (para)-Kählerian space form $(M, g, \nabla, J_{\pm})$.

(3) Conformal Weyl-Hamilton equations or partial differential equations (39) found by using the Theorem 5 with a generalized (para)-Kählerian space form $(M, g, \nabla, J_{\pm}^{*})$.

(4) It is seen that equations can be found without errors and without need to use the dynamic equations (15) and (17) for calculation of movement equations by these theorems.

(5) In this way, researchers can find the Lagrangian and Hamiltonian motion equations on any space/manifold with Theorems 4 and 5 correctly without need calculating using long operations.

(6) In future studies, researchers may be advised to use Theorems 4 and 5 for to know the route and their requirements of mechanical systems in defined space/manifold.

8. Discussion

(1) The geodesics of objects moving on the almost pseudo/para-Kähler Weyl manifolds were modelled using differential equations (19) and (39). These geodesics can be used for the positions and requirements of objects moving in space by the time.

(2) This study raised geodesics for about the non-linear orbits of the objects in the space by the help of revised features using Weyl's unified theorem on the almost pseudo/para-Kähler Weyl manifolds.

(3) The obtained equations are very important to explain problems about physics and other related fields. Because in Weyl's unified theory, the metrics are interpreted as the gravitational potential, as in general relativity, and the corresponding forms are interpreted as the electromagnetic potentials [31].

(4) In future studies, researchers may be advised an explanation of the rotational spatial mechanical-physical problems on the almost pseudo/para-Kähler Weyl manifolds.

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